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Generating Functions for Orthogonal Polynomials of A_2, C_2 and G_2

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Abstract: The generating functions of fourteen families of generalized Chebyshev polynomials related to rank two Lie algebras A_2, C_2 and G_2 are explicitly developed. There exist two classes of the orthogonal polynomials corresponding to the symmetric and antisymmetric orbit functions of each rank two algebra. The Lie algebras G_2 and C_2 admit two additional polynomial collections arising from their hybrid character functions. The admissible shift of the weight lattice permits the construction of a further four shifted polynomial classes of C_2 and directly generalizes formation of the classical univariate Chebyshev polynomials of the third and fourth kinds. Explicit evaluating formulas for each polynomial family are derived and linked to the incomplete exponential Bell polynomials.

Keywords: generating function; root system; orthogonal polynomial; Weyl group; Lie algebra

1. Introduction

The purpose of this article is to develop generating functions of fourteen types of bivariate generalized Chebyshev polynomials [1–3]. There exist two families of polynomials corresponding to the Lie algebra A_2, four to the algebra G_2 and eight to the algebra C_2. Explicit formulas for the polynomials are deduced from their generating functions.

The four kinds of the of classical univariate Chebyshev polynomials [4] constitute a fundamental part of polynomial numeric methods. Inherent relation of these polynomials to the standard trigonometric functions forms the cornerstone of their theoretical and practical applications. As cosine images of a finite part of the equidistant lattice [4], Chebyshev nodes play a special role. Orbit functions related to the crystallographic root systems of Weyl groups [5,6] serve as multidimensional generalizations of the trigonometric functions and induce specific multivariate versions of Chebyshev polynomials [2,3]. Symmetric and antisymmetric orbit functions occur as a standard tool in Lie theory and the form of the corresponding two kinds of polynomials, which appears already in [1], specializes for the algebra A_1 to univariate Chebyshev polynomials of the first and second kind.

For any root system with two root-lengths, the concept of a sign homomorphism produces two additional classes of hybrid character polynomials [3]. All four polynomial classes constitute special cases of the Heckman–Opdam polynomials [7]. Discrete orthogonality relations of all four sorts of orbit functions over distinct finite fragments of Weyl group invariant lattices are developed in [8–11]. Specific generalized cosine images of the finite fragments of multidimensional lattices form the sets of generalized Chebyshev nodes [3,12–16]. Moreover, analysis of intrinsic discrete orthogonality relations of orbit functions leads to a special type of admissible shifts of the weight lattices [17]. The admissible shift of the weight lattice doubles the number of polynomial families of C_2 and specializes for the algebra A_1 to the Chebyshev polynomials of the third and fourth kinds [4]. Generating functions of the multivariate Chebyshev polynomials are closely linked to the character generators of simple Lie groups.

During the last century, the approach of generating functions was developed to resolve many diverse problems in Lie theory and in the theory of finite groups. A wide range of applications of generating functions in Lie theory can be traced to the definition of the generating function for the characters of the representation [18]. Subsequently, the theory was developed further for the specific types of the Lie groups [19–22]. The generating functions have a unique capability to provide answers to questions that are inaccessible to any other methods. Typically, a generating function of a simple Lie group G is a rational function of several formal variables built to solve a series of analogous problems like decomposition of the product of two irreducible finite dimensional representations of G into the direct sum of them, or reduction of any finite irreducible representations of G to the direct sum of representations of a particular subgroup $G' \subset G$. A number of other problems in group representation theory are listed in the reference [23].

Developed into the power series, the coefficients of the series provide answers to infinite number of computational problems involving the same Lie group [18,23–25]. A practical difficulty often is the complexity of the generating functions for the higher ranks of G. So far, the generating functions practically for all problems are explicitly derived by hand computation. The derivation becomes particularly complicated when one wants to have the generating function in a positive form that also provides the integrity basis for each problem [26] or in polynomial form involving fundamental character functions. Direct calculation of generating functions as rational polynomial functions in fundamental characters is utilized in the present paper. With efficient tools for symbolic computation available in recent years, many more generating functions could be calculated.

Constructed generating functions and explicit formulas in the present article serve as theoretical and practical means for handling the corresponding bivariate generalized Chebyshev polynomials. Explicit evaluating formulas for the polynomials, derived from the explicit form of the generating functions, permit straightforward calculation and computer implementation, considerably more efficient than current recursive algorithms [14]. Cubature formulas for numerical integration, polynomial interpolation and approximation methods corresponding to several current cases of polynomials are recently intensively studied [3,12–16]. Since viability of these polynomial methods is a direct consequence of the discrete orthogonality relations of the underlying orbit functions, extensibility of these techniques to all fourteen cases of the studied bivariate polynomials is guaranteed.

The paper is organized as follows. In Section 2, the fourteen cases of the bivariate Chebyshev polynomials are explicitly constructed and the lowest reference polynomials listed for each case. In Section 3, generating functions are explicitly evaluated from their generic forms and tabulated for each case. In Section 4, the K-polynomials are introduced, their form for each algebra calculated and their utilization as components in evaluating formulas presented. Concluding remarks and follow-up questions are contained in Section 5.

2. Chebyshev Polynomials Associated with Root Systems

2.1. Four Kinds of Univariate Polynomials of A_1

Throughout the article, the standard scalar product of the Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$ is denoted by brackets $\langle\, , \,\rangle$. Recall that the four kinds of the classical Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ are for $n \in \mathbb{Z}^{\geq 0}$ and a real variable $x = \cos\theta$ defined as [4]

$$\begin{aligned} T_n(\cos\theta) &= \cos n\theta, \\ U_n(\cos\theta) &= \frac{\sin{(n+1)\theta}}{\sin\theta}, \\ V_n(\cos\theta) &= \frac{\cos{(n+\frac{1}{2})\theta}}{\cos\frac{1}{2}\theta}, \\ W_n(\cos\theta) &= \frac{\sin{(n+\frac{1}{2})\theta}}{\sin\frac{1}{2}\theta}. \end{aligned}$$

Suitable reformulating of the definitions of the four kinds of Chebyshev polynomials in terms of quantities related to the root system A_1 leads directly to generalized Chebyshev-like polynomials of rank two algebras. The cornerstone of this reformulation is the relation of the one-variable cosine and sine functions to the orbit functions of the algebra A_1, detailed in [27].

Integer multiples of the weight vector $\omega = 1/\sqrt{2} \in \mathbb{R}$ determine the weight lattice P, non-negative integer multiples of ω yield the set of dominant weights P^+. In the present notation, the symmetric orbit functions $\varphi_\lambda^{(0)}$ of A_1, parametrized by any $\lambda \in \mathbb{R}$, are of the form of the cosine functions,

$$\varphi_\lambda^{(0)}(z) = e^{2\pi i\langle\lambda,\,z\rangle} + e^{-2\pi i\langle\lambda,\,z\rangle} = 2\cos 2\pi\langle\lambda,\,z\rangle,$$

and the antisymmetric orbit functions $\varphi_\lambda^{(1)}$ become sine functions,

$$\varphi_\lambda^{(1)}(z) = e^{2\pi i\langle\lambda,\,z\rangle} - e^{-2\pi i\langle\lambda,\,z\rangle} = 2i\sin 2\pi\langle\lambda,\,z\rangle.$$

The four generalized ϱ-vectors of A_1, stemming from the admissible shift $\mu = \omega/2$ of the weight lattice in [17], are introduced by

$$\begin{aligned} \varrho^{(0,0)} &= 0, \\ \varrho^{(1,0)} &= \omega, \\ \varrho^{(0,1)} &= \frac{1}{2}\omega, \\ \varrho^{(1,1)} &= \frac{1}{2}\omega. \end{aligned}$$

The fundamental character function of A_1 is simplified as

$$\chi_\omega(z) = \frac{\varphi^{(1)}_{\omega+\varrho^{(1,0)}}(z)}{\varphi^{(1)}_{\varrho^{(1,0)}}(z)} = 2\cos 2\pi\langle\omega,\,z\rangle = 2\cos\theta, \tag{1}$$

where the substitution $\theta = 2\pi\langle\omega,\,z\rangle$ is used. The four types of Chebyshev polynomials $\mathbb{U}_\lambda^{(j,k)}$, labelled by dominant weights $\lambda \in P^+$ and with the fundamental character (1) as their common variable, are defined via relation

$$\mathbb{U}_\lambda^{(j,k)}(\chi_\omega(z)) = \frac{\varphi^{(j)}_{\lambda+\varrho^{(j,k)}}(z)}{\varphi^{(j)}_{\varrho^{(j,k)}}(z)}, \tag{2}$$

where the indices j,k are taking the values

$$A_1: \quad j \in \{0,1\},\, k \in \{0,1\}.$$

The four types of polynomials $\mathbb{U}_\lambda^{(j,k)}$, with $\lambda = n\omega = (n)$, $n \in \mathbb{Z}^{\geq 0}$ then directly become the four kinds of the classical Chebyshev polynomials,

$$\mathbb{U}_{(n)}^{(0,0)}(2x) = T_n(x),$$
$$\mathbb{U}_{(n)}^{(1,0)}(2x) = U_n(x),$$
$$\mathbb{U}_{(n)}^{(0,1)}(2x) = V_n(x),$$
$$\mathbb{U}_{(n)}^{(1,1)}(2x) = W_n(x).$$

Thus, the four corresponding generating functions $v^{(j,k)}$ of the polynomials $\mathbb{U}_{(n)}^{(j,k)}(x)$,

$$v^{(j,k)}(x,u) = \sum_{n=0}^{+\infty} \mathbb{U}_{(n)}^{(j,k)}(x)u^n, \tag{3}$$

are obtained by substitution $x \to x/2$ into the generating functions of the four kinds of Chebyshev polynomials,

$$v^{(0,0)}(x,u) = \frac{1-\frac{xu}{2}}{1-xu+u^2}, \tag{4}$$
$$v^{(1,0)}(x,u) = \frac{1}{1-xu+u^2}, \tag{5}$$
$$v^{(0,1)}(x,u) = \frac{1-u}{1-xu+u^2}, \tag{6}$$
$$v^{(1,1)}(x,u) = \frac{1+u}{1-xu+u^2}. \tag{7}$$

The four generating functions $v^{(j,k)}$ satisfy the following three symmetry relations,

$$v^{(0,0)}(-x,-u) = v^{(0,0)}(x,u),$$
$$v^{(1,0)}(-x,-u) = v^{(1,0)}(x,u),$$
$$v^{(0,1)}(-x,-u) = v^{(1,1)}(x,u),$$

which generate the parity relations for the polynomials $\mathbb{U}_{\lambda}^{(j,k)}$ of the form

$$\mathbb{U}_{(n)}^{(0,0)}(-x) = (-1)^n \mathbb{U}_{(n)}^{(0,0)}(x), \tag{8}$$
$$\mathbb{U}_{(n)}^{(1,0)}(-x) = (-1)^n \mathbb{U}_{(n)}^{(1,0)}(x), \tag{9}$$
$$\mathbb{U}_{(n)}^{(0,1)}(-x) = (-1)^n \mathbb{U}_{(n)}^{(1,1)}(x). \tag{10}$$

For the case A_1, define the K-polynomials of two variables $K_l(y_1,y_2)$ and $l \in \mathbb{Z}$ as

$$K_l(y_1,y_2) = \sum_{k=\lceil \frac{l}{2} \rceil}^{l} \frac{k!\, y_1^{2k-l} y_2^{l-k}}{(2k-l)!(l-k)!},$$

where $\lceil\ \rceil$ denotes the ceiling function. The several first K-polynomials of A_1 are given as

$$\begin{aligned}
K_0(y_1,y_2) =& 1,\\
K_1(y_1,y_2) =& y_1,\\
K_2(y_1,y_2) =& y_1^2+y_2,\\
K_3(y_1,y_2) =& y_1^3+2y_1y_2,\\
K_4(y_1,y_2) =& y_1^4+3y_1^2y_2+y_2^2,\\
K_5(y_1,y_2) =& y_1^5+4y_1^3y_2+3y_1y_2^2.
\end{aligned}$$

The explicit forms of the Chebyshev polynomials [4] are in terms of K-polynomials expressed as

$$\begin{aligned}
T_n\left(\frac{x}{2}\right) &= \mathbb{U}^{(0,0)}_{(n)}(x) = K_n(x,-1) - \frac{x}{2}K_{n-1}(x,-1),\\
U_n\left(\frac{x}{2}\right) &= \mathbb{U}^{(1,0)}_{(n)}(x) = K_n(x,-1),\\
V_n\left(\frac{x}{2}\right) &= \mathbb{U}^{(0,1)}_{(n)}(x) = K_n(x,-1) - K_{n-1}(x,-1),\\
W_n\left(\frac{x}{2}\right) &= \mathbb{U}^{(1,1)}_{(n)}(x) = K_n(x,-1) + K_{n-1}(x,-1).
\end{aligned}$$

2.2. Rank Two Root Systems and Weyl Groups

The ordered set of simple roots $\Delta = (\alpha_1, \alpha_2)$ of a simple Lie algebra of rank two is a collection of two vectors spanning a real two-dimensional Euclidean space $\mathbb{R}^2$ [28]. The simple roots of Δ form a basis of $\mathbb{R}^2$—they are fully specified by their lengths and the angle between them. The lengths of the simple roots and the angles between them are in accordance with the standard convention listed in Table 1. The coroots $\alpha_i^\vee$ are defined as $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i\rangle$, $i = 1, 2$. The Cartan matrix $C_{ij} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$ equivalently characterizes the root system Δ and the determinants

$$c = \det C \tag{11}$$

of rank two Cartan matrices are listed in Table 1. In addition to the α- and $\alpha^\vee$-bases, the weight ω-basis is defined by $\langle \alpha_i^\vee, \omega_j\rangle = \delta_{ij}$, $i, j \in \{1, 2\}$. The weight lattice P and the cone of dominant weights P^+ are of the form

$$P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2.$$

The dual weight $\omega^\vee$-basis is defined by $\left\langle \alpha_i, \omega_j^\vee \right\rangle = \delta_{ij}$, $i, j \in \{1, 2\}$, and the dual weight lattice $P^\vee$ is of the form

$$P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee.$$

The reflection r_α, $\alpha \in \Delta$, which fixes the hyperplane orthogonal to α and passes through the origin, is for $x \in \mathbb{R}^2$ explicitly written as $r_\alpha x = x - \langle \alpha, x\rangle \alpha^\vee$. The Weyl group W is a finite group generated by reflections $r_i \equiv r_{\alpha_i}$, $i = 1, 2$. Half of the number of elements of the Weyl group W is denoted by

$$p = \frac{|W|}{2}, \tag{12}$$

and the values of p are listed in Table 1.

Admissible shifts $\mu, \mu^\vee \in \mathbb{R}^2$ of the weight and dual weight lattices $P, P^\vee$ are defined as vectors which preserve the Weyl group invariance of the shifted lattices [17],

$$\begin{aligned}
W(\mu + P) &= \mu + P,\\
W(\mu^\vee + P^\vee) &= \mu^\vee + P^\vee.
\end{aligned}$$

For any irreducible crystallographic root system, all admissible shifts are classified in [17]. The admissible shifts of rank two root systems are listed in Table 1.

The two standard sign homomorphisms $\sigma^{(0)}, \sigma^{(1)} : W \to \{\pm 1\}$ are defined for any root system on $w \in W$ as

$$\sigma^{(0)}(w) = 1,$$
$$\sigma^{(1)}(w) = \det w.$$

The short sign homomorphism $\sigma^{(2)} : W \to \{\pm 1\}$ is defined for the root systems with two root-lengths C_2 and G_2 by its values on the generators r_α, $\alpha \in \Delta$. To the r_α of the short simple root, α is assigned the value -1; to the r_α of the long simple root, α is assigned 1. The long sign homomorphism $\sigma^{(3)}$ is given similarly, assigning the value -1 to the r_α of the long simple root α.

For all root systems, the generalized ϱ-vector $\varrho^{(0,0)}$ is defined as zero and the vector $\varrho^{(1,0)}$ as half of the sum of the positive roots,

$$\varrho^{(0,0)} = 0,$$
$$\varrho^{(1,0)} = \omega_1 + \omega_2.$$

The short and long ϱ-vectors $\varrho^{(2,0)}$, $\varrho^{(3,0)}$ which are halves of the sums of the positive short or long roots, respectively, are listed for C_2 and G_2 in Table 1.

For the root system C_2, which admits the non-trivial admissible shift $\mu = \frac{1}{2}\omega_2$, four additional minimal ϱ-vectors in the shifted weight lattice are given as

$$\varrho^{(0,1)} = \varrho^{(3,1)} = \frac{1}{2}\omega_2,$$
$$\varrho^{(1,1)} = \varrho^{(2,1)} = \omega_1 + \frac{1}{2}\omega_2.$$

Table 1. Simple roots of rank two irreducible root systems, short and long minimal weights $\varrho^{(2,0)}$, $\varrho^{(3,0)}$, admissible shifts μ, $\mu^\vee$ and the determinants of the Cartan matrices c.

	$\langle\alpha_1, \alpha_1\rangle$	$\langle\alpha_2, \alpha_2\rangle$	$\angle\alpha_1, \alpha_2$	p	c	$\varrho^{(2,0)}$	$\varrho^{(3,0)}$	μ	$\mu^\vee$
A_2	2	2	$\frac{2\pi}{3}$	3	3	–	–	–	–
C_2	1	2	$\frac{3\pi}{4}$	4	2	ω_1	ω_2	$\frac{1}{2}\omega_2$	$\frac{1}{2}\omega_1^\vee$
G_2	2	$\frac{2}{3}$	$\frac{5\pi}{6}$	6	1	ω_2	ω_1	–	–

2.3. Four Types of Orbit Functions

The four types of special functions which correspond to the Weyl groups are recalled [3,5,6,9]. Each of these special functions together with an admissible shift induces a family of orthogonal polynomials. Up to four types of normalized orbit functions $\varphi_\lambda^{(j)} : \mathbb{R}^2 \to \mathbb{C}$ are parametrized by $\lambda \in \mathbb{R}^2$ and given in general form as

$$\varphi_\lambda^{(j)}(z) = \sum_{w \in W} \sigma^{(j)}(w)\, e^{2\pi \mathrm{i} \langle w\lambda, z \rangle}, \quad z \in \mathbb{R}^2, \tag{13}$$

where $j = 0, 1$ for the root system A_2 and $j = 0, \ldots, 3$ for C_2 and G_2.

2.3.1. Orbit Functions of A_2

The two types of orbit functions of A_2 are explicitly evaluated for a point with coordinates in $\alpha^\vee$-basis (z_1, z_2) and a weight with coordinates in ω-basis (λ_1, λ_2),

$$\begin{aligned}\varphi^{(0)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&e^{2\pi \mathrm{i}(z_1\lambda_1+z_2\lambda_2)}+e^{-2\pi \mathrm{i}(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2)}+e^{2\pi \mathrm{i}(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2)}\\&+e^{-2\pi \mathrm{i}(z_1\lambda_2+z_2\lambda_1)}+e^{-2\pi \mathrm{i}(z_1\lambda_1+z_1\lambda_2-z_2\lambda_1)}+e^{2\pi \mathrm{i}(z_1\lambda_2-z_2\lambda_1-z_2\lambda_2)},\\ \varphi^{(1)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&e^{2\pi \mathrm{i}(z_1\lambda_1+z_2\lambda_2)}-e^{-2\pi \mathrm{i}(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2)}-e^{2\pi \mathrm{i}(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2)}\\&-e^{-2\pi \mathrm{i}(z_1\lambda_2+z_2\lambda_1)}+e^{-2\pi \mathrm{i}(z_1\lambda_1+z_1\lambda_2-z_2\lambda_1)}+e^{2\pi \mathrm{i}(z_1\lambda_2-z_2\lambda_1-z_2\lambda_2)}.\end{aligned}$$

2.3.2. Orbit Functions of C_2

The four types of orbit functions of C_2 are explicitly evaluated for a point with coordinates in $\alpha^\vee$-basis (z_1,z_2) and a weight with coordinates in ω-basis (λ_1,λ_2)

$$\begin{aligned}\varphi^{(0)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_1-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2))\\&+2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)),\\ \varphi^{(1)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_1-z_2\lambda_2))-2\cos(2\pi(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2))\\&-2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)),\\ \varphi^{(2)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&-2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_1-z_2\lambda_2))-2\cos(2\pi(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2))\\&+2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)),\\ \varphi^{(3)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&-2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_1-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1-z_2\lambda_1-z_2\lambda_2))\\&-2\cos(2\pi(z_1\lambda_1+2z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)).\end{aligned}$$

2.3.3. Orbit Functions of G_2

The four types of orbit functions of G_2 are explicitly evaluated for a point with coordinates in $\alpha^\vee$-basis (z_1,z_2) and a weight with coordinates in ω-basis (λ_1,λ_2)

$$\begin{aligned}\varphi^{(0)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2\cos(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-z_2\lambda_2))+2\cos(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))\\&+2\cos(2\pi(z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1-3z_2\lambda_1-z_2\lambda_2))\\&+2\cos(2\pi(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)),\end{aligned}$$

$$\begin{aligned}\varphi^{(1)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2\cos(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-z_2\lambda_2))-2\cos(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))\\&+2\cos(2\pi(z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))-2\cos(2\pi(z_1\lambda_1-3z_2\lambda_1-z_2\lambda_2))\\&-2\cos(2\pi(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2))+2\cos(2\pi(z_1\lambda_1+z_2\lambda_2)),\end{aligned}$$

$$\begin{aligned}\varphi^{(2)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2i(-\sin(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-z_2\lambda_2))+\sin(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))\\&+\sin(2\pi(z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))-\sin(2\pi(z_1\lambda_1-3z_2\lambda_1-z_2\lambda_2))\\&-\sin(2\pi(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2))+\sin(2\pi(z_1\lambda_1+z_2\lambda_2))),\end{aligned}$$

$$\begin{aligned}\varphi^{(3)}_{(\lambda_1,\lambda_2)}(z_1,z_2) =&2i(-\sin(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-z_2\lambda_2))-\sin(2\pi(2z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))\\&+\sin(2\pi(z_1\lambda_1+z_1\lambda_2-3z_2\lambda_1-2z_2\lambda_2))+\sin(2\pi(z_1\lambda_1-3z_2\lambda_1-z_2\lambda_2))\\&+\sin(2\pi(z_1\lambda_1+z_1\lambda_2-z_2\lambda_2))+\sin(2\pi(z_1\lambda_1+z_2\lambda_2))).\end{aligned}$$

2.4. Discrete Orthogonality of Orbit Functions

The fundamental domain F of the affine Weyl group W^{aff} and the dual fundamental domain $F^\vee$ of the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ are for the rank two cases triangles with their vertices explicitly given in [17]. For any sign homomorphism σ, the boundaries of the subsets of the fundamental domains

$F^{\sigma}(\mu) \subset F$ and $F^{\sigma\vee}(\mu^{\vee}) \subset F^{\vee}$, with values of the admissible shifts μ and $\mu^{\vee}$ either zero or taken from their classification in Table 1, follow from Equations (57) and (61) in [17]. The functions $\varepsilon : \mathbb{R}^2 \to \mathbb{N}$ and $h_M^{\vee} : \mathbb{R}^2 \to \mathbb{N}$, $M \in \mathbb{N}$ are determined by the orders of the affine stabilizers $\mathrm{Stab}_{W^{\mathrm{aff}}}(z)$ and $\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}\left(\frac{z}{M}\right)$ via relations

$$\varepsilon(z) = \frac{|W|}{|\mathrm{Stab}_{W^{\mathrm{aff}}}(z)|}, \quad h_M^{\vee}(z) = \left|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}\left(\frac{z}{M}\right)\right|. \tag{14}$$

The domains $F^{\sigma}(\mu)$ induce discrete point sets $F_M^{\sigma}(\mu, \mu^{\vee}) \subset F$,

$$F_M^{\sigma}(\mu, \mu^{\vee}) = \left[\frac{1}{M}(\mu^{\vee} + P^{\vee})\right] \cap F^{\sigma}(\mu) \tag{15}$$

and the domains $F^{\sigma\vee}(\mu^{\vee})$ determine the corresponding shifted weight sets

$$\Lambda_M^{\sigma}(\mu, \mu^{\vee}) = (\mu + P) \cap M F^{\sigma\vee}(\mu^{\vee}). \tag{16}$$

For any sign homomorphism $\sigma^{(j)}$ and any two labels $\lambda, \lambda' \in \Lambda_M^{\sigma}(\mu, \mu^{\vee})$, the discrete orthogonality relations of the rank two orbit functions are according to ([17] (Thm 4.1)) of the form

$$\sum_{z \in F_M^{\sigma^{(j)}}(\mu, \mu^{\vee})} \varepsilon(z) \varphi_{\lambda}^{(j)}(z) \overline{\varphi_{\lambda'}^{(j)}(z)} = c\, |W|\, M^2 h_M^{\vee}(\lambda)\, \delta_{\lambda,\lambda'}, \tag{17}$$

where c is the determinant of the Cartan matrix (11).

2.5. Orthogonal Polynomials

Common variables of families of orthogonal polynomials are for each root system defined via the standard character functions χ_{λ},

$$\chi_{\lambda}(z) = \frac{\varphi^{(1)}_{\lambda + \varrho^{(1,0)}}(z)}{\varphi^{(1)}_{\varrho^{(1,0)}}(z)}, \quad \lambda \in P^{+}. \tag{18}$$

The character functions are explicitly evaluated using the explicit forms of the antisymmetric functions $\varphi_{\lambda}^{(1)}$. The special role as variables of the orthogonal polynomials play fundamental characters corresponding to the fundamental weights $\chi_{\omega_1}, \chi_{\omega_2}$.

For a point with coordinates in $\alpha^{\vee}$-basis (z_1, z_2) and the fundamental weights ω_1 and ω_2, the fundamental character functions are explicitly calculated as

$$\begin{aligned}
A_2 : \chi_{\omega_1}(z_1, z_2) &= e^{2\pi i z_1} + e^{-2\pi i (z_1 - z_2)} + e^{-2\pi i z_2}, && (19)\\
\chi_{\omega_2}(z_1, z_2) &= e^{2\pi i z_2} + e^{2\pi i (z_1 - z_2)} + e^{-2\pi i z_1}, && (20)\\
C_2 : \chi_{\omega_1}(z_1, z_2) &= 2\cos 2\pi(z_1 - z_2) + 2\cos(2\pi z_1) && (21)\\
\chi_{\omega_2}(z_1, z_2) &= 1 + 2\cos 2\pi(2z_1 - z_2) + 2\cos(2\pi z_2), && (22)\\
G_2 : \chi_{\omega_1}(z_1, z_2) &= 2 + 2\cos 2\pi(z_1 - z_2) + 2\cos 2\pi(z_1 - 2z_2) + 2\cos(2\pi z_2)\\
&\quad + 2\cos 2\pi(2z_1 - 3z_2) + 2\cos 2\pi(z_1 - 3z_2) + 2\cos(2\pi z_1), && (23)\\
\chi_{\omega_2}(z_1, z_2) &= 1 + 2\cos 2\pi(z_1 - z_2) + 2\cos 2\pi(z_1 - 2z_2) + 2\cos(2\pi z_2). && (24)
\end{aligned}$$

The fourteen families of two-variable orthogonal polynomials $\mathbb{U}_{\lambda}^{(j,k)}(x_1, x_2)$, labelled uniformly by the dominant weights $\lambda \in P^{+}$, are induced by the relations

$$\mathbb{U}_\lambda^{(j,k)}(\chi_{\omega_1}(z),\chi_{\omega_2}(z)) = \frac{\varphi^{(j)}_{\lambda+\varrho^{(j,k)}}(z)}{\varphi^{(j)}_{\varrho^{(j,k)}}(z)}, \quad \lambda \in P^+, \tag{25}$$

with the indices j,k taking the values

$$\begin{aligned} A_2 :& \quad j \in \{0,1\},\, k=0, \\ C_2 :& \quad j \in \{0,1,2,3\},\, k \in \{0,1\}, \\ G_2 :& \quad j \in \{0,1,2,3\},\, k=0. \end{aligned} \tag{26}$$

2.5.1. Polynomials of A_2

The lowest orthogonal polynomials (25) of variables x_1, x_2, given by relations (19) and (20), are for the first kind $\mathbb{U}_\lambda^{(0,0)}$ of the following explicit form:

$$\begin{aligned} \mathbb{U}^{(0,0)}_{(0,0)}(x_1,x_2) &= 1, \\ \mathbb{U}^{(0,0)}_{(1,0)}(x_1,x_2) &= \frac{1}{3}x_1, \\ \mathbb{U}^{(0,0)}_{(0,1)}(x_1,x_2) &= \frac{1}{3}x_2, \\ \mathbb{U}^{(0,0)}_{(1,1)}(x_1,x_2) &= \frac{1}{6}x_1x_2 - \frac{1}{2}, \\ \mathbb{U}^{(0,0)}_{(2,1)}(x_1,x_2) &= \frac{1}{6}x_1^2x_2 - \frac{1}{3}x_2^2 - \frac{1}{6}x_1, \\ \mathbb{U}^{(0,0)}_{(1,2)}(x_1,x_2) &= \frac{1}{6}x_1x_2^2 - \frac{1}{3}x_1^2 - \frac{1}{6}x_2, \end{aligned}$$

and for the second kind $\mathbb{U}_\lambda^{(1,0)}$ are of the form

$$\begin{aligned} \mathbb{U}^{(1,0)}_{(0,0)}(x_1,x_2) &= 1, \\ \mathbb{U}^{(1,0)}_{(1,0)}(x_1,x_2) &= x_1, \\ \mathbb{U}^{(1,0)}_{(0,1)}(x_1,x_2) &= x_2, \\ \mathbb{U}^{(1,0)}_{(1,1)}(x_1,x_2) &= x_1x_2 - 1, \\ \mathbb{U}^{(1,0)}_{(2,1)}(x_1,x_2) &= x_1^2x_2 - x_2^2 - x_1, \\ \mathbb{U}^{(1,0)}_{(1,2)}(x_1,x_2) &= x_1x_2^2 - x_1^2 - x_2. \end{aligned}$$

2.5.2. Polynomials of C_2

The lowest orthogonal polynomials (25) of variables x_1, x_2, given by relations (21) and (22), are for the first kind $\mathbb{U}_\lambda^{(0,0)}$ of the following explicit form:

$$\begin{aligned} \mathbb{U}^{(0,0)}_{(0,0)}(x_1,x_2) &= 1, \\ \mathbb{U}^{(0,0)}_{(1,0)}(x_1,x_2) &= \frac{1}{4}x_1, \\ \mathbb{U}^{(0,0)}_{(0,1)}(x_1,x_2) &= \frac{1}{4}x_2 - \frac{1}{4}, \\ \mathbb{U}^{(0,0)}_{(1,1)}(x_1,x_2) &= \frac{1}{8}x_1x_2 - \frac{3}{8}x_1, \\ \mathbb{U}^{(0,0)}_{(2,1)}(x_1,x_2) &= \frac{1}{8}x_1^2x_2 - \frac{1}{8}x_1^2 - \frac{1}{4}x_2^2 - \frac{1}{4}x_2 + \frac{1}{2}, \\ \mathbb{U}^{(0,0)}_{(1,2)}(x_1,x_2) &= -\frac{1}{4}x_1^3 + \frac{1}{8}x_1x_2^2 + \frac{1}{8}x_1x_2 + \frac{1}{2}x_1, \end{aligned}$$

and for the second kind $\mathbb{U}_{\lambda}^{(1,0)}$ are of the form

$$
\begin{aligned}
&\mathbb{U}_{(0,0)}^{(1,0)}(x_1,x_2) = 1,\\
&\mathbb{U}_{(1,0)}^{(1,0)}(x_1,x_2) = x_1,\\
&\mathbb{U}_{(0,1)}^{(1,0)}(x_1,x_2) = x_2,\\
&\mathbb{U}_{(1,1)}^{(1,0)}(x_1,x_2) = x_1x_2 - x_1,\\
&\mathbb{U}_{(2,1)}^{(1,0)}(x_1,x_2) = x_1^2x_2 - x_1^2 - x_2^2 - x_2 + 1,\\
&\mathbb{U}_{(1,2)}^{(1,0)}(x_1,x_2) = -x_1^3 + x_1x_2^2 + x_1.
\end{aligned}
$$

The lowest polynomials of the short second kind $\mathbb{U}_{\lambda}^{(2,0)}$ are of the following explicit form:

$$
\begin{aligned}
&\mathbb{U}_{(0,0)}^{(2,0)}(x_1,x_2) = 1,\\
&\mathbb{U}_{(1,0)}^{(2,0)}(x_1,x_2) = x_1,\\
&\mathbb{U}_{(0,1)}^{(2,0)}(x_1,x_2) = \frac{1}{2}x_2 + \frac{1}{2},\\
&\mathbb{U}_{(1,1)}^{(2,0)}(x_1,x_2) = \frac{1}{2}x_1x_2 - \frac{1}{2}x_1,\\
&\mathbb{U}_{(2,1)}^{(2,0)}(x_1,x_2) = \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_2 + \frac{1}{2},\\
&\mathbb{U}_{(1,2)}^{(2,0)}(x_1,x_2) = -x_1^3 + \frac{1}{2}x_1x_2^2 + x_1x_2 + \frac{3}{2}x_1,
\end{aligned}
$$

and the polynomials of the long second kind $\mathbb{U}_{\lambda}^{(3,0)}$ are of the form

$$
\begin{aligned}
&\mathbb{U}_{(0,0)}^{(3,0)}(x_1,x_2) = 1,\\
&\mathbb{U}_{(1,0)}^{(3,0)}(x_1,x_2) = \frac{1}{2}x_1,\\
&\mathbb{U}_{(0,1)}^{(3,0)}(x_1,x_2) = x_2 - 1,\\
&\mathbb{U}_{(1,1)}^{(3,0)}(x_1,x_2) = \frac{1}{2}x_1x_2 - x_1,\\
&\mathbb{U}_{(2,1)}^{(3,0)}(x_1,x_2) = \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_1^2 - x_2^2 + 1,\\
&\mathbb{U}_{(1,2)}^{(3,0)}(x_1,x_2) = -\frac{1}{2}x_1^3 + \frac{1}{2}x_1x_2^2 - \frac{1}{2}x_1x_2 + x_1.
\end{aligned}
$$

The lowest polynomials of the third kind $\mathbb{U}_{\lambda}^{(0,1)}$ are of the following explicit form:

$$
\begin{aligned}
&\mathbb{U}_{(0,0)}^{(0,1)}(x_1,x_2) = 1,\\
&\mathbb{U}_{(1,0)}^{(0,1)}(x_1,x_2) = \frac{1}{2}x_1 - 1,\\
&\mathbb{U}_{(0,1)}^{(0,1)}(x_1,x_2) = -x_1 + x_2,\\
&\mathbb{U}_{(1,1)}^{(0,1)}(x_1,x_2) = -\frac{1}{2}x_1^2 + \frac{1}{2}x_1x_2 - \frac{1}{2}x_1 + 1,\\
&\mathbb{U}_{(2,1)}^{(0,1)}(x_1,x_2) = -\frac{1}{2}x_1^3 + \frac{1}{2}x_1^2x_2 + x_1x_2 - x_2^2 + \frac{1}{2}x_1 - x_2 + 1,\\
&\mathbb{U}_{(1,2)}^{(0,1)}(x_1,x_2) = -\frac{1}{2}x_1^3 - \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1x_2^2 + \frac{3}{2}x_1^2 - 1,
\end{aligned}
$$

and the polynomials of the fourth kind $\mathbb{U}_\lambda^{(1,1)}$ are of the form

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(1,1)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(1,1)}(x_1,x_2) &= x_1+1,\\
\mathbb{U}_{(0,1)}^{(1,1)}(x_1,x_2) &= x_1+x_2+1,\\
\mathbb{U}_{(1,1)}^{(1,1)}(x_1,x_2) &= x_1^2+x_1x_2-1,\\
\mathbb{U}_{(2,1)}^{(1,1)}(x_1,x_2) &= x_1^3+x_1^2x_2-x_1x_2-x_2^2-2x_1-2x_2,\\
\mathbb{U}_{(1,2)}^{(1,1)}(x_1,x_2) &= -x_1^3+x_1^2x_2+x_1x_2^2-2x_1^2+x_1x_2+1.
\end{aligned}$$

The lowest polynomials of the short fourth kind $\mathbb{U}_\lambda^{(2,1)}$ are of the following explicit form:

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(2,1)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(2,1)}(x_1,x_2) &= x_1-1,\\
\mathbb{U}_{(0,1)}^{(2,1)}(x_1,x_2) &= -x_1+x_2+1,\\
\mathbb{U}_{(1,1)}^{(2,1)}(x_1,x_2) &= -x_1^2+x_1x_2+1,\\
\mathbb{U}_{(2,1)}^{(2,1)}(x_1,x_2) &= -x_1^3+x_1^2x_2+x_1x_2-x_2^2+2x_1-2x_2,\\
\mathbb{U}_{(1,2)}^{(2,1)}(x_1,x_2) &= -x_1^3-x_1^2x_2+x_1x_2^2+2x_1^2+x_1x_2-1,
\end{aligned}$$

and the polynomials of the long fourth kind $\mathbb{U}_\lambda^{(3,1)}$ are of the form

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(3,1)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(3,1)}(x_1,x_2) &= \frac{1}{2}x_1+1,\\
\mathbb{U}_{(0,1)}^{(3,1)}(x_1,x_2) &= x_1+x_2,\\
\mathbb{U}_{(1,1)}^{(3,1)}(x_1,x_2) &= \frac{1}{2}x_1^2+\frac{1}{2}x_1x_2-\frac{1}{2}x_1-1,\\
\mathbb{U}_{(2,1)}^{(3,1)}(x_1,x_2) &= \frac{1}{2}x_1^3+\frac{1}{2}x_1^2x_2-x_1x_2-x_2^2-\frac{1}{2}x_1-x_2+1,\\
\mathbb{U}_{(1,2)}^{(3,1)}(x_1,x_2) &= -\frac{1}{2}x_1^3+\frac{1}{2}x_1^2x_2+\frac{1}{2}x_1x_2^2-\frac{3}{2}x_1^2+1.
\end{aligned}$$

2.5.3. Polynomials of G_2

The lowest orthogonal polynomials (25) of variables x_1, x_2, given by relations (23) and (24), are for the first kind $\mathbb{U}_\lambda^{(0,0)}$ of the following explicit form:

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(0,0)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(0,0)}(x_1,x_2) &= \frac{1}{6}x_1-\frac{1}{6}x_2-\frac{1}{6},\\
\mathbb{U}_{(0,1)}^{(0,0)}(x_1,x_2) &= \frac{1}{6}x_2-\frac{1}{6},\\
\mathbb{U}_{(1,1)}^{(0,0)}(x_1,x_2) &= \frac{1}{12}x_1x_2-\frac{1}{4}x_2^2+\frac{1}{4}x_1+\frac{1}{6}x_2+\frac{5}{12},\\
\mathbb{U}_{(2,1)}^{(0,0)}(x_1,x_2) &= -\frac{1}{6}x_2^4+\frac{1}{12}x_1^2x_2+\frac{1}{4}x_1x_2^2+\frac{1}{3}x_2^3+\frac{1}{12}x_1^2-\frac{1}{6}x_1x_2+\frac{1}{6}x_2^2-\frac{1}{4}x_1-\frac{1}{2}x_2-\frac{1}{6},\\
\mathbb{U}_{(1,2)}^{(0,0)}(x_1,x_2) &= \frac{1}{12}x_1x_2^2-\frac{1}{12}x_2^3-\frac{1}{6}x_1^2-\frac{1}{12}x_1x_2+\frac{1}{3}x_2^2-\frac{1}{6}x_1-\frac{1}{12}x_2-\frac{1}{6},
\end{aligned}$$

and for the second kind $\mathbb{U}_\lambda^{(1,0)}$ are of the form

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(1,0)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(1,0)}(x_1,x_2) &= x_1,\\
\mathbb{U}_{(0,1)}^{(1,0)}(x_1,x_2) &= x_2,\\
\mathbb{U}_{(1,1)}^{(1,0)}(x_1,x_2) &= x_1x_2 - x_2^2 + x_1 + 1,\\
\mathbb{U}_{(2,1)}^{(1,0)}(x_1,x_2) &= -x_2^4 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2 + 2x_2^2 - x_2 - 1,\\
\mathbb{U}_{(1,2)}^{(1,0)}(x_1,x_2) &= x_1x_2^2 - x_2^3 - x_1^2 + x_2^2 - x_1 + x_2.
\end{aligned}$$

The lowest polynomials of the short second kind $\mathbb{U}_\lambda^{(2,0)}$ are of the following explicit form:

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(2,0)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(2,0)}(x_1,x_2) &= \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2},\\
\mathbb{U}_{(0,1)}^{(2,0)}(x_1,x_2) &= x_2 + 1,\\
\mathbb{U}_{(1,1)}^{(2,0)}(x_1,x_2) &= \frac{1}{2}x_1x_2 - \frac{1}{2}x_2^2 + x_1 - \frac{1}{2}x_2,\\
\mathbb{U}_{(2,1)}^{(2,0)}(x_1,x_2) &= -x_2^4 + \frac{1}{2}x_1^2x_2 + 2x_1x_2^2 + \frac{1}{2}x_2^3 + \frac{1}{2}x_1^2 + x_1x_2 + \frac{3}{2}x_2^2 - x_1 - \frac{1}{2}x_2 - \frac{1}{2},\\
\mathbb{U}_{(1,2)}^{(2,0)}(x_1,x_2) &= \frac{1}{2}x_1x_2^2 - \frac{1}{2}x_2^3 - \frac{1}{2}x_1^2 + x_2 + \frac{1}{2},
\end{aligned}$$

and the polynomials of the long second kind $\mathbb{U}_\lambda^{(3,0)}$ are of the form

$$\begin{aligned}
\mathbb{U}_{(0,0)}^{(3,0)}(x_1,x_2) &= 1,\\
\mathbb{U}_{(1,0)}^{(3,0)}(x_1,x_2) &= x_1 - x_2 + 1,\\
\mathbb{U}_{(0,1)}^{(3,0)}(x_1,x_2) &= \frac{1}{2}x_2 - \frac{1}{2},\\
\mathbb{U}_{(1,1)}^{(3,0)}(x_1,x_2) &= \frac{1}{2}x_1x_2 - x_2^2 + \frac{1}{2}x_1 + x_2 + 1,\\
\mathbb{U}_{(2,1)}^{(3,0)}(x_1,x_2) &= -\frac{1}{2}x_2^4 + \frac{1}{2}x_1^2x_2 + \frac{3}{2}x_2^3 + \frac{1}{2}x_1^2 - \frac{1}{2}x_1x_2 + \frac{1}{2}x_1 - 2x_2,\\
\mathbb{U}_{(1,2)}^{(3,0)}(x_1,x_2) &= \frac{1}{2}x_1x_2^2 - \frac{1}{2}x_2^3 - x_1^2 + \frac{3}{2}x_2^2 - \frac{3}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}.
\end{aligned}$$

2.6. Discrete Orthogonality of Polynomials

The X-transform is a mapping $X : \mathbb{R}^2 \to \mathbb{C}^2$ defined for any $x \in \mathbb{R}^2$ as

$$X(x) = (\chi_{\omega_1}(x), \chi_{\omega_2}(x)).$$

For a non–trivial admissible shift $\mu \neq 0$, the point and label sets $\Omega_M^{(j,1)}(\mu^\vee)$ and $L_M^{(j,1)}(\mu^\vee)$ are defined as

$$\Omega_M^{(j,1)}(\mu^\vee) = X\left(F_M^{\sigma(j)}(\mu,\mu^\vee)\right), \quad L_M^{(j,1)}(\mu^\vee) = -\varrho^{(j,1)} + \Lambda_M^{\sigma(j)}(\mu,\mu^\vee) \tag{27}$$

and for $\mu = 0$ the point and label sets $\Omega_M^{(j,0)}(\mu^\vee)$ and $L_M^{(j,0)}(\mu^\vee)$ as

$$\Omega_M^{(j,0)}(\mu^\vee) = X\left(F_M^{\sigma(j)}(0,\mu^\vee)\right), \quad L_M^{(j,0)}(\mu^\vee) = -\varrho^{(j,0)} + \Lambda_M^{\sigma(j)}(0,\mu^\vee). \tag{28}$$

The restricted X-transform $X_M^{(j,k)}(\mu^\vee)$, which produces the transformed point sets $\Omega_M^{(j,k)}(\mu^\vee)$ via relations (27) and (28), is one-to-one [16] and therefore the numbers of points and labels coincide [17],

$$|\Omega_M^{(j,k)}(\mu^\vee)| = |L_M^{(j,k)}(\mu^\vee)|. \tag{29}$$

The discrete function $\widetilde{\varepsilon} : \Omega_M^{(j,k)}(\mu^\vee) \to \mathbb{N}$ is defined for any $y \in \Omega_M^{(j,k)}(\mu^\vee)$ by relation

$$\widetilde{\varepsilon}(y) = \varepsilon\left(\left(X_M^{(j,k)}(\mu^\vee)\right)^{-1} y\right).$$

The polynomial weight functions $w^{(j,k)}(x_1, x_2)$ are defined by

$$w^{(j,k)}(\chi_{\omega_1}(z), \chi_{\omega_2}(z)) = \left|\varphi^{(j)}_{\varrho^{(j,k)}}(z)\right|^2.$$

The weight functions $w^{(0,0)}(x_1, x_2)$ are constant,

$$w^{(0,0)}(x_1, x_2) = |W|^2 = 4p^2,$$

and the non-constant polynomial weight functions $w^{(j,k)}(x_1, x_2)$ are given explicitly as

$$\begin{aligned}
A_2 :\ & w^{(1,0)}(x_1, x_2) = x_1^2 x_2^2 - 4x_1^3 - 4x_2^3 + 18x_1 x_2 - 27,\\
C_2 :\ & w^{(1,0)}(x_1, x_2) = -4x_1^4 + x_1^2 x_2^2 + 22x_1^2 x_2 - 4x_2^3 - 7x_1^2 - 20x_2^2 - 12x_2 + 36,\\
& w^{(2,0)}(x_1, x_2) = 4x_1^2 - 16x_2 + 16,\\
& w^{(3,0)}(x_1, x_2) = -16x_1^2 + 4x_2^2 + 24x_2 + 36,\\
& w^{(0,1)}(x_1, x_2) = 8x_1 + 4x_2 + 12,\\
& w^{(1,1)}(x_1, x_2) = -2x_1^3 + x_1^2 x_2 + 3x_1^2 + 8x_1 x_2 - 4x_2^2 - 8x_1 - 8x_2 + 12,\\
& w^{(2,1)}(x_1, x_2) = 2x_1^3 + x_1^2 x_2 + 3x_1^2 - 8x_1 x_2 - 4x_2^2 + 8x_1 - 8x_2 + 12,\\
& w^{(3,1)}(x_1, x_2) = -8x_1 + 4x_2 + 12,\\
G_2 :\ & w^{(1,0)}(x_1, x_2) = -4x_2^5 + x_1^2 x_2^2 + 26x_1 x_2^3 - 7x_2^4 - 4x_1^3 - 38x_1^2 x_2 + 26x_1 x_2^2 + 32x_2^3 - 47x_1^2 - 58x_1 x_2\\
& \qquad - 10x_2^2 - 42x_1 - 28x_2 + 49,\\
& w^{(2,0)}(x_1, x_2) = -4x_2^2 + 16x_1 - 8x_2 + 28,\\
& w^{(3,0)}(x_1, x_2) = 16x_2^3 - 4x_1^2 - 40x_1 x_2 - 4x_2^2 - 40x_1 - 8x_2 + 28.
\end{aligned}$$

The discrete orthogonality of Weyl orbit functions (17) induces for all $\lambda, \lambda' \in L_M^{(j,k)}(\mu^\vee)$ the discrete orthogonality of the bivariate polynomials $\mathbb{U}_\lambda^{(j,k)}$,

$$\sum_{y \in \Omega_M^{(j,k)}(\mu^\vee)} \widetilde{\varepsilon}(y) w^{(j,k)}(y)\, \mathbb{U}_\lambda^{(j,k)}(y) \overline{\mathbb{U}_{\lambda'}^{(j,k)}(y)} = c\,|W|\, M^2\, h_M^\vee(\lambda + \varrho^{(j,k)}) \cdot \delta_{\lambda,\lambda'}.$$

3. Generating Functions

3.1. General Form of Generating Functions

The method for constructing generating functions of bivariate polynomials $\mathbb{U}_\lambda^{(j,k)}$ closely follows the method for the classical Chebyshev polynomials in Section 2.1. Taking two supplementary variables $u_1, u_2 \in \mathbb{R}$ and $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 = (\lambda_1, \lambda_2) \in P^+$ in ω-basis, the corresponding generating functions $v^{(j,k)}$ of $\mathbb{U}_{(\lambda_1,\lambda_2)}^{(j,k)}(x_1, x_2)$, with indices j, k given by (26), are defined standardly as

$$v^{(j,k)}(x_1, x_2, u_1, u_2) = \sum_{\lambda_1,\lambda_2=0}^{+\infty} \mathbb{U}_{(\lambda_1,\lambda_2)}^{(j,k)}(x_1, x_2) u_1^{\lambda_1} u_2^{\lambda_2}. \tag{30}$$

The families of polynomials $\mathbb{U}_{(\lambda_1,\lambda_2)}^{(j,k)}(x_1, x_2)$ are from the power series $v^{(j,k)}(x_1, x_2, u_1, u_2)$ generated by the standard differentiation

$$\mathbb{U}_{(\lambda_1,\lambda_2)}^{(j,k)}(x_1, x_2) = \frac{1}{\lambda_1!\lambda_2!} \frac{\partial^{\lambda_1}}{\partial u_1^{\lambda_1}} \frac{\partial^{\lambda_2}}{\partial u_2^{\lambda_2}} v^{(j,k)}(x_1, x_2, 0, 0). \tag{31}$$

The infinite sum in the defining relation (30) can be further evaluated. Considering $|u_1|, |u_2| < 1$ and using the formula for infinite geometric series, the following finite sum is obtained:

$$\begin{aligned} v^{(j,k)}(\chi_{\omega_1}(z), \chi_{\omega_2}(z), u_1, u_2) &= \sum_{\lambda_1,\lambda_2=0}^{+\infty} \frac{\varphi^{(j)}_{\lambda+\varrho^{(j,k)}}(z)}{\varphi^{(j)}_{\varrho^{(j,k)}}(z)} u_1^{\lambda_1} u_2^{\lambda_2} \\ &= \frac{1}{\varphi^{(j)}_{\varrho^{(j,k)}}(z)} \sum_{w\in W} \frac{\sigma^{(j)}(w)\, e^{2\pi \mathrm{i}\langle w\varrho^{(j,k)}, z\rangle}}{(1-u_1 e^{2\pi\mathrm{i}\langle w\omega_1, z\rangle})(1-u_2 e^{2\pi\mathrm{i}\langle w\omega_2, z\rangle})}. \end{aligned} \tag{32}$$

The functions in (32) need to be expressed as functions of

$$x_1 = \chi_{\omega_1}, \quad x_2 = \chi_{\omega_2}. \tag{33}$$

It appears that the form of the resulting generating functions $v^{(j,k)}$ is analogous to fractions (4)–(7). For each $v^{(j,k)}$, there is a numerator function $N^{(j,k)}$ of the form

$$N^{(j,k)}(x_1, x_2, u_1, u_2) = \sum_{k_1,k_2=0}^{p-1} \alpha^{(j,k)}_{(k_1,k_2)}(x_1, x_2) u_1^{k_1} u_2^{k_2}, \tag{34}$$

with polynomial coefficients $\alpha^{(j,k)}_{(k_1,k_2)}(x_1, x_2)$ and the parameter p given by (12). Two denominator functions D_1 and D_2, common for all $v^{(j,k)}$, are of the form

$$D_m(x_1, x_2, u_m) = 1 - \sum_{k=1}^{p} a^{(m)}_k\, u_m^k, \quad m = 1, 2. \tag{35}$$

The resulting general form of $v^{(j,k)}$ is

$$v^{(j,k)}(x_1, x_2, u_1, u_2) = \frac{N^{(j,k)}(x_1, x_2, u_1, u_2)}{D_1(x_1, x_2, u_1) D_2(x_1, x_2, u_2)}. \tag{36}$$

In the following, the explicit forms of the generating functions are listed. The explicit forms of all generating functions are straightforwardly verified by substituting (33) into (36) and comparing the result directly to (32).

3.2. The Lie Algebra A_2

The two numerator functions $N^{(0,0)}$ and $N^{(1,0)}$ are given as

$$\begin{aligned} 6N^{(0,0)}(x_1, x_2, u_1, u_2) =& 6 - 4x_2u_2 - 4x_1u_1 + 2x_1u_2^2 + (3x_1x_2 - 3)u_1u_2 \\ &+ 2x_2u_1^2 + (-2x_1^2 + 2x_2)u_1u_2^2 + (-2x_2^2 + 2x_1)u_1^2u_2 + (x_1x_2 - 3)u_1^2u_2^2, \end{aligned} \tag{37}$$

$$N^{(1,0)}(u_1, u_2) = 1 - u_1u_2, \tag{38}$$

and the denominator functions D_1, D_2 are

$$D_1(x_1, x_2, u_1) = 1 - x_1u_1 + x_2u_1^2 - u_1^3, \tag{39}$$

$$D_2(x_1, x_2, u_2) = 1 - x_2u_2 + x_1u_2^2 - u_2^3. \tag{40}$$

3.3. The Lie Algebra C_2

The four coefficients $\alpha^{(j,0)}_{(k_1,k_2)}$ from (34) of the four numerator functions $N^{(j,0)}$ are listed in Table 2 while coefficients $\alpha^{(j,1)}_{(k_1,k_2)}$ of the four numerator functions $N^{(j,1)}$ are listed in Table 3. From this table, one can see, for example, that the function $N^{(1,0)}$ is given as

$$N^{(1,0)}(x_1, u_1, u_2) = 1 + u_2 - x_1u_1u_2 + u_1^2u_2 + u_1^2u_2^2.$$

The two denominator functions D_1, D_2 are given as

$$
\begin{aligned}
D_1(x_1, x_2, u_1) =& 1 - x_1 u_1 + (x_2 + 1)u_1^2 - x_1 u_1^3 + u_1^4, \\
D_2(x_1, x_2, u_2) =& 1 + (-x_2 + 1)u_2 + (x_1^2 - 2x_2)u_2^2 + (-x_2 + 1)u_2^3 + u_2^4.
\end{aligned}
$$

Table 2. Coefficients $\alpha^{(j,0)}_{(k_1,k_2)}$ of the numerator functions (34) of C_2.

(k_1, k_2)	$8\alpha^{(0,0)}_{(k_1,k_2)}$	$\alpha^{(2,0)}_{(k_1,k_2)}$	$\alpha^{(3,0)}_{(k_1,k_2)}$	$\alpha^{(1,0)}_{(k_1,k_2)}$
$(0,0)$	8	1	1	1
$(0,1)$	$-6x_2 + 6$	$-\frac{1}{2}x_2 + \frac{3}{2}$	0	1
$(0,2)$	$4x_1^2 - 8x_2$	$-\frac{1}{2}x_2 + \frac{3}{2}$	-1	0
$(0,3)$	$-2x_2 + 2$	1	0	0
$(1,0)$	$-6x_1$	0	$-\frac{1}{2}x_1$	0
$(1,1)$	$5x_1x_2 - 7x_1$	$-x_1$	$-\frac{1}{2}x_1$	$-x_1$
$(1,2)$	$-4x_1^3 + 9x_1x_2 + x_1$	$\frac{1}{2}x_1x_2 - \frac{1}{2}x_1$	x_1	0
$(1,3)$	$2x_1x_2 - 4x_1$	$-x_1$	0	0
$(2,0)$	$4x_2 + 4$	-1	1	0
$(2,1)$	$2x_1^2 - 4x_2^2 - 2x_2 + 6$	x_2	$\frac{1}{2}x_1^2 - x_2 + 1$	1
$(2,2)$	$3x_1^2x_2 + x_1^2 - 6x_2^2 - 8x_2 + 6$	$-x_1^2 + \frac{3}{2}x_2 + \frac{3}{2}$	$-x_2$	1
$(2,3)$	$2x_1^2 - 2x_2^2 - 2x_2 + 4$	$\frac{1}{2}x_2 + \frac{1}{2}$	0	0
$(3,0)$	$-2x_1$	0	0	0
$(3,1)$	$2x_1x_2 - 4x_1$	0	$-\frac{1}{2}x_1$	0
$(3,2)$	$-2x_1^3 + 5x_1x_2 + x_1$	0	$\frac{1}{2}x_1$	0
$(3,3)$	$x_1x_2 - 3x_1$	0	0	0

Table 3. Coefficients $\alpha^{(j,1)}_{(k_1,k_2)}$ of the numerator functions (34) of C_2.

(k_1, k_2)	$\alpha^{(0,1)}_{(k_1,k_2)}$	$\alpha^{(1,1)}_{(k_1,k_2)}$	$\alpha^{(2,1)}_{(k_1,k_2)}$	$\alpha^{(3,1)}_{(k_1,k_2)}$
$(0,0)$	1	1	1	1
$(0,1)$	$-x_1 + 1$	$x_1 + 2$	$-x_1 + 2$	$x_1 + 1$
$(0,2)$	$x_1 - 1$	1	1	$-x_1 - 1$
$(0,3)$	-1	0	0	-1
$(1,0)$	$-\frac{1}{2}x_1 - 1$	1	-1	$-\frac{1}{2}x_1 + 1$
$(1,1)$	$\frac{1}{2}x_1^2 - x_1 + x_2$	$-x_1 - x_2$	$-x_1 + x_2$	$-\frac{1}{2}x_1^2 - x_1 - x_2$
$(1,2)$	$-x_1^2 + \frac{1}{2}x_1 + x_2$	$-x_1 - x_2$	$-x_1 + x_2$	$x_1^2 + \frac{1}{2}x_1 - x_2$
$(1,3)$	$x_1 - 1$	1	-1	$x_1 + 1$
$(2,0)$	$\frac{1}{2}x_1 + 1$	0	0	$-\frac{1}{2}x_1 + 1$
$(2,1)$	$\frac{1}{2}x_1^2 - \frac{1}{2}x_1x_2 - x_1 - x_2 + 2$	1	1	$\frac{1}{2}x_1^2 + \frac{1}{2}x_1x_2 + x_1 - x_2 + 2$
$(2,2)$	$\frac{1}{2}x_1^2 + \frac{1}{2}x_1x_2 - 2x_2 + 1$	$x_1 + 2$	$-x_1 + 2$	$\frac{1}{2}x_1^2 - \frac{1}{2}x_1x_2 - 2x_2 + 1$
$(2,3)$	$\frac{1}{2}x_1 - x_2$	1	1	$-\frac{1}{2}x_1 - x_2$
$(3,0)$	-1	0	0	1
$(3,1)$	$-\frac{1}{2}x_1 + x_2$	0	0	$-\frac{1}{2}x_1 - x_2$
$(3,2)$	$-\frac{1}{2}x_1^2 + x_2$	0	0	$\frac{1}{2}x_1^2 - x_2$
$(3,3)$	$\frac{1}{2}x_1 - 1$	0	0	$\frac{1}{2}x_1 + 1$

3.4. The Lie Algebra G_2

The coefficients $\alpha^{(0,0)}_{(k_1,k_2)}$ of the numerator function are listed in Table 4, the coefficients $\alpha^{(1,0)}_{(k_1,k_2)}$, $\alpha^{(2,0)}_{(k_1,k_2)}$ are listed in Table 5 and the coefficients $\alpha^{(3,0)}_{(k_1,k_2)}$ are listed in Table 6. The two denominator functions D_1, D_2 are given as

$$\begin{aligned} D_1(x_1,x_2,u_1) =& 1+(-x_1+x_2+1)u_1+(x_2^3-3x_1x_2-2x_1-x_2+1)u_1^2 \\ &+(2x_2^3-x_1^2-4x_1x_2-x_2^2-4x_1-2x_2+1)u_1^3+(x_2^3-3x_1x_2-2x_1-x_2+1)u_1^4 \\ &+(-x_1+x_2+1)u_1^5+u_1^6, \\ D_2(x_1,x_2,u_2) =& 1+(-x_2+1)u_2+(x_1+1)u_2^2+(-x_2^2+2x_1+1)u_2^3 \\ &+(x_1+1)u_2^4+(-x_2+1)u_2^5+u_2^6. \end{aligned}$$

Table 4. Coefficients $\alpha^{(0,0)}_{(k_1,k_2)}$ of the numerator function (34) of G_2.

(k_1,k_2)	$12\,\alpha^{(0,0)}_{(k_1,k_2)}$
$(0,0)$	12
$(0,1)$	$-10x_2+10$
$(0,2)$	$8x_1+8$
$(0,3)$	$-6x_2^2+12x_1+6$
$(0,4)$	$4x_1+4$
$(0,5)$	$-2x_2+2$
$(1,0)$	$-10x_1+10x_2+10$
$(1,1)$	$9x_1x_2-11x_2^2-5x_1+2x_2+13$
$(1,2)$	$2x_2^3-8x_1^2+3x_1x_2-x_2^2+x_1+2x_2+9$
$(1,3)$	$6x_1x_2^2-6x_2^3-12x_1^2+10x_1x_2-4x_2^2+6x_1+2x_2+12$
$(1,4)$	$-4x_1^2+4x_1x_2+2x_2^2-2x_2+8$
$(1,5)$	$2x_1x_2-2x_2^2-2x_1-2x_2+4$
$(2,0)$	$8x_2^3-24x_1x_2-16x_1-8x_2+8$
$(2,1)$	$-8x_2^4+25x_1x_2^2+7x_2^3-2x_1^2-6x_1x_2+7x_2^2-17x_1-11x_2+9$
$(2,2)$	$7x_1x_2^3-x_2^4-21x_1^2x_2+3x_1x_2^2+11x_2^3-19x_1^2-29x_1x_2-5x_2^2-13x_1-7x_2+10$
$(2,3)$	$-6x_2^5+30x_1x_2^3-35x_1^2x_2+11x_1x_2^2+16x_2^3-27x_1^2-35x_1x_2-11x_2^2-12x_1-2x_2+11$
$(2,4)$	$4x_1x_2^3-12x_1^2x_2-x_1x_2^2+3x_2^3-8x_1^2-8x_1x_2-x_2^2-7x_1+x_2+5$
$(2,5)$	$-2x_2^4+6x_1x_2^2+2x_2^3-x_1x_2+3x_2^2-9x_1-6x_2-1$
$(3,0)$	$12x_2^3-6x_1^2-24x_1x_2-6x_2^2-24x_1-12x_2+6$
$(3,1)$	$-12x_2^4+6x_1^2x_2+23x_1x_2^2+19x_2^3-4x_1^2-2x_1x_2+7x_2^2-23x_1-23x_2+5$
$(3,2)$	$13x_1x_2^3-3x_2^4-6x_1^3-26x_1^2x_2+2x_1x_2^2+13x_2^3-30x_1^2-40x_1x_2-17x_1-13x_2+11$
$(3,3)$	$-10x_2^5+5x_1^2x_2^2+40x_1x_2^3+5x_2^4-10x_1^3-42x_1^2x_2+14x_1x_2^2+26x_2^3-47x_1^2-62x_1x_2$ $-14x_2^2-16x_1-16x_2+17$
$(3,4)$	$9x_1x_2^3-3x_2^4-4x_1^3-18x_1^2x_2+4x_1x_2^2+9x_2^3-20x_1^2-28x_1x_2+2x_2^2-11x_1-9x_2+9$
$(3,5)$	$-4x_2^4+2x_1^2x_2+7x_1x_2^2+7x_2^3-2x_1x_2+3x_2^2-7x_1-11x_2+1$
$(4,0)$	$4x_2^3-12x_1x_2-8x_1-4x_2+4$
$(4,1)$	$-4x_2^4+12x_1x_2^2+4x_2^3-3x_1x_2+5x_2^2-13x_1-10x_2+1$
$(4,2)$	$4x_1x_2^3-12x_1^2x_2-x_1x_2^2+3x_2^3-8x_1^2-8x_1x_2-x_2^2-7x_1+x_2+5$
$(4,3)$	$-4x_2^5+20x_1x_2^3-23x_1^2x_2+7x_1x_2^2+12x_2^3-19x_1^2-25x_1x_2-9x_2^2-12x_1+9$
$(4,4)$	$3x_1x_2^3-x_2^4-9x_1^2x_2+3x_1x_2^2+7x_2^3-11x_1^2-13x_1x_2-5x_2^2-9x_1-3x_2+6$
$(4,5)$	$-2x_2^4+7x_1x_2^2+x_2^3-2x_1^2+x_2^2-5x_1+x_2+3$
$(5,0)$	$-2x_1+2x_2+2$
$(5,1)$	$2x_1x_2-2x_2^2-2x_1-2x_2+4$
$(5,2)$	$-2x_1^2+2x_1x_2+2x_2^2-4x_2+6$
$(5,3)$	$2x_1x_2^2-2x_2^3-4x_1^2+2x_1x_2+2x_1-2x_2+8$
$(5,4)$	$2x_2^3-2x_1^2-3x_1x_2-x_2^2+x_1-4x_2+3$
$(5,5)$	$x_1x_2-3x_2^2+3x_1+2x_2+5$

Table 5. Coefficients $\alpha^{(1,0)}_{(k_1,k_2)}$, $\alpha^{(2,0)}_{(k_1,k_2)}$ of the numerator functions (34) of G_2.

(k_1,k_2)	$\alpha^{(2,0)}_{(k_1,k_2)}$	$\alpha^{(1,0)}_{(k_1,k_2)}$
$(0,0)$	1	1
$(0,1)$	2	1
$(0,2)$	$-x_2+1$	0
$(0,3)$	2	0
$(0,4)$	1	0
$(1,0)$	$-\frac{1}{2}x_1+\frac{3}{2}x_2+\frac{1}{2}$	x_2+1
$(1,1)$	$-x_2^2-\frac{1}{2}x_1+\frac{5}{2}x_2+\frac{3}{2}$	$-x_2^2+x_1+x_2+2$
$(1,2)$	$x_1x_2-x_2^2+x_2+1$	x_1+1
$(1,3)$	$-2x_1+x_2+2$	$-x_2+1$
$(1,4)$	$-x_1+x_2+2$	1
$(2,0)$	$-\frac{1}{2}x_1x_2+\frac{1}{2}x_2^2+\frac{1}{2}x_1+\frac{1}{2}$	x_2+1
$(2,1)$	$\frac{1}{2}x_1x_2^2+\frac{1}{2}x_2^3-\frac{1}{2}x_1^2-3x_1x_2-\frac{3}{2}x_1-\frac{3}{2}x_2+1$	$-x_2^2+x_2+1$
$(2,2)$	$-x_2^4+3x_1x_2^2+x_2^3-\frac{1}{2}x_1^2-\frac{1}{2}x_1x_2+x_2^2-x_1-\frac{1}{2}x_2+\frac{3}{2}$	$x_1x_2-x_2^2+x_1+2x_2+1$
$(2,3)$	$2x_2^3-\frac{11}{2}x_1x_2-\frac{3}{2}x_2^2-\frac{7}{2}x_1-x_2+\frac{5}{2}$	$-x_2^2+x_2+1$
$(2,4)$	$x_2^3-3x_1x_2-\frac{5}{2}x_1+\frac{1}{2}x_2+\frac{3}{2}$	x_2+1
$(3,0)$	$\frac{1}{2}x_1x_2-\frac{1}{2}x_2^2-\frac{1}{2}x_1-\frac{1}{2}$	1
$(3,1)$	$-\frac{1}{2}x_1x_2^2+\frac{3}{2}x_2^3-\frac{3}{2}x_1x_2-x_2^2-\frac{5}{2}x_1-2x_2-\frac{1}{2}$	$-x_2+1$
$(3,2)$	$-x_2^4+\frac{1}{2}x_1^2x_2+2x_1x_2^2+\frac{3}{2}x_2^3-\frac{1}{2}x_1^2+\frac{1}{2}x_2^2-2x_1-\frac{3}{2}x_2+\frac{1}{2}$	x_1+1
$(3,3)$	$\frac{1}{2}x_1x_2^2+\frac{5}{2}x_2^3-2x_1^2-\frac{13}{2}x_1x_2-x_2^2-\frac{11}{2}x_1-2x_2+\frac{5}{2}$	$-x_2^2+x_1+x_2+2$
$(3,4)$	$2x_2^3-x_1^2-\frac{9}{2}x_1x_2-\frac{1}{2}x_2^2-\frac{7}{2}x_1-2x_2+\frac{3}{2}$	x_2+1
$(4,0)$	$\frac{1}{2}x_1-\frac{3}{2}x_2-\frac{1}{2}$	0
$(4,1)$	$-\frac{1}{2}x_1x_2+\frac{3}{2}x_2^2-\frac{1}{2}x_1-x_2-\frac{1}{2}$	0
$(4,2)$	$\frac{1}{2}x_1^2-\frac{1}{2}x_1x_2-x_1-\frac{3}{2}x_2-\frac{1}{2}$	0
$(4,3)$	$-\frac{1}{2}x_1x_2^2+\frac{3}{2}x_2^3+\frac{1}{2}x_1^2-3x_1x_2-\frac{5}{2}x_1-\frac{1}{2}x_2+1$	1
$(4,4)$	$x_2^3-\frac{5}{2}x_1x_2-\frac{1}{2}x_2^2-\frac{5}{2}x_1-x_2+\frac{1}{2}$	1
$(5,0)$	-1	0
$(5,1)$	x_2	0
$(5,2)$	$-x_1-x_2$	0
$(5,3)$	$x_2^2-\frac{3}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}$	0
$(5,4)$	$-\frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}$	0

Table 6. Coefficients $\alpha^{(3,0)}_{(k_1,k_2)}$ of the numerator function (34) of G_2.

(k_1,k_2)	$\alpha^{(3,0)}_{(k_1,k_2)}$
$(0,0)$	1
$(0,1)$	$-\frac{1}{2}x_2+\frac{1}{2}$
$(0,2)$	$\frac{1}{2}x_2-\frac{1}{2}$
$(0,3)$	-1
$(1,0)$	2
$(1,1)$	$-\frac{1}{2}x_2^2+x_1-x_2+\frac{5}{2}$
$(1,2)$	$\frac{1}{2}x_2^3-\frac{3}{2}x_1x_2+\frac{3}{2}x_1-\frac{3}{2}x_2+1$
$(1,3)$	$-\frac{1}{2}x_1x_2+\frac{3}{2}x_1-\frac{3}{2}x_2+\frac{1}{2}$
$(1,4)$	$\frac{1}{2}x_2^2-x_2+\frac{1}{2}$
$(1,5)$	$-\frac{1}{2}x_2+\frac{1}{2}$
$(2,0)$	$-x_1+x_2+1$
$(2,1)$	$x_1x_2-\frac{3}{2}x_2^2+x_2+\frac{3}{2}$
$(2,2)$	$\frac{1}{2}x_2^3-x_1^2-x_2^2+x_2+\frac{1}{2}$
$(2,3)$	$\frac{1}{2}x_1x_2^2-\frac{1}{2}x_2^3-x_1^2+x_1x_2-\frac{1}{2}x_2^2+\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}$
$(2,4)$	$-\frac{1}{2}x_2^3+x_1x_2+x_2^2+\frac{1}{2}$
$(2,5)$	$\frac{1}{2}x_2^2-x_1-x_2-\frac{1}{2}$
$(3,0)$	2
$(3,1)$	$-\frac{3}{2}x_2+\frac{3}{2}$
$(3,2)$	$-\frac{1}{2}x_2^2+2x_1+x_2+\frac{3}{2}$
$(3,3)$	$\frac{1}{2}x_1x_2-2x_2^2+\frac{5}{2}x_1+\frac{3}{2}x_2+\frac{3}{2}$
$(3,4)$	$-\frac{1}{2}x_2^3+\frac{3}{2}x_1x_2+\frac{1}{2}x_1+\frac{3}{2}x_2+1$
$(3,5)$	$\frac{1}{2}x_2^2-x_1-x_2-\frac{1}{2}$
$(4,0)$	1
$(4,1)$	$-x_2+1$
$(4,2)$	x_1+1
$(4,3)$	$-x_2^2+2x_1+2$
$(4,4)$	$x_1-\frac{1}{2}x_2+\frac{3}{2}$
$(4,5)$	$-\frac{1}{2}x_2+\frac{1}{2}$

4. Explicit Formulas for Polynomials

4.1. General Form of Evaluating Formulas

The goal of this section is to derive the explicit formulas of the two-variable orthogonal polynomials from their generating functions. The explicit formulas are derived by generalizing an evaluation procedure for Chebyshev polynomials of the first and the second kind [4] and using the Bell polynomials. The incomplete exponential Bell polynomials $B_{l,k}$ are defined [29,30] for a sequence of variables $(t_1, t_2, \dots)$ and a set of indices $j_1, j_2, \dots \in \mathbb{Z}^{\geq 0}$ as

$$B_{l,k}(t_1, t_2, \dots) = \sum_{\substack{j_1+j_2+\dots=k \\ j_1+2j_2+\dots=l}} \frac{l!}{j_1! j_2! \cdots} \left(\frac{t_1}{1!}\right)^{j_1} \left(\frac{t_2}{2!}\right)^{j_2} \cdots . \tag{41}$$

The sequence of K-polynomials of p variables $K_l(y_1, \dots, y_p)$, $l \in \mathbb{Z}$, with the number p associated to the Weyl group of each rank two algebra by relation (12), are introduced via the Bell polynomials $B_{l,k}$ for $l \in \mathbb{Z}^{\geq 0}$ as

$$K_l(y_1, \dots, y_p) = \frac{1}{l!} \sum_{k=0}^{l} k! B_{l,k}(1! y_1, 2! y_2, \dots, p! y_p, 0, 0, \dots), \tag{42}$$

and for $l \in \mathbb{Z}^{<0}$ as

$$K_l(y_1, \dots, y_p) = 0. \tag{43}$$

Using the K-polynomials, the general form of the evaluating formulas is for each class of two-variable polynomials determined in the following theorem.

Theorem 1. *The two-variable polynomials (25) are of the form*

$$\mathbb{U}^{(j,k)}_{(\lambda_1,\lambda_2)}(x_1, x_2) = \sum_{k_1,k_2=0}^{p-1} \alpha^{(j,k)}_{(k_1,k_2)} K_{\lambda_1-k_1}(a_1^{(1)}, \dots, a_p^{(1)}) K_{\lambda_2-k_2}(a_1^{(2)}, \dots, a_p^{(2)}), \tag{44}$$

with the numerator coefficients $\alpha^{(j,k)}_{(k_1,k_2)}$ determined by (34) and the denominator coefficients $a_k^{(m)}$ defined by (35).

Proof. Applying the Faà di Bruno's formula [29] to expand each function $1/D_m$ as power series of u_m, the following expansions involving K-polynomials (42) are obtained:

$$\frac{1}{D_m} = \sum_{l=0}^{+\infty} K_l(a_1^{(m)}, \dots, a_p^{(m)}) u_m^l, \quad m = 1, 2.$$

Taking into account the convention (43), the generating function of the form (36) is then expanded into the power series as

$$\begin{aligned}
\frac{N^{(j,k)}}{D_1 D_2} &= N^{(j,k)} \sum_{l_1,l_2=0}^{+\infty} K_{l_1}(a_1^{(1)}, \dots, a_p^{(1)}) K_{l_2}(a_1^{(2)}, \dots, a_p^{(2)}) u_1^{l_1} u_2^{l_2} \\
&= \sum_{l_1,l_2=0}^{+\infty} \sum_{k_1,k_2=0}^{p-1} \alpha^{(j,k)}_{(k_1,k_2)} K_{l_1}(a_1^{(1)}, \dots, a_p^{(1)}) K_{l_2}(a_1^{(2)}, \dots, a_p^{(2)}) u_1^{l_1+k_1} u_2^{l_2+k_2} \\
&= \sum_{\lambda_1,\lambda_2=0}^{+\infty} \sum_{k_1,k_2=0}^{p-1} \alpha^{(j,k)}_{(k_1,k_2)} K_{\lambda_1-k_1}(a_1^{(1)}, \dots, a_p^{(1)}) K_{\lambda_2-k_2}(a_1^{(2)}, \dots, a_p^{(2)}) u_1^{\lambda_1} u_2^{\lambda_2}.
\end{aligned} \tag{45}$$

Comparing the resulting power series expansion (45) to defining relation (30) yields the result. □

The K-polynomials are for each rank two case further explicitly evaluated in the following sections.

4.2. K-Polynomials of A_2

For the case A_2, the K-polynomials of three variables $K_l(y_1, y_2, y_3)$ are brought to the compact form

$$K_l(y_1,y_2,y_3) = \sum_{k=\lceil \frac{l}{3} \rceil}^{l} \sum_{j_3=\max\{0,l-2k\}}^{\lfloor \frac{l-k}{2} \rfloor} \frac{k!\, y_1^{2k-l+j_3} y_2^{l-k-2j_3} y_3^{j_3}}{(2k-l+j_3)!(l-k-2j_3)!j_3!},$$

where $\lceil\,\rceil$ and $\lfloor\,\rfloor$ denote the ceiling and the floor function, respectively. The several first K-polynomials of A_2 are given as

$$\begin{aligned}
K_0(y_1,y_2,y_3) =&1,\\
K_1(y_1,y_2,y_3) =&y_1,\\
K_2(y_1,y_2,y_3) =&y_1^2+y_2,\\
K_3(y_1,y_2,y_3) =&y_1^3+2y_1y_2+y_3,\\
K_4(y_1,y_2,y_3) =&y_1^4+3y_1^2y_2+2y_1y_3+y_2^2,\\
K_5(y_1,y_2,y_3) =&y_1^5+4y_1^3y_2+3y_1^2y_3+3y_1y_2^2+2y_2y_3,\\
K_6(y_1,y_2,y_3) =&y_1^6+5y_1^4y_2+4y_1^3y_3+6y_1^2y_2^2+6y_1y_2y_3+y_2^3+y_3^2.
\end{aligned}$$

Example 1. *Utilization of K-polynomials in evaluation formula (44) is demonstrated for the polynomials* $\mathbb{U}^{(0,0)}_{(2,3)}$ *and* $\mathbb{U}^{(1,0)}_{(2,3)}$ *of* A_2*. Taking into account the numerator coefficients in (37), (38) and denominator coefficients (39) and (40), the polynomial* $\mathbb{U}^{(0,0)}_{(2,3)}$ *is calculated as*

$$\begin{aligned}
6\mathbb{U}^{(0,0)}_{(2,3)}(x_1,x_2) =&6K_2(x_1,-x_2,1)K_3(x_2,-x_1,1)-4x_2K_2(x_1,-x_2,1)K_2(x_2,-x_1,1)\\
&-4x_1K_1(x_1,-x_2,1)K_3(x_2,-x_1,1)+2x_1K_2(x_1,-x_2,1)K_1(x_2,-x_1,1)\\
&+(3x_1x_2-3)K_1(x_1,-x_2,1)K_2(x_2,-x_1,1)+2x_2K_0(x_1,-x_2,1)K_3(x_2,-x_1,1)\\
&+(-2x_1^2+2x_2)K_1(x_1,-x_2,1)K_1(x_2,-x_1,1)+(-2x_2^2+2x_1)K_0(x_1,-x_2,1)K_2(x_2,-x_1,1)\\
&+(x_1x_2-3)K_0(x_1,-x_2,1)K_1(x_2,-x_1,1)\\
=&x_1^2x_2^3-3x_1^3x_2-2x_2^4+6x_1x_2^2+3x_1^2-7x_2,
\end{aligned}$$

and similarly the polynomial $\mathbb{U}^{(1,0)}_{(2,3)}$ *is calculated as*

$$\begin{aligned}
\mathbb{U}^{(1,0)}_{(2,3)}(x_1,x_2) =&K_2(x_1,-x_2,1)K_3(x_2,-x_1,1)-K_1(x_1,-x_2,1)K_2(x_2,-x_1,1)\\
=&x_1^2x_2^3-2x_1^3x_2-x_2^4+x_1x_2^2+2x_1^2-x_2.
\end{aligned}$$

4.3. K-Polynomials of C_2

For the case C_2, the K-polynomials of four variables $K_l(y_1,y_2,y_3,y_4)$ are of the form

$$K_l(y_1,y_2,y_3,y_4) = \sum_{k=\lceil \frac{l}{4} \rceil}^{l} \sum_{j_3=0}^{\lfloor \frac{l-k}{2} \rfloor} \sum_{j_4=\max\{0,\lceil \frac{l-2k-j_3}{2} \rceil\}}^{\lfloor \frac{l-k-2j_3}{3} \rfloor} \frac{k!\, y_1^{2k-l+j_3+2j_4} y_2^{l-k-2j_3-3j_4} y_3^{j_3} y_4^{j_4}}{(2k-l+j_3+2j_4)!(l-k-2j_3-3j_4)!j_3!j_4!}.$$

The several first K-polynomials of C_2 are given as

$$\begin{aligned}
K_0(y_1,y_2,y_3,y_4) =&1,\\
K_1(y_1,y_2,y_3,y_4) =&y_1,\\
K_2(y_1,y_2,y_3,y_4) =&y_1^2+y_2,\\
K_3(y_1,y_2,y_3,y_4) =&y_1^3+2y_1y_2+y_3,\\
K_4(y_1,y_2,y_3,y_4) =&y_1^4+3y_1^2y_2+2y_1y_3+y_2^2+y_4,\\
K_5(y_1,y_2,y_3,y_4) =&y_1^5+4y_1^3y_2+3y_1^2y_3+3y_1y_2^2+2y_1y_4+2y_2y_3,\\
K_6(y_1,y_2,y_3,y_4) =&y_1^6+5y_1^4y_2+4y_1^3y_3+6y_1^2y_2^2+3y_1^2y_4+6y_1y_2y_3+y_2^3+2y_2y_4+y_3^2.
\end{aligned}$$

4.4. K-Polynomials of G_2

For the case G_2, the K-polynomials of six variables $K_l(y_1, y_2, y_3, y_4, y_5, y_6)$ are of the form

$$K_l(y_1, y_2, y_3, y_4, y_5, y_6) = \sum_{k, j_3, j_4, j_5, j_6} \frac{k!\, y_1^{2k-l+j_3+2j_4+3j_5+4j_6} y_2^{l-k-2j_3-3j_4-4j_5-5j_6} y_3^{j_3} y_4^{j_4} y_5^{j_5} y_6^{j_6}}{(2k-l+j_3+2j_4+3j_5+4j_6)!(l-k-2j_3-3j_4-4j_5-5j_6)!j_3!j_4!j_5!j_6!}$$

with the indices in the sum taking values

$$\begin{aligned} k &= \left\lceil \frac{l}{6} \right\rceil \dots l, \\ j_3 &= 0 \dots \left\lfloor \frac{l-k}{2} \right\rfloor, \\ j_4 &= 0 \dots \left\lfloor \frac{l-k}{3} \right\rfloor, \\ j_5 &= 0 \dots \left\lfloor \frac{l-k}{4} \right\rfloor, \\ j_6 &= \max\left\{0, \left\lceil \frac{l-2k-j_3-2j_4-3j_5}{4} \right\rceil \right\} \dots \left\lfloor \frac{l-k-2j_3-3j_4-4j_5}{5} \right\rfloor. \end{aligned}$$

The several first K-polynomials of G_2 are given as

$$\begin{aligned} K_0(y_1, y_2, y_3, y_4, y_5, y_6) &= 1, \\ K_1(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1, \\ K_2(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1^2 + y_2, \\ K_3(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1^3 + 2y_1y_2 + y_3, \\ K_4(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1^4 + 3y_1^2y_2 + 2y_1y_3 + y_2^2 + y_4, \\ K_5(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1^5 + 4y_1^3y_2 + 3y_1^2y_3 + 3y_1y_2^2 + 2y_1y_4 + 2y_2y_3 + y_5, \\ K_6(y_1, y_2, y_3, y_4, y_5, y_6) &= y_1^6 + 5y_1^4y_2 + 4y_1^3y_3 + 6y_1^2y_2^2 + 3y_1^2y_4 + 6y_1y_2y_3 + y_2^3 + 2y_1y_5 + 2y_2y_4 + y_3^2 + y_6. \end{aligned}$$

5. Conclusions

- The explicit evaluating formulas (44) comprise the family of p-variate case-dependent polynomials K_l. The construction of the K-polynomials from truncation formula (42) relies on solving the two indices' equations inside definition (41) of Bell polynomials $B_{l,k}$ with respect to j_1 and j_2,

$$\begin{aligned} j_1 + j_2 + \cdots + j_p &= k, \\ j_1 + 2j_2 + \cdots + pj_p &= l. \end{aligned}$$

  Compared to calculation by a direct differentiation from (31) and by recursive formulas, the Formula (44) indeed represents efficient and straightforward means of evaluation of any given polynomial $\mathbb{U}_\lambda^{(j,k)}$.
- Two distinct renormalizations of polynomials $\mathbb{U}_\lambda^{(j,k)}$, inherited from normalizations of the underlying orbit functions, are mainly used throughout the literature [2,3,31–33]. Between the normalized orbit functions (13), summed over the entire Weyl group W, and orbit functions $\widehat{\varphi}_\lambda^{(j)}$, added over the group orbit $O(\lambda)$ only, holds the following relation:

$$\varphi_\lambda^{(j)} = h_\lambda \widehat{\varphi}_\lambda^{(j)},$$

  where $h_\lambda = |\mathrm{Stab}_W \lambda|$ denotes the order of the stabilizer of $\lambda \in \mathbb{R}^2$ in the group W. Thus, the two polynomials $\mathbb{U}_\lambda^{(j,k)}$ and the polynomials $\widehat{\mathbb{U}}_\lambda^{(j,k)}$ induced by $\widehat{\varphi}_\lambda^{(j)}$ are intertwined as

$$\mathbb{U}_\lambda^{(j,k)} = \frac{h_{\lambda+\varrho^{(j,k)}}}{h_{\varrho^{(j,k)}}} \widehat{\mathbb{U}}_\lambda^{(j,k)}.$$

- Three principal choices of the variables of the polynomials $\mathbb{U}_\lambda^{(j,k)}$ and $\widehat{\mathbb{U}}_\lambda^{(j,k)}$ and their corresponding generating functions are available. The present fundamental character variables x_1 and x_2, the fundamental normalized symmetric orbit functions variables $c_1 = \varphi_{(1,0)}^{(0)}$, $c_2 = \varphi_{(0,1)}^{(0)}$ and the fundamental symmetric orbit functions variables $\widehat{c}_1 = \widehat{\varphi}_{(1,0)}^{(0)}$, $\widehat{c}_2 = \widehat{\varphi}_{(0,1)}^{(0)}$ satisfy the following linear transformations:

$$\begin{aligned} A_2: \ c_1 =& 2\widehat{c}_1 = 2x_1, \\ c_2 =& 2\widehat{c}_2 = 2x_2, \\ C_2: \ c_1 =& 2\widehat{c}_1 = 2x_1, \\ c_2 =& 2\widehat{c}_2 = 2x_2 - 2, \\ G_2: \ c_1 =& 2\widehat{c}_1 = 2x_1 - 2x_2 - 2, \\ c_2 =& 2\widehat{c}_2 = 2x_2 - 2. \end{aligned}$$

- Generating functions represent an efficient tool for analyzing the symmetries of the generated polynomials. The symmetries of the C_2 generating functions

$$\begin{aligned} v^{(j,0)}(-x_1, x_2, -u_1, u_2) &= v^{(j,0)}(x_1, u_1, u_1, u_2), \\ v^{(0,1)}(-x_1, x_2, -u_1, u_2) &= v^{(3,1)}(x_1, u_1, u_1, u_2), \\ v^{(1,1)}(-x_1, x_2, -u_1, u_2) &= v^{(2,1)}(x_1, u_1, u_1, u_2), \end{aligned}$$

  generalize the parity properties of the classical Chebyshev polynomials (8)–(10) for the case C_2 in the following six relations:

$$\begin{aligned} \mathbb{U}^{(j,0)}_{(\lambda_1,\lambda_2)}(-x_1, x_2) &= (-1)^{\lambda_1}\mathbb{U}^{(j,0)}_{(\lambda_1,\lambda_2)}(x_1, x_2), \\ \mathbb{U}^{(0,1)}_{(\lambda_1,\lambda_2)}(-x_1, x_2) &= (-1)^{\lambda_1}\mathbb{U}^{(3,1)}_{(\lambda_1,\lambda_2)}(x_1, x_2), \\ \mathbb{U}^{(1,1)}_{(\lambda_1,\lambda_2)}(-x_1, x_2) &= (-1)^{\lambda_1}\mathbb{U}^{(2,1)}_{(\lambda_1,\lambda_2)}(x_1, x_2). \end{aligned}$$

  Similar parity properties do not appear for the cases A_2 and G_2.
- The advent of generating functions in the representation theory of Lie groups can be traced to papers of Cayley [34] and Sylvester and Franklin [35], which predate the emergence of the representation theory of Lie groups. In the terminology that is challenging to a contemporary reader, they calculated the generating functions for the structure of polynomial rings, where the variables are transforming under irreducible representations of $SU(2)$ of dimensions $2, 3, \dots, 13$. A group theoretical 'interpretation' of their results came a century later [26]. Invaluable information provided by such generating function is about the existence, degrees and structure of the syzygies that are present in such rings.
- The current fourteen generating functions of bivariate polynomials include for completeness the five bivariate cases from [31–33]. The present calculation procedure as well as the procedure in [31–33] is based on case-by-case analysis of the given Lie algebra. Expressing the general form of character generators [19] in rational polynomial form potentially yields the polynomial generating functions $v^{(1,0)}$ for any case. Specific polynomial form of character generators together with the polynomial generating function of the remaining polynomial classes deserve further study. Even a more challenging and deep problems poses formation of polynomial generating functions for the entire class of Heckman–Opdam polynomials [7].

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