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The Gauss Map and the Third Laplace-Beltrami Operator of the Rotational Hypersurface in 4-Space

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Abstract: We study and examine the rotational hypersurface and its Gauss map in Euclidean four-space \mathbb{E}^4 . We calculate the Gauss map, the mean curvature and the Gaussian curvature of the rotational hypersurface and obtain some results. Then, we introduce the third Laplace–Beltrami operator. Moreover, we calculate the third Laplace–Beltrami operator of the rotational hypersurface in \mathbb{E}^4 . We also draw some figures of the rotational hypersurface.

Keywords: four-space; rotational hypersurface; Gauss map; Gaussian curvature; mean curvature; the third Laplace–Beltrami operator

1. Introduction

When we focus on the rotational characters in the literature, we meet Arslan et al. [1,2], Arvanitoyeorgos et al. [3], Chen [4,5], Dursun and Turgay [6], Kim and Turgay [7], Takahashi [8], and many others.

Magid, Scharlach and Vrancken [9] introduced the affine umbilical surfaces in four-space. Vlachos [10] considered hypersurfaces in \mathbb{E}^4 with the harmonic mean curvature vector field. Scharlach [11] studied the affine geometry of surfaces and hypersurfaces in four-space. Cheng and Wan [12] considered complete hypersurfaces of four-space with constant mean curvature.

General rotational surfaces in \mathbb{E}^4 were introduced by Moore [13,14]. Ganchev and Milousheva [15] considered these kinds of surfaces in the Minkowski four-space. They classified completely the minimal rotational surfaces and those consisting of parabolic points. Arslan et al. [2] studied generalized rotation surfaces in \mathbb{E}^4 . Moreover, Dursun and Turgay [6] studied minimal and pseudo-umbilical rotational surfaces in \mathbb{E}^4 .

In [16], Dillen, Fastenakels and Van der Veken studied the rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and proved a criterion for a hypersurface of one of these spaces to be a rotation hypersurface. They classified minimal, flat rotation hypersurfaces and normally flat rotation hypersurfaces in the Euclidean and Lorentzian space containing $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, respectively. Senoussi and Bekkar [17] studied the Laplace operator using the fundamental forms I , II and III of the helicoidal surfaces in \mathbb{E}^3 .

For the characters of ruled (helicoid) and rotational surfaces, please see Bour's theorem in [18]. Do Carmo and Dajczer [19] showed the existence of surfaces isometric to helicoidal ones by using Bour's theorem [18]. Güler [20] studied a helicoidal surface with a light-like profile curve using Bour's theorem in Minkowski geometry. Furthermore, Hieu and Thang [21] studied helicoidal surfaces by Bour's theorem in four-space. Choi et al. [22] studied helicoidal surfaces and their Gauss map in Minkowski three-space. See also [23–26]. Güler, Magid and Yaylı [27] studied the Laplace–Beltrami operator of a helicoidal hypersurface in \mathbb{E}^4 .

In the present paper, we consider the rotational hypersurface with three-parameters and its Gauss map in Euclidean four-space \mathbb{E}^4 . We give some basic notions of the four-dimensional Euclidean geometry in Section 2. We give the definition of a rotational hypersurface, and then, we calculate the mean and the Gaussian curvatures of such a hypersurface in Section 3. In Section 4, we obtain the mean and the Gaussian curvatures of the Gauss map of the hypersurface. Moreover, we introduce the third Laplace–Beltrami operator and calculate it in \mathbb{E}^4 in Section 5. Finally, we give a conclusion in the last section.

2. Curvatures in \mathbb{E}^4

We identify a vector \vec{x} with its transpose. Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in \mathbb{E}^4 . The triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$ and $\vec{z} = (z_1, z_2, z_3, z_4)$ is defined by

$$\vec{x} \times \vec{y} \times \vec{z} = \begin{pmatrix} x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2 \\ -x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3z_1y_4 - y_1x_4z_3 + x_4y_3z_1 \\ x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1 \\ -x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1 \end{pmatrix}.$$

For a hypersurface \mathbf{M} in four-space, the first and the second fundamental form matrices are as follows: $\mathbf{I} = (g_{ij})_{3 \times 3}$, $\mathbf{II} = (h_{ij})_{3 \times 3}$, where $1 \leq i, j \leq 3$, $g_{11} = \mathbf{M}_u \cdot \mathbf{M}_u$, $g_{12} = \mathbf{M}_u \cdot \mathbf{M}_v$, ..., $g_{33} = \mathbf{M}_w \cdot \mathbf{M}_w$, $h_{11} = \mathbf{M}_{uu} \cdot \mathbf{e}$, $h_{12} = \mathbf{M}_{uv} \cdot \mathbf{e}$, ..., $h_{33} = \mathbf{M}_{ww} \cdot \mathbf{e}$, " · " means the Euclidean dot product; some partial differentials that we represent are $\mathbf{M}_u = \frac{\partial \mathbf{M}}{\partial u}$, $\mathbf{M}_{uw} = \frac{\partial^2 \mathbf{M}}{\partial u \partial w}$; and

$$\mathbf{e} = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|} \tag{1}$$

is the Gauss map (i.e., the unit normal vector). The product of the matrices $(g_{ij})^{-1}$ and (h_{ij}) gives the matrix of the shape operator $\mathbf{S} = \frac{1}{\det \mathbf{I}} (s_{ij})_{3 \times 3}$. Here, $g_{ij}^{-1} \cdot h_{ij} = \frac{s_{ij}}{\det \mathbf{I}}$ are the elements of \mathbf{S} . Therefore, the formulas of the Gaussian curvature and the mean curvature are given by

$$K = \det(\mathbf{S}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}, \tag{2}$$

and

$$H = \frac{1}{3} \text{tr}(\mathbf{S}), \tag{3}$$

respectively.

3. Rotational Hypersurface in \mathbb{E}^4

We define the rotational hypersurface in \mathbb{E}^4 . Let $\gamma : I \subset \mathbb{R} \rightarrow \Pi$ be a curve in a plane Π in \mathbb{E}^4 and ℓ be a line in Π .

Definition 1. A rotational hypersurface M^3 in \mathbb{E}^4 is defined by rotating a curve γ around a line ℓ . In this case, γ and ℓ are called the profile curve and the axis of M^3 , resp.

We now describe a rotational hypersurface M^3 of \mathbb{E}^4 more precisely. Without loss of generality, we may assume that the straight line ℓ is the line spanned by the vector $(0, 0, 0, 1)^t$. The orthogonal matrix $\mathbf{Z}(v, w)$ that fixes the above vector is

$$\mathbf{Z}(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v, w \in \mathbb{R}.$$

The matrix \mathbf{Z} can be found by solving the equations

$$\mathbf{Z}\ell = \ell, \quad \mathbf{Z}^t\mathbf{Z} = \mathbf{Z}\mathbf{Z}^t = I_4, \quad \det \mathbf{Z} = 1,$$

simultaneously. Since the axis of rotation ℓ is the x_4 -axis of \mathbb{E}^4 , the profile curve can be put as follows

$$\gamma(u) = (u, 0, 0, \varphi(u)),$$

where $\varphi(u)$ is a differentiable function, $u \in I$. Therefore, the rotational hypersurface, which is spanned by the vector $(0, 0, 0, 1)$ in \mathbb{E}^4 , is given by

$$\mathbf{R}(u, v, w) = \mathbf{Z}(v, w) \cdot \gamma(u)^t,$$

where $u, v, w \in \mathbb{R}$. Therefore, we obtain the parametrization of M^3

$$\mathbf{R}(u, v, w) = \begin{pmatrix} u \cos v \cos w \\ u \sin v \cos w \\ u \sin w \\ \varphi(u) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{4}$$

Theorem 1. *The Gaussian curvature K and the mean curvature H of the rotational hypersurface (4) are given as follows, respectively,*

$$K = -\frac{\varphi'^2 \varphi''}{u^3 (1 + \varphi'^2)^3}, \quad \text{and} \quad H = -\frac{u\varphi'' + 2\varphi'^3 + 2\varphi'}{3u (1 + \varphi'^2)^{3/2}},$$

where $u \in \mathbb{R}$, $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$.

Proof. Using the first partial differentials of (4) with respect to u, v, w , we get the first quantities as follows

$$\mathbf{I} = \begin{pmatrix} 1 + \varphi'^2 & 0 & 0 \\ 0 & u^2 \cos^2 w & 0 \\ 0 & 0 & u^2 \end{pmatrix}.$$

We have

$$\det \mathbf{I} = u^4 (1 + \varphi'^2) \cos^2 w.$$

Using the second partial differentials of (4) with respect to u, v, w , the second quantities are given as follows

$$\mathbf{II} = \begin{pmatrix} -\frac{\varphi''}{\mathbf{W}} & 0 & 0 \\ 0 & -\frac{u\varphi' \cos^2 w}{\mathbf{W}} & 0 \\ 0 & 0 & -\frac{u\varphi'}{\mathbf{W}} \end{pmatrix},$$

where

$$\det \mathbf{II} = -\frac{u^2 \varphi'^2 \varphi'' \cos^2 w}{\mathbf{W}^3},$$

and $\mathbf{W} = \sqrt{1 + \varphi'^2}$. \square

The Gauss map of the rotational hypersurface is given by

$$\mathbf{e}_R = \frac{1}{\mathbf{W}} \begin{pmatrix} \varphi' \cos v \cos w \\ \varphi' \sin v \cos w \\ \varphi' \sin w \\ -1 \end{pmatrix}. \tag{5}$$

See Figures 1 and 2 for some different projections and symmetries, from four-space to three-spaces of the Gauss map of the rotational hypersurface.

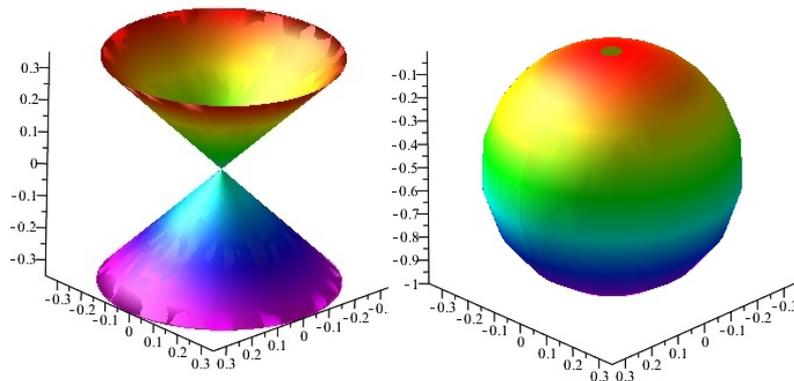


Figure 1. Projections of e_R : (left) into $x_1x_2x_3$ -space; (right) into $x_1x_2x_4$ -space; $\varphi = u^4, w = \pi/3$.

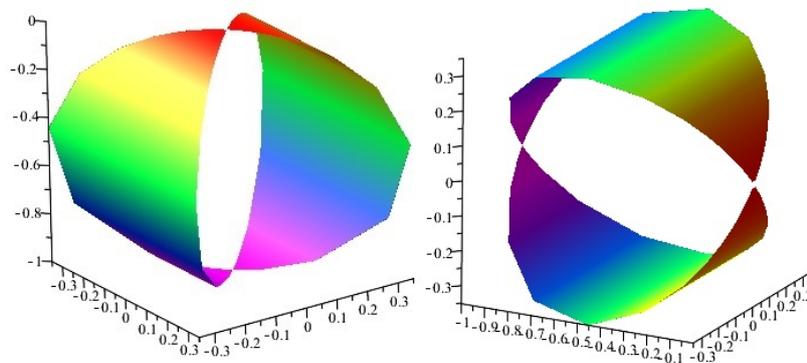


Figure 2. Projections of e_R : (left) into $x_1x_3x_4$ -space; (right) into $x_2x_3x_4$ -space; $\varphi = u^4, w = \pi/3$.

Thus, the shape operator of the rotational hypersurface is obtained as

$$S = \begin{pmatrix} -\frac{\varphi''}{w^3} & 0 & 0 \\ 0 & -\frac{\varphi'}{uw} & 0 \\ 0 & 0 & -\frac{\varphi'}{uw} \end{pmatrix}.$$

From these, we obtain the Gaussian curvature K and the mean curvature H of the rotational hypersurface as follows

$$K = -\frac{\varphi'^2\varphi''}{u^2w^5} \text{ and } H = -\frac{u\varphi'' + 2\varphi'^3 + 2\varphi'}{3uw^3}.$$

Therefore, we have the following corollaries:

Corollary 1. Let $R : M^3 \rightarrow \mathbb{E}^4$ be an isometric immersion given by (4). Then, M^3 has constant Gaussian curvature iff

$$\varphi'^4\varphi''^2 - Cu^4(1 + \varphi'^2)^5 = 0.$$

Corollary 2. Let $\mathbf{R} : M^3 \rightarrow \mathbb{E}^4$ be an isometric immersion given by (4). Then, M^3 has constant mean curvature (CMC) iff

$$(u\varphi'' + 2\varphi'^3 + 2\varphi')^2 - 9Cu^2(1 + \varphi'^2)^3 = 0.$$

Corollary 3. Let $\mathbf{R} : M^3 \rightarrow \mathbb{E}^4$ be an isometric immersion given by (4). Then, M^3 has zero Gaussian curvature iff

$$\varphi(u) = c_1u + c_2, \text{ or } \varphi(u) = c_1.$$

Proof. Solving the second order differential equation $K = 0$, i.e.,

$$\varphi'^2\varphi'' = 0,$$

we get the solution. \square

Corollary 4. Let $\mathbf{R} : M^3 \rightarrow \mathbb{E}^4$ be an isometric immersion given by (4). Then, M^3 has zero mean curvature iff

$$\varphi(u) = \pm c_1 \int \frac{du}{\sqrt{u^4 - c_1}} + c_2.$$

Proof. When we solve the second order differential equation $H = 0$, i.e.,

$$u\varphi'' + 2\varphi'^3 + 2\varphi' = 0,$$

we get the solution. Taking $z = \varphi'$, $z' = \varphi''$, we have

$$\Rightarrow uz' + 2z^3 + 2z = 0$$

$$\Rightarrow \frac{z^2 u^4}{z^2 + 1} = c_1^2$$

$$\Rightarrow \varphi(u) = \pm c_1 \int \frac{du}{\sqrt{u^4 - c_1}}.$$

Therefore, the solutions of $\varphi(u)$ are given, by using Mathematica, as follows

$$\varphi(u) = \pm \frac{ic_1 \sqrt{\frac{1}{u^4 - c_1}} \sqrt{1 - \frac{u^4}{c_1}} \text{EllipticF}\left[i \operatorname{arg} \sinh \left[\sqrt{-\frac{1}{\sqrt{c_1}}} u\right], -1\right]}{\sqrt{-\frac{1}{\sqrt{c_1}}}} + c_2,$$

and, by using Maple, we get for the solutions $\varphi(u)$:

$$\varphi(u) = \pm \frac{c_1 \sqrt{1 + \frac{u^2}{\sqrt{c_1}}} \sqrt{1 + \frac{u^2}{\sqrt{c_1}}} \text{EllipticF}\left[u \sqrt{-\frac{1}{\sqrt{c_1}}} u, i\right]}{\sqrt{-\frac{1}{\sqrt{c_1}}} \sqrt{u^4 - c_1}} + c_2,$$

where $\text{EllipticF}[\phi, m]$ gives the elliptic integral of the first kind $F(\phi | m)$. \square

Corollary 5. Let $\mathbf{R} : M^3 \rightarrow \mathbb{E}^4$ be an isometric immersion given by (4). If M^3 has zero mean curvature, then we get

$$\varphi(u) = \int_1^\infty \frac{du}{\sqrt{u^4 - 1}} < \frac{\pi}{2},$$

where $u \in [1, \infty)$.

Proof. For the third kind improper integral $\int_1^\infty \frac{du}{\sqrt{u^4-1}}$, we get

$$\int_1^\infty \frac{du}{\sqrt{u^4-1}} < \int_1^\infty \frac{du}{u\sqrt{u^2-1}}.$$

Taking $u = \cosh x$, we have

$$\int_1^\infty \frac{du}{u\sqrt{u^2-1}} = \int_0^\infty \frac{dx}{\cosh x} = \frac{\pi}{2}.$$

□

4. Gauss Map

Next, we calculate the curvatures of the Gauss map (5) of the rotational hypersurface (4).

Theorem 2. *The Gaussian curvature and the mean curvature of the Gauss map (5) of the rotational hypersurface (4) are given as follows*

$$K = -1 \text{ and } H = -1,$$

respectively,

Proof. Using the first partial differentials of (5), we get the first quantities as follows

$$\mathbf{I} = \begin{pmatrix} \frac{\varphi'^2}{\mathbf{W}^4} & 0 & 0 \\ 0 & \frac{\varphi'^2 \cos^2 w}{\mathbf{W}^2} & 0 \\ 0 & 0 & \frac{\varphi'^2}{\mathbf{W}^2} \end{pmatrix}.$$

We have

$$\det \mathbf{I} = \frac{\varphi'^4 \varphi'^2 \cos^2 w}{\mathbf{W}^8},$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$, $\varphi'' = \frac{d^2\varphi}{du^2}$. Using the second partial differentials of (5), we have the second quantities as follows

$$\mathbf{II} = \begin{pmatrix} -\frac{\varphi'^2}{\mathbf{W}^4} & 0 & 0 \\ 0 & -\frac{\varphi'^2 \cos^2 w}{\mathbf{W}^2} & 0 \\ 0 & 0 & -\frac{\varphi'^2}{\mathbf{W}^2} \end{pmatrix},$$

where

$$\det \mathbf{II} = -\frac{\varphi'^4 \varphi'^2 \cos^2 w}{\mathbf{W}^8}.$$

The Gauss map of the Gauss map (5) of the rotational hypersurface (4) is

$$\mathbf{e}(e_R) = \frac{1}{\mathbf{W}} \begin{pmatrix} \varphi' \cos v \cos w \\ \varphi' \sin v \cos w \\ \varphi' \sin w \\ -1 \end{pmatrix}. \tag{6}$$

The shape operator of (6) is

$$\mathbf{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, we obtain the Gaussian curvature and the mean curvature of (5) as $K = -1$, and $H = -1$. \square

5. The Third Laplace–Beltrami Operator

Next, we introduce the third Laplace–Beltrami operator to the four-space. Then, we apply it for the hypersurface (4). See [28] for the Laplace–Beltrami operator in three-space.

The inverse of the matrix

$$(e_{ij}) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

is as follows

$$\frac{1}{e} \begin{pmatrix} e_{22}e_{33} - e_{23}e_{32} & -(e_{12}e_{33} - e_{13}e_{32}) & e_{12}e_{23} - e_{13}e_{22} \\ -(e_{21}e_{33} - e_{31}e_{23}) & e_{11}e_{33} - e_{13}e_{31} & -(e_{11}e_{23} - e_{21}e_{13}) \\ e_{21}e_{32} - e_{22}e_{31} & -(e_{11}e_{32} - e_{12}e_{31}) & e_{11}e_{22} - e_{12}e_{21} \end{pmatrix},$$

where

$$\begin{aligned} e &= \det(e_{ij}) \\ &= e_{11}e_{22}e_{33} - e_{11}e_{23}e_{32} + e_{12}e_{31}e_{23} - e_{12}e_{21}e_{33} + e_{21}e_{13}e_{32} - e_{13}e_{22}e_{31}. \end{aligned}$$

Definition 2. The third Laplace–Beltrami operator Δ^{III} of hypersurface \mathbf{M} is as follows

$$\Delta^{\text{III}}\phi = \frac{1}{\sqrt{e}} \sum_{i,j=1}^3 \frac{\partial}{\partial x^i} \left(\sqrt{e} e^{ij} \frac{\partial \phi}{\partial x^j} \right), \tag{7}$$

where $(e^{ij}) = (e_{kl})^{-1}$ and $e = \det(e_{ij})$ and $\phi = \phi(x^1, x^2, x^3) |_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^3$) is a smooth function of class C^3 . We can write $\Delta^{\text{III}}\phi$ as follows

$$\frac{1}{\sqrt{e}} \left[\begin{aligned} &\frac{\partial}{\partial x^1} \left(\sqrt{e} e^{11} \frac{\partial \phi}{\partial x^1} \right) - \frac{\partial}{\partial x^1} \left(\sqrt{e} e^{12} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^1} \left(\sqrt{e} e^{13} \frac{\partial \phi}{\partial x^3} \right) \\ &- \frac{\partial}{\partial x^2} \left(\sqrt{e} e^{21} \frac{\partial \phi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\sqrt{e} e^{22} \frac{\partial \phi}{\partial x^2} \right) - \frac{\partial}{\partial x^2} \left(\sqrt{e} e^{23} \frac{\partial \phi}{\partial x^3} \right) \\ &+ \frac{\partial}{\partial x^3} \left(\sqrt{e} e^{31} \frac{\partial \phi}{\partial x^1} \right) - \frac{\partial}{\partial x^3} \left(\sqrt{e} e^{32} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(\sqrt{e} e^{33} \frac{\partial \phi}{\partial x^3} \right) \end{aligned} \right].$$

Clearly, we can write the matrix of the third fundamental form

$$\text{III} = \begin{pmatrix} X & Y & U \\ Y & Z & J \\ U & J & O \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} X &= \mathbf{e}_u \cdot \mathbf{e}_u, Y = \mathbf{e}_u \cdot \mathbf{e}_v, U = \mathbf{e}_u \cdot \mathbf{e}_w, \\ Z &= \mathbf{e}_v \cdot \mathbf{e}_v, J = \mathbf{e}_v \cdot \mathbf{e}_w, O = \mathbf{e}_w \cdot \mathbf{e}_w. \end{aligned}$$

Here, \mathbf{e} is the Gauss map (i.e., the unit normal vector). More precisely, we get

$$\text{III}^{-1} = \frac{1}{\det \text{III}} \begin{pmatrix} OZ - J^2 & JU - OY & JY - UZ \\ JU - OY & OX - U^2 & UY - JX \\ JY - UZ & UY - JX & XZ - Y^2 \end{pmatrix},$$

where

$$\det \mathbf{III} = (XZ - Y^2)O - J^2X + 2JUY - U^2Z.$$

Hence, using a function $\phi = \phi(u, v, w)$, we specifically obtain

$$\Delta^{\mathbf{III}}\phi = \frac{1}{\sqrt{|\det \mathbf{III}|}} \left[\begin{array}{l} \frac{\partial}{\partial u} \left(\frac{(OZ - J^2)\phi_u - (JU - OY)\phi_v + (JY - UZ)\phi_w}{\sqrt{|\det \mathbf{III}|}} \right) \\ - \frac{\partial}{\partial v} \left(\frac{(JU - OY)\phi_u - (OX - U^2)\phi_v + (UY - JX)\phi_w}{\sqrt{|\det \mathbf{III}|}} \right) \\ + \frac{\partial}{\partial w} \left(\frac{(JY - UZ)\phi_u - (UY - JX)\phi_v + (XZ - Y^2)\phi_w}{\sqrt{|\det \mathbf{III}|}} \right) \end{array} \right]. \quad (9)$$

We continue our calculations to find the third Laplace–Beltrami operator $\Delta^{\mathbf{III}}\mathbf{R}$ of the rotational hypersurface \mathbf{R} using (9) in (4).

The third Laplace–Beltrami operator of the hypersurface parametrized by (4) is given by

$$\Delta^{\mathbf{III}}\mathbf{R} = \frac{1}{\sqrt{|\det \mathbf{III}|}} \left(\frac{\partial}{\partial u} \mathbf{U} - \frac{\partial}{\partial v} \mathbf{V} + \frac{\partial}{\partial w} \mathbf{W} \right),$$

where

$$\begin{aligned} \mathbf{U} &= \frac{(OZ - J^2) \mathbf{R}_u - (JU - OY) \mathbf{R}_v + (JY - UZ) \mathbf{R}_w}{\sqrt{|\det \mathbf{III}|}}, \\ \mathbf{V} &= \frac{(JU - OY) \mathbf{R}_u - (OX - U^2) \mathbf{R}_v + (UY - JX) \mathbf{R}_w}{\sqrt{|\det \mathbf{III}|}}, \\ \mathbf{W} &= \frac{(JY - UZ) \mathbf{R}_u - (UY - JX) \mathbf{R}_v + (XZ - Y^2) \mathbf{R}_w}{\sqrt{|\det \mathbf{III}|}}. \end{aligned}$$

Here, using the parametrization (4), we get $Y = U = J = 0$. Therefore, we have the following

$$\begin{aligned} \mathbf{U} &= \frac{OZ}{\sqrt{|\det \mathbf{III}|}} \mathbf{R}_u = \left(\frac{\varphi'^4 \cos^2 w}{(1 + \varphi'^2)^2 \sqrt{|\det \mathbf{III}|}} \right) \mathbf{R}_u, \\ \mathbf{V} &= -\frac{OX}{\sqrt{|\det \mathbf{III}|}} \mathbf{R}_v = \left(-\frac{\varphi'^2 \varphi''^2}{(1 + \varphi'^2)^3 \sqrt{|\det \mathbf{III}|}} \right) \mathbf{R}_v, \\ \mathbf{W} &= \frac{XZ}{\sqrt{|\det \mathbf{III}|}} \mathbf{R}_w = \left(\frac{\varphi'^2 \varphi''^2 \cos^2 w}{(1 + \varphi'^2)^3 \sqrt{|\det \mathbf{III}|}} \right) \mathbf{R}_w, \end{aligned}$$

where

$$\det \mathbf{III} = \frac{\varphi'^4 \varphi''^2 \cos^2 w}{(1 + \varphi'^2)^4}.$$

We obtain the vectors

$$\begin{aligned} \mathbf{U} &= \frac{\varphi'^2 \cos w}{(1 + \varphi'^2)^2 \varphi''} \begin{pmatrix} \cos v \cos w \\ \sin v \cos w \\ \sin w \\ \varphi' \end{pmatrix}, \\ \mathbf{V} &= -\frac{\varphi''}{(1 + \varphi'^2) \cos w} \begin{pmatrix} -u \sin v \cos w \\ u \cos v \cos w \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\mathbf{W} = \frac{\varphi'' \cos w}{1 + \varphi'^2} \begin{pmatrix} -u \cos v \sin w \\ -u \sin v \sin w \\ u \cos w \\ 0 \end{pmatrix}.$$

Taking differentials with respect to u, v and w on $\mathbf{U}, \mathbf{V}, \mathbf{W}$, respectively, we get

$$\frac{\partial}{\partial u} (\mathbf{U}) = \frac{\partial}{\partial u} \left(\frac{\varphi'^2}{(1 + \varphi'^2)^2 \varphi''} \right) \begin{pmatrix} \cos v \cos^2 w \\ \sin v \cos^2 w \\ \sin v \cos w \\ \varphi' \cos w \end{pmatrix} + \frac{\varphi'^2}{(1 + \varphi'^2)^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos w \end{pmatrix},$$

$$\frac{\partial}{\partial v} (\mathbf{V}) = -\frac{\varphi''}{(1 + \varphi'^2)} \begin{pmatrix} -u \cos v \\ -u \sin v \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial w} (\mathbf{W}) = \frac{\varphi''}{1 + \varphi'^2} \begin{pmatrix} u \cos v \cos 2w \\ u \sin v \cos 2w \\ -u \sin 2w \\ 0 \end{pmatrix}.$$

Therefore, we obtain the following theorem:

Theorem 3. *The third Laplace–Beltrami operator of the hypersurface (4) is given by*

$$\begin{aligned} \Delta^{\text{III}}\mathbf{R}_1 &= \eta \cdot \{ [-4\varphi' \varphi'' + 2\varphi' (1 + \varphi'^2) \varphi''^2 - (1 + \varphi'^2) \varphi'''] \cos v \cos^2 w \\ &\quad + (1 + \varphi'^2) u \cos v \varphi'^3 + (1 + \varphi'^2) u \cos v \cos 2w \varphi'^3 \}, \end{aligned}$$

$$\begin{aligned} \Delta^{\text{III}}\mathbf{R}_2 &= \eta \cdot \{ [-4\varphi' \varphi'' + 2\varphi' (1 + \varphi'^2) \varphi''^2 - (1 + \varphi'^2) \varphi'''] \sin v \cos^2 w \\ &\quad + (1 + \varphi'^2) u \sin v \varphi'^3 + (1 + \varphi'^2) u \sin v \cos 2w \varphi'^3 \}, \end{aligned}$$

$$\begin{aligned} \Delta^{\text{III}}\mathbf{R}_3 &= \eta \cdot \{ [-4\varphi' \varphi'' + 2\varphi' (1 + \varphi'^2) \varphi''^2 - (1 + \varphi'^2) \varphi'''] \sin v \cos w \\ &\quad - (1 + \varphi'^2) u \sin 2w \varphi'^3 \}, \end{aligned}$$

$$\begin{aligned} \Delta^{\text{III}}\mathbf{R}_4 &= \eta \cdot \{ [-4\varphi' \varphi'' + 2\varphi' (1 + \varphi'^2) \varphi''^2 - (1 + \varphi'^2) \varphi'''] \cos w \varphi' \\ &\quad + \cos w \varphi'^2 \varphi'^2 \}, \end{aligned}$$

where

$$\Delta^{\text{III}}\mathbf{R} = (\Delta^{\text{III}}\mathbf{R}_1, \Delta^{\text{III}}\mathbf{R}_2, \Delta^{\text{III}}\mathbf{R}_3, \Delta^{\text{III}}\mathbf{R}_4),$$

and

$$\eta = \eta(u, w) = (\varphi'^2 \varphi'^3 \cos w)^{-1}.$$

6. Conclusions

When the rotational hypersurface \mathbf{R} has the equation $\Delta^{\text{III}}\mathbf{R} = \mathbf{0}$, i.e., the rotational hypersurface (4) is **III**-minimal, then we have to solve the system of equation

$$\Delta^{\text{III}}\mathbf{R}_i = 0,$$

where $1 \leq i \leq 4$. In fact, the **III**-minimal hypersurface of the four-space is a very interesting problem. It is challenging to solve the above equations.

When $\varphi \neq c = \text{const.}$ or $\varphi \neq c_1u + c_2$, and $\theta_2 \neq \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$, we get $\Delta^{\text{III}}\mathbf{R} \neq \mathbf{0}$. Hence, the rotational hypersurface (4) is not **III**-minimal.

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