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# Symmetric Identities for $(P, Q)$-Analogue of Tangent Zeta Function

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**Abstract:** The goal of this paper is to define the $(p, q)$-analogue of tangent numbers and polynomials by generalizing the tangent numbers and polynomials and Carlitz-type $q$-tangent numbers and polynomials. We get some explicit formulas and properties in conjunction with $(p, q)$-analogue of tangent numbers and polynomials. We give some new symmetric identities for $(p, q)$-analogue of tangent polynomials by using $(p, q)$-tangent zeta function. Finally, we investigate the distribution and symmetry of the zero of $(p, q)$-analogue of tangent polynomials with numerical methods.

**Keywords:** tangent numbers; tangent polynomials; Carlitz-type $q$-tangent numbers; Carlitz-type $q$-tangent polynomials; $(p, q)$-analogue of tangent numbers and polynomials; $(p, q)$-analogue of tangent zeta function; symmetric identities; zeros

**MSC:** 11B68; 11S40; 11S80

## 1. Introduction

The field of the special polynomials such as tangent polynomials, Bernoulli polynomials, Euler polynomials, and Genocchi polynomials is an expanding area in mathematics (see [1–16]). Many generalizations of these polynomials have been studied (see [1,3–9,11–18]). Srivastava [14] developed some properties and $q$-extensions of the Euler polynomials, Bernoulli polynomials, and Genocchi polynomials. Choi, Anderson and Srivastava have discussed $q$-extension of the Riemann zeta function and related functions (see [5,17]). Dattoli, Migliorati and Srivastava derived a generalization of the classical polynomials (see [6]).

It is the purpose of this paper to introduce and investigate a new some generalizations of the Carlitz-type $q$-tangent numbers and polynomials, $q$-tangent zeta function, Hurwiz $q$-tangent zeta function. We call them Carlitz-type $(p, q)$-tangent numbers and polynomials, $(p, q)$-tangent zeta function, and Hurwitz $(p, q)$-tangent zeta function. The structure of the paper is as follows: In Section 2 we define Carlitz-type $(p, q)$-tangent numbers and polynomials and derive some of their properties involving elementary properties, distribution relation, property of complement, and so on. In Section 3, by using the Carlitz-type $(p, q)$-tangent numbers and polynomials, $(p, q)$-tangent zeta function and Hurwitz $(p, q)$-tangent zeta function are defined. We also contains some connection formulae between the Carlitz-type $(p, q)$-tangent numbers and polynomials and the $(p, q)$-tangent zeta function, Hurwitz $(p, q)$-tangent zeta function. In Section 4 we give several symmetric identities about $(p, q)$-tangent zeta function and Carlitz-type $(p, q)$-tangent polynomials and numbers. In the following Section, we investigate the distribution and symmetry of the zero of Carlitz-type $(p, q)$-tangent polynomials using a computer. Our paper ends with Section 6, where the conclusions and future developments of this work are presented. The following notations will be used throughout this paper.

- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical tangent numbers $T_n$ and tangent polynomials $T_n(x)$ are defined by the following generating functions (see [19])

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \quad (|2t| < \pi), \tag{1}$$

and

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad (|2t| < \pi). \tag{2}$$

respectively. Some interesting properties of basic extensions and generalizations of the tangent numbers and polynomials have been worked out in [11,12,18–20]. The $(p,q)$-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

It is clear that $(p,q)$-number contains symmetric property, and this number is $q$-number when $p = 1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$. Since $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p}}$, we observe that $(p,q)$-numbers and $p$-numbers are different. In other words, by substituting $q$ by $\frac{q}{p}$ in the definition $q$-number, we cannot have $(p,q)$-number. Duran, Acikgoz and Araci [7] introduced the $(p,q)$-analogues of Euler polynomials, Bernoulli polynomials, and Genocchi polynomials. Araci, Duran, Acikgoz and Srivastava developed some properties and relations between the divided differences and $(p,q)$-derivative operator (see [1]). The $(p,q)$-analogues of tangent polynomials were described in [20]. By using $(p,q)$-number, we construct the Carlitz-type $(p,q)$-tangent polynomials and numbers, which generalized the previously known tangent polynomials and numbers, including the Carlitz-type $q$-tangent polynomials and numbers. We begin by recalling here the Carlitz-type $q$-tangent numbers and polynomials (see [18]).

**Definition 1.** *For any complex $x$ we define the Carlitz-type $q$-tangent polynomials, $T_{n,q}(x)$, by the equation*

$$F_q(t,x) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+x]_q t}. \tag{3}$$

The numbers $T_{n,q}(0)$ are called the Carlitz-type $q$-tangent numbers and are denoted by $T_{n,q}$. Based on this idea, we generalize the Carlitz-type $q$-tangent number $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$. It follows that we define the following $(p,q)$-analogues of the the Carlitz-type $q$-tangent number $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$. In the next section we define the $(p,q)$-analogue of tangent numbers and polynomials. After that we will obtain some their properties.

## 2. $(p,q)$-Analogue of Tangent Numbers and Polynomials

Firstly, we construct $(p,q)$-analogue of tangent numbers and polynomials and derive some of their relevant properties.

**Definition 2.** *For $0 < q < p \leq 1$, the Carlitz-type $(p,q)$-tangent numbers $T_{n,p,q}$ and polynomials $T_{n,p,q}(x)$ are defined by means of the generating functions*

$$F_{p,q}(t) = \sum_{n=0}^{\infty} T_{n,p,q} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]_{p,q} t}, \tag{4}$$

*and*

$$F_{p,q}(t,x) = \sum_{n=0}^{\infty} T_{n,p,q}(x)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+x]_{p,q}t}, \tag{5}$$

*respectively.*

Setting $p = 1$ in (4) and (5), we can obtain the corresponding definitions for the Carlitz-type $q$-tangent numbers $T_{n,q}$ and $q$-tangent polynomials $T_{n,q}(x)$ respectively. Obviously, if we put $p = 1$, then we have

$$T_{n,p,q}(x) = T_{n,q}(x), \quad T_{n,p,q} = T_{n,q}.$$

Putting $p = 1$, we have

$$\lim_{q\to 1} T_{n,p,q}(x) = T_n(x), \quad \lim_{q\to 1} T_{n,p,q} = T_n.$$

**Theorem 1.** *For $n \in \mathbb{N} \cup \{0\}$, one has*

$$T_{n,p,q} = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}p^{2(n-l)}}. \tag{6}$$

**Proof.** By (4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,p,q}\frac{t^n}{n!} &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]_{p,q}t} \\ &= \sum_{n=0}^{\infty} \left( [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}p^{2(n-l)}} \right) \frac{t^n}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result (6). □

If we put $p = 1$ in Theorem 1, we obtain (cf. [18])

$$T_{n,q} = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}}.$$

Next, we construct the Carlitz-type $(h,p,q)$-tangent polynomials $T^{(h)}_{n,p,q}(x)$. Define the Carlitz-type $(h,p,q)$-tangent polynomials $T^{(h)}_{n,p,q}(x)$ by

$$T^{(h)}_{n,p,q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m p^{hm} [2m+x]^n_{p,q}. \tag{7}$$

**Theorem 2.** *For $n \in \mathbb{N} \cup \{0\}$, one has*

$$\begin{aligned}T_{n,p,q}(x) &= [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1+q^{2l+1}p^{2(n-l)+h}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m+x]^n_{p,q}.\end{aligned}$$

**Proof.** By (5), we obtain

$$T_{n,p,q}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1+q^{2l+1}p^{2(n-l)}}. \tag{8}$$

Again, by using (5) and (8), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,p,q}(x)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+q^{2l+1}p^{2(n-l)}}\right)\frac{t^n}{n!} \\
&= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+x]_{p,q}t}.
\end{aligned} \tag{9}$$

Since $[x+2y]_{p,q} = p^{2y}[x]_{p,q} + q^x[2y]_{p,q}$, we have

$$T_{n,p,q}(x) = [2]_q \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left(\frac{1}{p-q}\right)^l\frac{1}{1+q^{2k+1}p^{2(n-k)}}. \tag{10}$$

By using (9) and (10), $(p,q)$-number, and the power series expansion of $e^{xt}$, we give Theorem 2. □

Furthermore, by (7) and Theorem 2, we have

$$T_{n,p,q}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}T_{l,p,q}^{(2n-2l)},$$

$$T_{n,p,q}(x+y) = \sum_{l=0}^{n}\binom{n}{l}p^{xl}q^{y(n-l)}[y]_{p,q}^{l}T_{n-l,p,q}^{(2l)}.$$

From (4) and (5), we can derive the following properties of the Carlitz-type tangent numbers $T_{n,p,q}$ and polynomials $T_{n,p,q}(x)$. So, we choose to omit the details involved.

**Proposition 1.** *For any positive integer $n$, one has*

(1) $T_{n,p,q}(x) = \dfrac{[2]_q}{[2]_{q^m}}[m]_{p,q}^n \sum_{a=0}^{m-1}(-1)^a q^a T_{n,p^m,q^m}\left(\frac{2a+x}{m}\right), (m = odd).$

(2) $T_{n,p^{-1},q^{-1}}(2-x) = (-1)^n p^n q^n T_{n,p,q}(x).$

**Theorem 3.** *For $n \in \mathbb{N}\cup\{0\}$, one has*

$$qT_{n,p,q}(2) + T_{n,p,q} = \begin{cases} [2]_q, & \text{if } n = 0,\\ 0, & \text{if } n \neq 0.\end{cases}$$

**Theorem 4.** *If $n$ is a positive integer, then we have*

$$\sum_{l=0}^{n-1}(-1)^l q^l [2l]_{p,q}^m = \frac{(-1)^{n+1}q^n T_{m,p,q}(2n) + T_{m,p,q}}{[2]_q}.$$

**Proof.** By (4) and (5), we get

$$-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}q^{l+n}e^{[2l+2n]_{p,q}t} + [2]_q\sum_{l=0}^{\infty}(-1)^l q^l e^{[2l]_{p,q}t} = [2]_q\sum_{l=0}^{n-1}(-1)^l q^l e^{[2l]_{p,q}t}. \tag{11}$$

Hence, by (4), (5) and (11), we have

$$
\begin{aligned}
&(-1)^{n+1}q^n \sum_{m=0}^{\infty} T_{m,p,q}(2n)\frac{t^m}{m!} + \sum_{m=0}^{\infty} T_{m,p,q}\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left([2]_q \sum_{l=0}^{n-1}(-1)^l q^l [2l]_{p,q}^m\right)\frac{t^m}{m!}.
\end{aligned}
$$

Equating coefficients of $\frac{t^m}{m!}$ gives Theorem 4. □

## 3. $(p,q)$-Analogue of Tangent Zeta Function

Using Carlitz-type $(p,q)$-tangent numbers and polynomials, we define the $(p,q)$-tangent zeta function and Hurwitz $(p,q)$-tangent zeta function. These functions have the values of the Carlitz-type $(p,q)$-tangent numbers $T_{n,p,q}$, and polynomials $T_{n,p,q}(x)$ at negative integers, respectively. From (4), we note that

$$
\begin{aligned}
\left.\frac{d^k}{dt^k}F_{p,q}(t)\right|_{t=0} &= [2]_q \sum_{m=0}^{\infty}(-1)^n q^m [2m]_{p,q}^k \\
&= T_{k,p,q}, (k \in \mathbb{N}).
\end{aligned}
$$

From the above equation, we construct new $(p,q)$-tangent zeta function as follows:

**Definition 3.** *We define the $(p,q)$-tangent zeta function for $s \in \mathbb{C}$ with $Re(s) > 0$ by*

$$\zeta_{p,q}(s) = [2]_q \sum_{n=1}^{\infty}\frac{(-1)^n q^n}{[2n]_{p,q}^s}. \tag{12}$$

Notice that $\zeta_{p,q}(s)$ is a meromorphic function on $\mathbb{C}$(cf.7). Remark that, if $p = 1, q \to 1$, then $\zeta_{p,q}(s) = \zeta_T(s)$ which is the tangent zeta function (see [19]). The relationship between the $\zeta_{p,q}(s)$ and the $T_{k,p,q}$ is given explicitly by the following theorem.

**Theorem 5.** *Let $k \in \mathbb{N}$. We have*

$$\zeta_{p,q}(-k) = T_{k,p,q}.$$

Please note that $\zeta_{p,q}(s)$ function interpolates $T_{k,p,q}$ numbers at non-negative integers. Similarly, by using Equation (5), we get

$$\left.\frac{d^k}{dt^k}F_{p,q}(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m [2m+x]_{p,q}^k \tag{13}$$

and

$$\left.\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} T_{n,p,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} = T_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{14}$$

Furthermore, by (13) and (14), we are ready to construct the Hurwitz $(p,q)$-tangent zeta function.

**Definition 4.** *For $s \in \mathbb{C}$ with $Re(s) > 0$ and $x \notin \mathbb{Z}_0^-$, we define*

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty}\frac{(-1)^n q^n}{[2n+x]_{p,q}^s}. \tag{15}$$

Obverse that the function $\zeta_{p,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. We note that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q}(s,x) = \zeta_T(s,x)$ which is the Hurwitz tangent zeta function (see [19]). The function

$\zeta_{p,q}(-k,x)$ interpolates the numbers $T_{k,p,q}(x)$ at non-negative integers. Substituting $s=-k$ with $k \in \mathbb{N}$ into (15), and using Theorem 2, we easily arrive at the following theorem.

**Theorem 6.** *Let $k \in \mathbb{N}$. One has*

$$\zeta_{p,q}(-k,x) = T_{k,p,q}(x).$$

## 4. Some Symmetric Properties About $(P,Q)$-Analogue of Tangent Zeta Function

Our main objective in this section is to obtain some symmetric properties about $(p,q)$-tangent zeta function. In particular, some of these symmetric identities are also related to the Carlitz-type $(p,q)$-tangent polynomials and the alternate power sums. To end this section, we focus on some symmetric identities containing the Carlitz-type $(p,q)$-tangent zeta function and the alternate power sums.

**Theorem 7.** *Let $w_1$ and $w_2$ be positive odd integers. Then we have*

$$\begin{aligned}&[2]_{q^{w_1}}[w_1]^s_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\zeta_{p^{w_2},q^{w_2}}\left(s, w_1x+\frac{2w_1 i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_2]^s_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\zeta_{p^{w_1},q^{w_1}}\left(s, w_2x+\frac{2w_2 j}{w_1}\right).\end{aligned}$$

**Proof.** For any $x, y \in \mathbb{C}$, we observe that $[xy]_{p,q} = [x]_{p^y,q^y}[y]_{p,q}$. By substituting $w_1x+\frac{2w_1 i}{w_2}$ for $x$ in Definition 4, replace $p$ by $p^{w_2}$ and replace $q$ by $q^{w_2}$, respectively, we derive

$$\begin{aligned}\zeta_{p^{w_2},q^{w_2}}\left(s, w_1x+\frac{2w_1 i}{w_2}\right) &= [2]_{q^{w_2}}\sum_{n=0}^{\infty}\frac{(-1)^n q^{w_2 n}}{[w_1x+\frac{2w_1 i}{w_2}+2n]^s_{p^{w_2},q^{w_2}}}\\&=[2]_{q^{w_2}}[w_2]^s_{p,q}\sum_{n=0}^{\infty}\frac{(-1)^n q^{w_2 n}}{[w_1w_2x+2w_1 i+2w_2 n]^s_{p,q}}.\end{aligned}$$

Since for any non-negative integer $m$ and positive odd integer $w_1$, there exist unique non-negative integer $r$ such that $m = w_1 r + j$ with $0 \le j \le w_1 - 1$. Thus, this can be written as

$$\begin{aligned}&\zeta_{p^{w_2},q^{w_2}}\left(s, w_1x+\frac{2w_1 i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_2]^s_{p,q}\sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty}\frac{(-1)^{w_1r+j}q^{w_2(w_1r+j)}}{[2w_2(w_1r+j)+w_1w_2x+2w_1i]^s_{p,q}}\\&=[2]_{q^{w_2}}[w_2]^s_{p,q}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{w_1r+j}q^{w_2(w_1r+j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]^s_{p,q}}.\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}&[2]_{q^{w_1}}[w_1]^s_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\zeta_{p^{w_2},q^{w_2}}\left(s, w_1x+\frac{2w_1 i}{w_2}\right)\\&=[2]_{q^{w_1}}[2]_{q^{w_2}}[w_1]^s_{p,q}[w_2]^s_{p,q}\\&\qquad\times\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{r+i+j}q^{(w_1w_2r+w_1i+w_2j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]^s_q}.\end{aligned}\tag{16}$$

From the similar method, we can have that

$$\begin{aligned}\zeta_{p^{w_1},q^{w_1}}\left(s, w_2x+\frac{2w_2j}{w_1}\right) &= [2]_{q^{w_1}}\sum_{n=0}^{\infty}\frac{(-1)^n q^{w_1n}}{[w_2x+\frac{2w_2j}{w_1}+2n]^s_{p^{w_1},q^{w_1}}}\\ &= [2]_{q^{w_1}}[w_1]^s_{p,q}\sum_{n=0}^{\infty}\frac{(-1)^n q^{w_1n}}{[w_1w_2x+2w_2j+2w_1n]^s_{p,q}}.\end{aligned}$$

After some calculations in the above, we have

$$\begin{aligned}&[2]_{q^{w_2}}[w_2]^s_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}\zeta^{(h)}_{p^{w_1},q^{w_1}}\left(s, w_2x+\frac{2w_2j}{w_1}\right)\\ &= [2]_{q^{w_1}}[2]_{q^{w_2}}[w_1]^s_{p,q}[w_2]^s_{p,q}\\ &\quad\times\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{r+i+j}q^{(w_1w_2r+w_1i+w_2j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]^s_{p,q}}.\end{aligned}\tag{17}$$

Thus, from (16) and (17), we obtain the result. □

**Corollary 1.** *For $s\in\mathbb{C}$ with $Re(s)>0$, we have*

$$\zeta_{p,q}(s, w_1x) = [w_1]^{-s}_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^j \zeta_{p^{w_1},q^{w_1}}\left(s, \frac{x+2j}{w_1}\right).$$

**Proof.** Let $w_2=1$ in Theorem 7. Then we immediately get the result. □

Next, we also derive some symmetric identities for Carlitz-type $(p,q)$-tangent polynomials by using $(p,q)$-tangent zeta function.

**Theorem 8.** *Let $w_1$ and $w_2$ be any positive odd integers. The following multiplication formula holds true for the Carlitz-type $(p,q)$-tangent polynomials:*

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}T_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{2w_1i}{w_2}\right)\\ &= [2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}T_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{2w_2j}{w_1}\right).\end{aligned}$$

**Proof.** By substituting $T_{n,p,q}(x)$ for $\zeta_{p,q}(s,x)$ in Theorem 7, and using Theorem 6, we can find that

$$\begin{aligned}&[2]_{q^{w_1}}[w_1]^{-n}_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}\zeta_{p^{w_2},q^{w_2}}\left(-n, w_1x+\frac{2w_1i}{w_2}\right)\\ &= [2]_{q^{w_1}}[w_1]^{-n}_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}T_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{2w_1i}{w_2}\right),\end{aligned}\tag{18}$$

and

$$\begin{aligned}&[2]_{q^{w_2}}[w_2]^{-n}_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}\zeta_{p^{w_1},q^{w_1}}\left(-n, w_2x+\frac{2w_2j}{w_1}\right)\\ &= [2]_{q^{w_2}}[w_2]^{-n}_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}T_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{2w_2j}{w_1}\right).\end{aligned}\tag{19}$$

Thus, by (18) and (19), this concludes our proof. □

Considering $w_1 = 1$ in the Theorem 8, we obtain as below equation.

$$T_{n,p,q}(x) = \frac{[2]_q}{[2]_{q^{w_2}}}[w_2]^n_{p,q}\sum_{j=1}^{w_2-1}(-1)^j q^j T_{n,p^{w_2},q^{w_2}}\left(\frac{x+2j}{w_2}\right).$$

Furthermore, by applying the addition theorem for the Carlitz-type $(h,p,q)$-tangent polynomials $T^{(h)}_{n,p,q}(x)$, we can obtain the following theorem.

**Theorem 9.** *Let $w_1$ and $w_2$ be any positive odd integers. Then one has*

$$\begin{aligned}&[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_q[w_1]^{n-l}_{p,q}p^{w_1w_2xl}T^{(2l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{T}_{n,l,p^{w_2},q^{w_2}}(w_1)\\&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}T^{(2l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\mathcal{T}_{n,l,p^{w_1},q^{w_1}}(w_2).\end{aligned}$$

**Proof.** From Theorem 8, we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}T_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{2w_1i}{w_2}\right)\\&=[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{l=0}^{n}\binom{n}{l}q^{2w_1(n-l)i}p^{w_1w_2xl}\\&\qquad\times T^{(2l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l[2i]^l_{p^{w_1},q^{w_1}}\\&=[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l p^{w_1w_2xl}T^{(2l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\\&\qquad\times\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}q^{2(n-l)w_1i}[2i]^l_{p^{w_1},q^{w_1}}.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}T_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{2w_1i}{w_2}\right)\\&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}T^{(2l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\mathcal{T}_{n,l,p^{w_1},q^{w_1}}(w_2),\end{aligned}\tag{20}$$

and

$$\begin{aligned}&[2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}T_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{2w_2j}{w_1}\right)\\&=[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_q[w_1]^{n-l}_{p,q}p^{w_1w_2xl}T^{(2l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{T}_{n,l,p^{w_2},q^{w_2}}(w_1).\end{aligned}\tag{21}$$

where $\mathcal{T}_{n,l,p,q}(k) = \sum_{i=0}^{k-1}(-1)^i q^{(1+2n-2l)i}[2i]^l_{p,q}$ is called as the alternate power sums. Thus, the theorem can be established by (20) and (21). $\square$

## 5. Zeros of the Carlitz-Type $(P, Q)$-Tangent Polynomials

The purpose of this section is to support theoretical predictions using numerical experiments and to discover new exciting patterns for zeros of the Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x)$. We propose some conjectures by numerical experiments. The first values of the $T_{n,p,q}(x)$ are given by

$$T_{0,p,q}(x) = 1,$$
$$T_{1,p,q}(x) = -\frac{-p^x - p^x q^3 + q^x + p^2 q^{1+x}}{(p-q)(1+p^2 q)(1-q+q^2)},$$

$$T_{2,p,q}(x) = \frac{p^{2x} + p^{2+2x}q^3 + p^{2x}q^5 + p^{2+2x}q^8 - 2p^x q^x + q^{2x} - 2p^{4+x}q^{1+x}}{(p-q)^2(1+p^4 q)(1+p^2 q^3)(1-q+q^2-q^3+q^4)}$$
$$- \frac{2p^x q^{5+x} - 2p^{4+x}q^{6+x} + p^4 q^{1+2x} + p^2 q^{3+2x} + p^6 q^{4+2x}}{(p-q)^2(1+p^4 q)(1+p^2 q^3)(1-q+q^2-q^3+q^4)}.$$

Tables 1 and 2 present the numerical results for approximate solutions of real zeros of $T_{n,p,q}(x)$. The numbers of zeros of $T_{n,p,q}(x)$ are tabulated in Table 1 for a fixed $p = \frac{1}{2}$ and $q = \frac{1}{10}$.

**Table 1.** Numbers of real and complex zeros of $T_{n,p,q}(x)$, $p = \frac{1}{2}, q = \frac{1}{10}$.

| Degree $n$ | Real Zeros | Complex Zeros |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 2 | 0 |
| 3 | 1 | 2 |
| 4 | 2 | 2 |
| 5 | 1 | 4 |
| 6 | 2 | 4 |
| 7 | 1 | 6 |
| 8 | 2 | 6 |
| 9 | 1 | 8 |
| 10 | 2 | 8 |
| 11 | 1 | 10 |
| 12 | 2 | 10 |
| 13 | 1 | 12 |
| 14 | 2 | 12 |
| ⋮ | ⋮ | ⋮ |
| 30 | 2 | 28 |

**Table 2.** Numerical solutions of $T_{n,p,q}(x) = 0$, $p = \frac{1}{2}, q = \frac{1}{10}$.

| Degree $n$ | $x$ |
|---|---|
| 1 | 0.0147214 |
| 2 | –0.0451666, 0.0490316 |
| 3 | 0.0737013 |
| 4 | –0.0782386, 0.0906197 |
| 5 | 0.102727 |
| 6 | –0.0935042, 0.111767 |

The use of computer has made it possible to identify the zeros of the Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x)$. The zeros of the Carlitz-type $(p, q)$-tangent polynomials $T_{n,p,q}(x)$ for $x \in \mathbb{C}$ are plotted in Figure 1.

In Figure 1(top-left), we choose $n = 10, p = 1/2$ and $q = 1/10$. In Figure 1(top-right), we choose $n = 20, p = 1/2$ and $q = 1/10$. In Figure 1(bottom-left), we choose $n = 30, p = 1/2$ and $q = 1/10$. In Figure 1(bottom-right), we choose $n = 40, p = 1/2$ and $q = 1/10$. It is amazing that the structure of the real roots of the Carlitz-type $(p,q)$-tangent polynomials $T_{n,p,q}(x)$ is regular. Thus, theoretical prediction on the regular structure of the real roots of the Carlitz-type $(p,q)$-tangent polynomials $T_{n,p,q}(x)$ is await for further study (Table 1). Next, we have obtained the numerical solution satisfying Carlitz-type $(p,q)$-tangent polynomials $T_{n,p,q}(x) = 0$ for $x \in \mathbb{R}$. The numerical solutions are tabulated in Table 2 for a fixed $p = \frac{1}{2}$ and $q = \frac{1}{10}$ and various value of $n$.

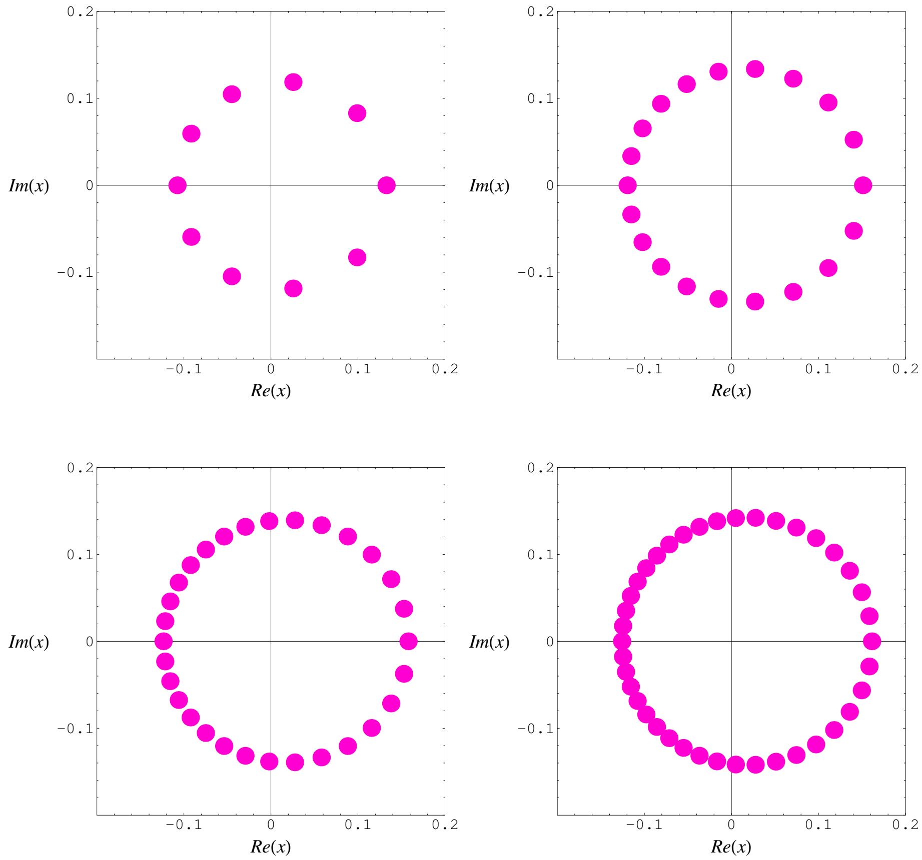


**Figure 1.** Zeros of $T_{n,p,q}(x)$.

## 6. Conclusions and Future Developments

This study constructed the Carlitz-type $(p,q)$-tangent numbers and polynomials. We have derived several formulas for the Carlitz-type $(h,q)$-tangent numbers and polynomials. Some interesting symmetric identities for Carlitz-type $(p,q)$-tangent polynomials are also obtained. Moreover, the results of [18] can be derived from ours as special cases when $q = 1$. By numerical experiments, we will make a series of the following conjectures:

**Conjecture 1.** *Prove or disprove that* $T_{n,p,q}(x), x \in \mathbb{C}$, *has* $Im(x) = 0$ *reflection symmetry analytic complex functions. Furthermore,* $T_{n,p,q}(x)$ *has* $Re(x) = a$ *reflection symmetry for* $a \in \mathbb{R}$.

Many more values of $n$ have been checked. It still remains unknown if the conjecture holds or fails for any value $n$ (see Figure 1).

**Conjecture 2.** *Prove or disprove that* $T_{n,p,q}(x) = 0$ *has n distinct solutions.*

In the notations: $R_{T_{n,p,q}(x)}$ denotes the number of real zeros of $T_{n,p,q}(x)$ lying on the real plane $Im(x) = 0$ and $C_{T_{n,p,q}(x)}$ denotes the number of complex zeros of $T_{n,p,q}(x)$. Since $n$ is the degree of the polynomial $T_{n,p,q}(x)$, we get $R_{T_{n,p,q}(x)} = n - C_{T_{n,p,q}(x)}$ (see Tables 1 and 2).

**Conjecture 3.** *Prove or disprove that*

$$R_{T_{n,p,q}(x)} = \begin{cases} 1, & \text{if } n = \text{odd}, \\ 2, & \text{if } n = \text{even}. \end{cases}$$

We expect that investigations along these directions will lead to a new approach employing numerical method regarding the research of the Carlitz-type $(p,q)$-tangent polynomials $T_{n,p,q}(x)$ which appear in applied mathematics, and mathematical physics (see [11,18–20]).

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