symmetry

MDPI

Article

On Special Kinds of Involute and Evolute Curves in 4-Dimensional Minkowski Space

Muhammad Hanif [1,*], Zhong Hua Hou [1] and Kottakkaran Sooppy Nisar [2]

1 School of Mathematical Science, Dalian University of Technology, Dalian 116024, China; zhhou@dlut.edu.cn
2 Department of Mathematics, Prince Sattam Bin Abdulaziz University, 11991 Alkharj, Saudi Arabia; n.sooppy@psau.edu.sa
* Correspondence: hanif@mail.dlut.edu.cn; Tel.: +86-150-0411-0375

Received: 18 June 2018; Accepted: 22 July 2018; Published: 2 August 2018

check for updates

Abstract: Recently, extensive research has been done on evolute curves in Minkowski space-time. However, the special characteristics of curves demand advanced level observations that are lacking in existing well-known literature. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space. We consider (1,3)-evolute curves with respect to the casual characteristics of the (1,3)-normal plane that are spanned by the principal normal and the second binormal of the vector fields and the (0,2)-evolute curve that is spanned by the tangent and first binormal of the given curve. We restrict our investigation of (1,3)-evolute curves to the (1,3)-normal plane in four-dimensional Minkowski space. This research contribution obtains a necessary and sufficient condition for the curve possessing the generalized evolute as well as the involute curve. Furthermore, the Cartan null curve is also discussed in detail.

Keywords: evolute; involute curves; mate curves; minkowski space

MSC: 53A04; 53A35

1. Introduction

In the theory of curves, one of the important and interesting problems is the characterization of regular curves, in particular, the involute–evolute of a given curve. Evolutes and involutes (also known as evolvents) were studied by C. Huygens [1]. According to D. Fuchs [2], an involute of a given curve is a curve to which all tangents of the given curve are normal. He also defined the equation for an enveloping curve of the family of normal planes for a space curve. Suleyman and Seyda [3] determined the concept of parallel curves, which means that if the evolute exists, then the evolute of the parallel arc will also exist and the involute will coincide with the evolute. Brewster and David [4] stated that a curve is composed of two arcs with a common evolute, and the common evolute of two arcs must be a curve with only one tangent in each direction. In general, the evolute of a regular curve has singularities, and these points correspond to vertices. Emin and Suha [5] determined that an evolute Frenet apparatus can be formed by an involute apparatus in four dimensional Euclidean space, so, in this way, another orthonormal of the same space can be obtained. Shyuichi Izumiya [6] defined evolutes as the loci of singularities of space-like parallels and geometric properties of non-singular space-like hyper surfaces corresponding to the singularities of space-like parallels or evolutes. Takami Sato [7] investigated the singularities and geometric properties of pseudo-spherical evolutes of curves on a space-like surface in three-dimensional Minkowski-space. Marcos Craizer [8] stated that the iteration of involutes generates a pair of sequences of curves with respect to the Minkowski metric and its dual.

According to Boaventura Nolasco and Rui Pacheco [9], correspondence between plane curves and null curves in Minkowski three-space exists. He also described the geometry of null curves in terms of the curvature of the corresponding plane curves. M. Turgut and S. Yilmaz [10] obtained the Frenet apparatus of a given curve by defining the space-like involute–evolute curve couple in Minkowski space-time. Some researchers have investigated evolute curves and their characterization in Minkowski space [11–16] as well as in Euclidean space. Many researchers have dealt with evolute–involute curves, but no research has been carried out on the Cartan null curve. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space. We obtained necessary and sufficient conditions for the curve possessing a generalized evolute as well as an involute.

2. Preliminaries

Consider the Minkowski space-time, (E_1^4, G), where $E_1^4 = \{y = (y_1, y_2, y_3, y_4) | y_i \in R\}$ and $G = -dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2$. For any $M = (m_1, m_2, m_3, m_4)$ and $N = (n_1, n_2, n_3, n_4) \in T_yE$. We denote $M \cdot N = G(M, N) = m_1n_1 + m_2n_2 + m_3n_3 + m_4n_4$. Let I be an open interval in R and $\alpha : I \to E_1^4$ be a regular curve in E_1^4 that is parameterized by the arc length parameter, s, and $\{T, N, B_1, B_2\}$ is the moving Frenet frame along α, consisting of the tangent vector, T; the principal normal vector, N; the first binormal vector, B_1, and the second binormal vector, B_2, respectively, so that $T \wedge N \wedge B_1 \wedge B_2$ coincides with the standard orientation of E_1^4. Then, $T \cdot T = \epsilon_1, N \cdot N = \epsilon_2, B_1 \cdot B_1 = \epsilon_3, B_2 \cdot B_2 = \epsilon_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1, \epsilon_i \in \{1, -1\}, i \in \{1, 2, 3, 4\}$.

In particular, the following conditions hold: $T \cdot N = T \cdot B_1 = T \cdot B_2 = N \cdot B_1 = N \cdot B_2 = B_1 \cdot B_2 = 0$.

In accordance with reference [17], the Frenet–Serret formula for α in E_1^4 is given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 & 0 \\ 0 & -\epsilon_2 k_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3 k_3 \\ 0 & 0 & -\epsilon_3 k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{1}$$

We introduce some methodologies in this paper. At any point of α, the plane spanned by $\{T, B_1\}$ is called the (0,2)-tangent plane of α. The plane spanned by $\{N, B_2\}$ is called the (1,3)-normal plane of α.

Let $\alpha : I \to E_1^4$ and $\alpha^* : I \to E_1^4$ be two regular curves in E_1^4, where s is the arc-length parameter of α. Denote $s^* = f(s)$ to be the arc-length parameters of α^*. For any $s \in I$, if the (0,2)-tangent plane of α at $\alpha(s)$ coincides with the (1,3)-normal plane at $\alpha^*(s)$ of α^*, then α^* is called the (0,2)-involute curve of α in E_1^4 and α is called the (1,3)-evolute curve of α^* in E_1^4.

An arbitrary curve, $\alpha(s)$ in E_1^4, can locally be space-like, time-like, or null (light-like) if all of its velocity vectors, $\alpha'(s)$, are respectively space-like, time-like, or null [18]. A null curve, α, is parametrized by the pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ [19]. On the other hand, a nonnull curve, α, is parametrized by the arc-length parameter, s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. In accordance with references [19,20], if α is null Cartan curve, the Cartan Frenet frame is given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & -k_1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2}$$

where $k_1(s) = 0$ if $\alpha(s)$ is a null straight line or $k_1(s) = 1$ in all other cases. In this case, the next conditions hold: $T \cdot T = B_1 \cdot B_1 = 0, N \cdot N = B_2 \cdot B_2 = 1, T \cdot N = T \cdot B_2 = N \cdot B_1 = N \cdot B_2 = B_1 \cdot B_2 = 0,$ $T \cdot B_1 = 1$.

3. The (0,2)-Involute Curve of a Given Curve in E_1^4

In this section, we proceed to study the existence and expression of the (0,2)-involute curve of a given curve in E_1^4.

Theorem 1. *Let $\alpha : I \to E_1^4$ be a regular curve parameterized by arc-length s so that k_1, k_2 and k_3 are not zero. If α possesses the (0,2)-involute mate curve, $\alpha^*(s) = \alpha(s) + (\phi_0 - s)T(s) + \varphi B_1(s)$, with $\varphi \neq 0$, then k_1, k_2 and k_3 satisfy*

$$\frac{k_2}{k_1} = \tau, \frac{k_3}{k_1} = t_1(\tau + \epsilon_1\epsilon_3 t_2), \tau = \frac{\phi_0 - s + \varphi t_1^2 t_2}{\varphi(1 - \epsilon_1\epsilon_3 t_1^2)},$$

where ϕ_0, φ, t_1 and t_2 are given constants. Moreover, the three curvatures of α^* are given by

$$k_1^* = -\frac{\epsilon_1\epsilon_4\epsilon_2^* f t_3^2}{\varphi(\tau + \epsilon_1\epsilon_3 t_2)}, k_2^* = \frac{f(\epsilon_4\epsilon_3^* t_2\tau - \epsilon_2\epsilon_3^* t_2^2 - \epsilon_1\epsilon_4^* t_3^2)}{\varphi t_1(\tau + \epsilon_1\epsilon_3 t_2)}, k_3^* = -\frac{\epsilon_4\epsilon_4^* f}{\varphi t_1},$$

where $f \neq 0$. The associated Frenet frame are given by

$$T^* = f t_3(t_1 N + B_2), N^* = f(T + t_2 B_1), B_1^* = g t_3(-N + t_1 B_2), B_2^* = f(-t_2 T + B_1).$$

Proof. Let $\alpha : I \to E_1^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. Suppose that $\alpha^* : I \to E_1^4$ is the (0,2)-involute curve of α. $\{T^*, N^*, B_1^*, B_2^*\}$ is the Frenet frame along α^* and k_1^*, k_2^* and k_3^* are the curvatures of β^*. Then

span $\{T, B_1\} = span\{N^*, B_2^*\}$, span $\{N, B_2\} = span\{T^*, B_1^*\}$.

Moreover, α^* can be expressed as

$$\alpha^*(s) = \alpha(s) + \phi(s)T(s) + \varphi(s)B_1, \tag{3}$$

where $\phi(s)$ and $\varphi(s)$ are C^∞ functions on I.

By differentiating (3) with respect to s and using the Frenet formula (1), we get

$$f'T^* = (1 + \phi')T(s) + \varphi'(s)B_1 + \epsilon_2(\phi k_1 - \varphi k_2)N - \epsilon_1\epsilon_2\epsilon_3\varphi k_3 B_2. \tag{4}$$

Taking the inner product on both sides of (4) with T and B_1, respectively, we get $1 + \phi' = 0$ and $\varphi' = 0$, which implies that φ is constant and $\phi = \varphi_0 - s$, where ϕ_0 is the integration constant. So, (4) turns into

$$f'T^* = \epsilon_2(\phi k_1 - \varphi k_2)N - \epsilon_1\epsilon_2\epsilon_3\varphi k_3 B_2. \tag{5}$$

If we denote

$$\mu = \frac{\epsilon_2(\phi k_1 - \varphi k_2)}{f'}, \nu = \frac{-\epsilon_1\epsilon_2\epsilon_3\varphi k_3}{f'}, \tag{6}$$

then (5) turns into

$$T^* = \mu N + \nu B_2, \mu^2 + \nu^2 = 1. \tag{7}$$

Case 1: $\varphi \neq 0$. In this case, $\nu \neq 0$. $\frac{\mu}{\nu} = t_1$ implies that $\mu = t_1\nu$ and

$$\epsilon_2(\phi k_1 - \varphi k_2) = -\epsilon_1\epsilon_2\epsilon_3\varphi t_1 k_3, f' = -\epsilon_1\epsilon_2\epsilon_3\varphi\nu^{-1}k_3, \nu^2 = \frac{1}{1 + t_1^2}. \tag{8}$$

By differentiating (7) with respect to s and using the Frenet formula (1), we get

$$\epsilon_2^* f' k_1^* N^* = \mu' N - \epsilon_1\mu k_1 T + \nu' B_2 + \epsilon_3(\mu k_2 - \nu k_3)B_1. \tag{9}$$

By taking the inner product from both sides of (9) with N and B_2, respectively, we get $\mu' = 0$ and $\nu' = 0$, which implies that μ and ν are constants. So, (9) turns into

$$\epsilon_2^* f' k_1^* N^* = -\epsilon_1\mu k_1 T + \epsilon_3(\mu k_2 - \nu k_3)B_1. \tag{10}$$

Denote

$$f = -\frac{\epsilon_1 \nu t_1 k_1}{\epsilon_2^* f' k_1^*}, g = \frac{\epsilon_3 \nu (t_1 k_2 - k_3)}{\epsilon_2^* f' k_1^*}, \tag{11}$$

then (10) turns into

$$N^* = fT + gB_1, f^2 + g^2 = 1. \tag{12}$$

$\frac{g}{f} = t_2$ implies that $g = t_2 f$ and

$$t_1 t_2 k_1 = -\epsilon_1 \epsilon_3 (t_1 k_2 - k_3), f^2 = \frac{1}{1 + t_2^2}. \tag{13}$$

From Equations (8) and (13), we have

$$\tau := \frac{k_2}{k_1} = \frac{\frac{\phi}{\varphi} + t_1^2 t_2}{1 - \epsilon_1 \epsilon_3 t_1^2}, \frac{k_3}{k_1} = t_1(\tau + \epsilon_1 \epsilon_3 t_2). \tag{14}$$

$\frac{\nu}{f} = t_3$ implies that $\nu = t_3 f$. From (11), we get

$$f' k_1^* = -\epsilon_1 \epsilon_2^* t_1 t_3 k_1, t_3^2 = \frac{1 + t_2^2}{1 + t_1^2}. \tag{15}$$

By differentiating (12) with respect to s using the Frenet formula (1), we get

$$-\epsilon_1^* f' k_1^* T^* + \epsilon_3^* f' k_2^* B_1^* = f' T + \epsilon_2 (f k_1 - g k_2) N + g' B_1 - \epsilon_1 \epsilon_2 \epsilon_3 g k_3 B_2. \tag{16}$$

By taking inner product on both side of (16) by T and B_1 respectively, we get $f' = 0$ and $g' = 0$, which implies that f and g are constants. In this case, (16) turns into

$$\epsilon_3^* f' k_2^* B_1^* = \epsilon_1^* f' k_1^* T^* + \epsilon_2 f (k_1 - t_2 k_2) N - \epsilon_1 \epsilon_2 \epsilon_3 g t_2 k_3 B_2. \tag{17}$$

By substituting (7) and (15) into (17), we get

$$f' k_2^* B_1^* = f k_1 (\epsilon_4 \epsilon_3^* t_2 \tau - \epsilon_2 \epsilon_3^* t_2^2 - \epsilon_1 \epsilon_4^* t_3^2)(-N + t_1 B_2). \tag{18}$$

From (18), we may choose that

$$B_1^* = -\epsilon_4 \nu N + \epsilon_2 \mu B_2, f' k_2^* = t_3^{-1} k_1 (\epsilon_4 \epsilon_3^* t_2 \tau - \epsilon_2 \epsilon_3^* t_2^2 - \epsilon_1 \epsilon_4^* t_3^2). \tag{19}$$

By differentiating (19) about s and using the Frenet formula (1), we get

$$-\epsilon_2^* f' k_2^* N^* - \epsilon_1^* \epsilon_2^* \epsilon_3^* f' k_3^* B_2^* = \epsilon_1 \epsilon_4 \nu k_1 T - (\epsilon_3 \epsilon_4 \nu k_2 + \epsilon_2 \epsilon_3 \mu k_3) B_1, \tag{20}$$

from which we obtain

$$\epsilon_4^* f' k_3^* B_2^* = (\epsilon_2^* f f' k_2^* + \epsilon_1 \epsilon_4 \nu k_1) T + (\epsilon_2^* g f' k_2^* - \epsilon_3 \epsilon_4 \nu k_2 - \epsilon_2 \epsilon_3 \mu k_3) B_1 = -t_3^{-1} k_1 (\tau + \epsilon_1 \epsilon_3 t_2)(-gT + f B_1). \tag{21}$$

From (21), we may choose that

$$B_2^* = -gT + f B_1, f' k_3^* = -\epsilon_4^* t_3^{-1} k_1 (\tau + \epsilon_1 \epsilon_3 t_2). \tag{22}$$

From Equations (14), (15), (18) and (22), we can easily acquire our theorem. □

Case 2: If $\varphi = 0$, we have the following theorem.

Theorem 2. *Let $\alpha : I \to E_1^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. If α possesses the (0,2)-involute mate curve $\alpha^* = \alpha(s) + (\phi_0 - s)T(s)$, then k_1 and k_2 satisfy*

$$gk_1 + fk_2 = 0, \tag{23}$$

where ϕ_0, f, and g are given constants.

Moreover, the three curvatures of α^* are given by

$$k_1^* = \frac{1}{\epsilon_1\epsilon_2\epsilon_2^* f(s-\phi_0)}, k_2^* = \frac{-\epsilon_4\epsilon_3^* g k_3}{\epsilon_2(s-\phi_0)k_1}, k_3^* = \frac{\epsilon_1^*\epsilon_4 f k_3}{\epsilon_2(s-\phi_0)k_1}.$$

The associated Frenet frames are given by

$$T^* = -N, N^* = fT + gB_1, B_1^* = -B_2, B_2^* = -gT + fB_1.$$

In this case, (4) turns into

$$\alpha^*(s) = \alpha(s) + (\phi_0 - s)T(s). \tag{24}$$

By differentiating (24) with respect to s and using the Frenet Formula (1), we get

$$f'T^* = \epsilon_2(\phi_0 - s)k_1 N, \tag{25}$$

from which we may assume that

$$f' = \epsilon_2(s - \phi_0)k_1, T^* = -N. \tag{26}$$

By differentiating the second equation of (26) about s and using the Frenet Formula (1), we get

$$\epsilon_2^* f' k_1^* N^* = \epsilon_1 k_1 T - \epsilon_3 k_2 B_1.$$

Suppose that

$$N^* = fT + gB_1, f = \frac{\epsilon_1 k_1}{\epsilon_2^* f' k_1^*}, g = -\frac{\epsilon_3 k_2}{\epsilon_2^* f' k_1^*}, f^2 + g^2 = 1. \tag{27}$$

It follows that

$$\frac{k_2}{k_1} = -\frac{g}{f}\epsilon_1\epsilon_3. \tag{28}$$

By differentiating (27) about s, we obtain that f and g are constants:

$$\epsilon_3^* f' k_2^* B_1^* = \epsilon_1^* f' k_1^* T^* + \epsilon_2(fk_1 - gk_2)N - \epsilon_1\epsilon_2\epsilon_3 g k_3 B_2 = -\epsilon_2 g(\tfrac{g}{f}k_1 + k_2)N + \epsilon_4 g k_3 B_2 = \epsilon_4 g k_3 B_2. \tag{29}$$

Suppose that

$$B_1^* = -B_2, f' k_2^* = -\epsilon_4\epsilon_3^* g k_3. \tag{30}$$

By differentiating (30) about s, we obtain

$$\epsilon_4^* f' k_3^* B_2^* = \epsilon_2 f' k_2^* N^* + \epsilon_3 k_3 B_1 = -\epsilon_1^*\epsilon_4 k_3[fgT - (1-g^2)B_1] = -\epsilon_1^*\epsilon_4 f k_3(gT - fB_1). \tag{31}$$

Suppose that

$$T^* \wedge N^* \wedge B_1^* \wedge B_2^* = T \wedge N \wedge B_1 \wedge B_2, \tag{32}$$

then

$$B_2^* = -gT + fB_1, f' k_3^* = \epsilon_1^*\epsilon_4 f k_3. \tag{33}$$

From Equations (27), (30) and (33), we have achieved the desired theorem.

Remark 1. *Theorems 1 and 2 are quite different.*

4. The (1,3)-Evolute Curve of a Given Curve in E_1^4

In this section, we want to study the (1,3)-evolute curve of a given curve in E_1^4.

Theorem 3. *Let $\alpha : I \to E_1^4$ be a regular curve with arc length parameter s so that k_1, k_2 and k_3 are not zero, If α possesses the (1,3)-evolute mate curve, $\alpha^*(s) = \alpha(s) + \frac{1}{ik_1(s)}[iN(s) + jB_2(s)] - \frac{1}{k_3(s)}B_2(s)$, then k_1, k_2 and k_3 satisfy $\epsilon_1 ik_1 + \epsilon_3(jk_2 - jk_3) = 0$, where i and j are given constants. Three curvatures of α^* are given by*

$$k_1^* = -\epsilon_1\epsilon_2^*\frac{\sqrt{2}(ik_1)}{f'}, k_2^* = \frac{\sqrt{2}[\epsilon_4\epsilon_3^*k_3/(2i) - \epsilon_1\epsilon_4^*jk_1]}{f'}, k_3^* = -\sqrt{2}k_3/(2if'), f' = (1/ik_1).$$

The associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = (T + B_1)/\sqrt{2}, B_1^* = -jN + iB_2, B_2^* = (-T + B_1)/\sqrt{2}.$$

Proof. Let $\alpha : I \to E_1^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. Let $\alpha^* : I \to E_1^4$ be the (1,3)-evolute curve of α. $\{T^*, N^*, B_1^*, B_2^*\}$ is the Frenet frame along α^* and k_1^*, k_2^* and k_3^* are the curvatures of α^*. Then,

$$span\{T, B_1\} = span\{N^*, B_2^*\}, span\{N, B_2\} = span\{T^*, B_1^*\}. \tag{34}$$

Moreover, α^* can be expressed as

$$\alpha^*(s) = \alpha(s) + p(s)N(s) + q(s)B_2, \tag{35}$$

where $p(s)$ and $q(s)$ are C^∞ functions on I.

Differentiating (35) with respect to s using Frenet Formula (1), we get

$$T^*f' = (1 - p\epsilon_1k_1)T + p^{'}N + q^{'}B_2 + \epsilon_3(pk_2 - qk_3)B_1, \tag{36}$$

By taking the inner product from both sides of (36) with T and B_1, respectively, we get

$$f^{'}T^* = p^{'}N + q^{'}B_2, p = \frac{1}{\epsilon_1k_1}, q = \frac{\epsilon_1k_2}{k_1k_3}. \tag{37}$$

Denote

$$i = \frac{u'}{f'}, j = \frac{v'}{f'}, \tag{38}$$

then (37) turns into

$$T^* = iN + jB_2, i^2 + j^2 = 1. \tag{39}$$

By differentiating (39) with respect to s and using the Frenet formula (1), we get

$$\epsilon_2^*f'k_1^*N^* = i\prime N - \epsilon_1ik_1T + j'B_2 + \epsilon_3(ik_2 - jk_3)B_1. \tag{40}$$

By taking inner product on both sides of (40) with N and B_2 respectively, we get $i' = o$ and $j' = 0$, which implies that i and j are constants.

From (38), we obtain

$$p = if + p_0 = \frac{1}{\epsilon_1k_1}, q = jf + q_0 = \frac{\epsilon_1k_2}{k_1k_3}. \tag{41}$$

Moreover, (40) turns into

$$\epsilon_2^*f'k_1^*N^* = -\epsilon_1ik_1T + \epsilon_3(ik_2 - jk_3)B_1. \tag{42}$$

Denote
$$r=-\frac{\epsilon_1 ik_1}{\epsilon_2^* f'k_1^*}, t=\frac{\epsilon_3(ik_2-jk_3)}{\epsilon_2^* f'k_1^*}. \tag{43}$$
then (42) turns into
$$N^*=rT+tB_1, f'k_1^*=-\epsilon_1\epsilon_2^* r^{-1}ik_1, r^2+t^2=1. \tag{44}$$
Moreover, we have
$$-\epsilon_1\epsilon_2^* tik_1+\epsilon_3(rik_2-rjk_3)=0. \tag{45}$$

Case 1: $t\neq 0$. By differentiating (44) about s and using the Frenet Formula (1), we get
$$-\epsilon_1^* f'k_1^*T^*+\epsilon_3^* f'k_2^*B_1^*=\epsilon_2(rk_1-tk_2)N-\epsilon_1\epsilon_2\epsilon_3 tk_3B_2+r'T+t'B_1. \tag{46}$$

By taking inner product on both sides of (46) with T and B_1 respectively, we get $r'=0$ and $t'=0$, which implies that r and t are constants. In this case, (46) turns into
$$f'k_2^*B_1^*=(\frac{\epsilon_2\epsilon_3^* r^2-\epsilon_1\epsilon_4^* i^2}{r}k_1-\epsilon_2\epsilon_3^* tk_2)N-(\epsilon_4\epsilon_3^* tk_3+\epsilon_1\epsilon_4^*\frac{ij}{r}k_1)B_2. \tag{47}$$

Denote
$$\sigma=(f'k_2^*)^{-1}(\frac{\epsilon_2\epsilon_3^* r^2-\epsilon_1\epsilon_4^* i^2}{r}k_1-\epsilon_2\epsilon_3^* tk_2), \varsigma=(f'k_2^*)^{-1}(\epsilon_4\epsilon_3^* tk_3+\epsilon_1\epsilon_4^*\frac{ij}{r}k_1), \tag{48}$$
then (47) turns into
$$B_1^*=\sigma N+\varsigma B_2, \sigma^2+\varsigma^2=1. \tag{49}$$
Since $T^*\perp B_1^*$, it follows from (40) and (50) that $\frac{\sigma}{\varsigma}=-\frac{j}{i}$, which implies that
$$\epsilon_1 ik_1+\epsilon_3(ik_2-jk_3)=0. \tag{50}$$

From (45) and (50), we can see that
$$ik_2-jk_3=-\epsilon_1\epsilon_3 ik_1, (\epsilon_1 r-\epsilon_1\epsilon_1^* t)ik_1=0. \tag{51}$$

Since $t\neq 0$, it follows from (51) that $t=r$. Hence, (49) turns into
$$B_1^*=-jN+iB_2, f'k_2^*=\epsilon_1\epsilon_4^*\frac{j}{r}k_1+\epsilon_4\epsilon_3^*\frac{t}{i}k_3. \tag{52}$$

By differentiating (52) about s using (1), we get
$$-\epsilon_2^* f'k_2^*N^*+\epsilon_4^* f'k_3^*B_2^*=\epsilon_1 jk_1T-\epsilon_3(jk_2+ik_3)B_1, \tag{53}$$
from which we obtain
$$f'k_3^*B_2^*=\epsilon_2^* f'k_2^*N^*+\epsilon_1 jk_1T-\epsilon_3(jk_2+ik_3)B_1, -\epsilon_4\epsilon_3^*\epsilon_4^*\frac{t^2}{i}k_3(-T+B_1). \tag{54}$$

It follows from (54) that
$$B_2^*=-tT+rB_1, f'k_3^*=-\epsilon_4\epsilon_3^*\epsilon_4^*\frac{t}{i}k_3. \tag{55}$$

From (43), (52) and (55), we can easily acquire our desired theorem. □

Case 2: If $t=0$, we have the following theorem.

Theorem 4. *Let $\alpha : I \to E_1^4$ be a regular curve parameterized by arc-length s so that k_1, k_2 and k_3 are not zero. If α possesses the (1,3)-evolute mate curve, $\alpha^*(s) = \alpha(s) + \frac{1}{ik_1(s)}[iN(s) + jB_2(s)]$, then k_2 and k_3 satisfy $ik_2 - jk_3 = 0$, where i and j are given constants. Moreover, the three curvatures of α^* are given by*

$$k_1^* = -\epsilon_1\epsilon_2^* ik_1/f', k_2^* = -\epsilon_1\epsilon_2^* jk_1/f', k_3^* = -\epsilon_3 i^{-1}k_3/f', f' = (\frac{1}{ik_1}). \tag{56}$$

The associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = T, B_1^* = -jN + iB_2, B_2^* = B_1. \tag{57}$$

Proof. For this case, we may suppose that

$$N^* = T, f'k_1^* = -\epsilon_1\epsilon_2^* ik_1, \epsilon_3(ik_2 - jk_3) = 0. \tag{58}$$

From (41) and the third equation of (58), we acquire

$$p = i(f + f_0) = \frac{\epsilon_1}{k_1} q = j(f + f_0) = \frac{\epsilon_1 j}{ik_1}. \tag{59}$$

By differentiating (58) about s and using (1), we get

$$-\epsilon_1^* f'k_1^* T^* + \epsilon_3^* f'k_2^* B_1^* = \epsilon_2 k_1 N. \tag{60}$$

It follows that we may choose

$$B_1^* = -jN + iB_2, f'k_2^* = -\epsilon_1\epsilon_2^* jk_1. \tag{61}$$

By differentiating (61) about s using the Frenet Formula (1) and third equation of (58), we get

$$B_2^* = B_1, f'k_3^* = -\epsilon_3(jk_2 + ik_3) = \epsilon_3 i^{-1}k_3. \tag{62}$$

From (58), (61) and (62), we can easily acquire our desired theorem. □

Remark 2. *Theorems 3 and 4 are quite different.*

5. The (1,3)-Evolute Curve of a Cartan Null Curve in E_1^4

In this section, we proceed to study the existence and expression of the (1,3)-evolute curve of a given Cartan null curve in E_1^4. At any point of α, the plane spanned by $\{N, B_2\}$ is called the (1,3)-normal plane of α.

Let $\alpha : I \to E_1^4$ and $\alpha^* : I \to E_1^4$ be two regular curves in E_1^4, where s is the arc-length parameter of α. Denote $s^* = f(s)$ to be the arc-length parameters of α^*. For any $s \in I$, if the (0,2)-tangent plane of α at $\alpha(s)$ coincides with the (1,3)-normal plane at $\alpha^*(s)$ of α^*, then α^* is called the (0,2)-involute curve of α in E_1^4 and α is called the (1,3)-evolute curve of α^* in E_1^4.

Theorem 5. *Let $\alpha : I \to E_1^4$ be a null Cartan curve with arc length parameter s so that $k_1 = 1$, and k_2 k_3 are not zero, if α possesses the (1,3)-evolute mate curve, $\alpha^*(s) = \alpha(s) + \frac{1}{i(s)}[iN(s) + jB_2(s)] - \frac{1}{k_3(s)}B_2(s)$, then k_1, k_2 and k_3 satisfy $i + ik_2 - jk_3 = 0$, where i and j are given constants. Three curvatures of α^* are given by*

$$k_1^* = -\frac{\sqrt{2}(i)}{f'}, k_2^* = \frac{\sqrt{2}[k_3/(2i) - j]}{f'}, k_3^* = -\sqrt{2}k_3/(2if'), f' = (1/i). \tag{63}$$

Moreover, the associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = (T + B_1)/\sqrt{2}, B_1^* = -jN + iB_2, B_2^* = (-T + B_1)/\sqrt{2}. \quad (64)$$

Proof. Let $\alpha : I \to E_1^4$ be a Cartan null curve parameterized by the pseudo-arc parameter s with curvatures $k_1 = 1$, and k_2 and k_3 are not zero. Let $\alpha^* : I \to E_1^4$ be the (1,3)-evolute curve of α. Denote $\{T^*, N^*, B_1^*, B_2^*\}$ as the Frenet frame along α^* and k_1^*,k_2^* and k_3^* as the curvatures of α^*. Then

$$span\{T, B_1\} = span\{N^*, B_2^*\}, span\{N, B_2\} = span\{T^*, B_1^*\}. \quad (65)$$

Moreover, α^* can be expressed as

$$\alpha^*(s) = \alpha(s) + p(s)N(s) + q(s)B_2, \quad (66)$$

where $p(s)$ and $q(s)$ are C^∞ functions on I. By differentiating (66) with respect to s using the Frenet Formula (2), we get

$$T^* f' = (1 + pk_2 - qk_3)T + p^{'}N + q^{'}B_2 - pB_1. \quad (67)$$

By taking the inner product on both sides of (67) with T and B_1, respectively, we get

$$f^{'}T^* = p^{'}N + q^{'}B_2, p = 1, q = \frac{k_2}{k_3}. \quad (68)$$

Denote

$$i = \frac{p'}{f'}, j = \frac{q'}{f'}, \quad (69)$$

then (68) turns into

$$T^* = iN + jB_2, i^2 + j^2 = 1. \quad (70)$$

By differentiating (70) with respect to s and using the Frenet formula (2), we get

$$f'k_1^* N^* = i'N - iB_1 + j'B_2 + (ik_2 - jk_3)T. \quad (71)$$

By taking the inner product on both sides of (71) with N and B_2 respectively, we get $i' = o$ and $j' = 0$ which implies that i and j are constants. From (69), we get

$$p = if + p_0 = 1, q = jf + q_0 = \frac{k_2}{k_3}. \quad (72)$$

Moreover, (71) turns into

$$f'k_1^* N^* = -iB_1 + (ik_2 - jk_3)T. \quad (73)$$

Denote

$$r = -\frac{i}{f'k_1^*}, t = \frac{(ik_2 - jk_3)}{f'k_1^*}, \quad (74)$$

then (73) turns into

$$N^* = rB_1 + tT, f'k_1^* = -r^{-1}i, r^2 + t^2 = 1. \quad (75)$$

Moreover,

$$ti + rik_2 - rjk_3 = 0. \quad (76)$$

Case 1: $t \neq 0$. By differentiating (75) about s and using the Frenet Formula (2), we get

$$-f'k_1^* T^* + f'k_2^* B_1^* = (tk_1 - rk_2)N + rk_3 B_2 + r'T + t'B_1. \quad (77)$$

By taking the inner product from both sides of (77) with T and B_1 respectively, we get $r' = 0$ and $t' = 0$, which implies that r and t are constants. In this case, (77) turns into

$$f'k_2^* B_1^* = (\frac{r^2 - i^2}{r} - tk_2)N - (tk_3 + \frac{ij}{r})B_2. \tag{78}$$

Denote

$$\sigma = (f'k_2^*)^{-1}(\frac{r^2 - i^2}{r} - tk_2), \varsigma = (f'k_2^*)^{-1}(tk_3 - \frac{ij}{r}), \tag{79}$$

then (78) turns into

$$B_1^* = \sigma N + \varsigma B_2, \sigma^2 + \varsigma^2 = 1. \tag{80}$$

Since $T^* \perp B_1^*$, it follows from (70) and (80) that $\frac{\sigma}{\varsigma} = -\frac{j}{i}$, which implies that

$$i + ik_2 - jk_3 = 0. \tag{81}$$

From (76) and (81), we can see that

$$ik_2 - jk_3 = -i, (t - r)i = 0. \tag{82}$$

Since $t \neq 0$, it follows from (82) that $t = r$.
Hence (80) turns into

$$B_1^* = -j + iB_2, f'k_2^* = -\frac{j}{r} + \frac{t}{i}k_3. \tag{83}$$

By differentiating (83) about s using (2), we get

$$-f'k_2^* N^* + f'k_3^* B_2^* = jB_1 - (jk_2 + ik_3)T. \tag{84}$$

From which we have

$$f'k_3^* B_2^* = f'k_2^* N^* + jB_1 - (jk_2 + ik_3)T, -\frac{t^2}{i}k_3(-T + B_1). \tag{85}$$

It follows from (85) that

$$B_2^* = -tT + rB_1, f'k_3^* = -\frac{t}{i}k_3. \tag{86}$$

From (74), (83) and (86), we easily acquire our desired theorem. □

Case 2: For $t = 0$, we have the following theorem.

Theorem 6. *Let $\alpha : I \to E_1^4$ be a null Cartan curve with arc-length parameter s so that $k_1 = 1$, k_2 and k_3 are not zero. If α possesses the (1,3)-evolute mate curve, $\alpha^*(s) = \alpha(s) + \frac{1}{ik_1(s)}[iN(s) + jB_2(s)]$, then k_2 and k_3 satisfy $ik_2 - jk_3 = 0$, where i and j are given constants. Moreover, the three curvatures of α^* are given by*

$$k_1^* = -i/f', k_2^* = -j/f', k_3^* = i^{-1}k_3/f', f' = (\frac{1}{i}). \tag{87}$$

The associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = T, B_1^* = -jN + iB_2, B_2^* = B_1. \tag{88}$$

Proof. For this case, we may suppose that

$$N^* = T, f'k_1^* = -i, ik_2 - jk_3 = 0. \tag{89}$$

Moreover, from (72) and the third equation of (89), we get

$$p = i(f + f_0) = \frac{1}{k_1} q = j(f + f_0) = \frac{j}{i}. \tag{90}$$

By differentiating (89) about s and using (2), we get

$$f' k_1^* T^* + f' k_2^* B_1^* = k_1 N. \tag{91}$$

It follows that we can choose

$$B_1^* = -jN + iB_2, f' k_2^* = -j. \tag{92}$$

By differentiating (92) about s using the Frenet Formula (2) and third equation of (89), we get

$$B_2^* = B_1, f' k_3^* = -(jk_2 + ik_3) = -i^{-1} k_3. \tag{93}$$

From (89), (92) and (93), we can easily acquire our desired theorem. □

Remark 3. *Theorems 5 and 6 are quite different.*

Condition 2:

Theorem 7. *Let $\alpha : I \to E_1^4$ be a null Cartan curve with arc length parameter s so that $k_1 = 1$, and k_2 k_3 are not zero if α possesses the (1,3)-evolute mate curve, $\alpha^*(s) = \alpha(s) + \frac{1}{i(s)k_1}[iN(s) + jB_2(s)] - \frac{1}{k_3(s)} B_2(s)$. Then, k_1, k_2 and k_3 satisfy $ik_1 + i - jk_3 = 0$, where i and j are given constants. Three curvatures of α^* are given, as follows:*

$$k_1^* = -\frac{\sqrt{2}(ik_1)}{f'}, k_2^* = \frac{\sqrt{2}[k_3/(2i) - jk_1]}{f'}, k_3^* = -\sqrt{2}k_3/(2if'), f' = (1/ik_1). \tag{94}$$

The associated Frenet Frame are given by

$$T^* = iN + jB_2, N^* = (T + B_1)/\sqrt{2}, B_1^* = -jN + iB_2, B_2^* = (-T + B_1)/\sqrt{2}. \tag{95}$$

Proof. Let $\alpha : I \to E_1^4$ be a Cartan null curve parametrized by pseudo-arc parameter s with curvatures $k_2 = 1$, and k_1 and k_3 are not zero. Let $\alpha^* : I \to E_1^4$ be the (1,3)-evolute curve of α. Denote $\{T^*, N^*, B_1^*, B_2^*\}$ as the Frenet frame along α^* and k_1^*, k_2^* and k_3^* as the curvatures of α^*. Then,

$$span\{T, B_1\} = span\{N^*, B_2^*\}, span\{N, B_2\} = span\{T^*, B_1^*\}. \tag{96}$$

Moreover, α^* can be expressed as

$$\alpha^*(s) = \alpha(s) + p(s)N(s) + q(s)B_2, \tag{97}$$

where $p(s)$ and $q(s)$ are C^∞ functions on I. By differentiating (97) with respect to s using the Frenet Formula (2), we get

$$T^* f' = (1 + p - qk_3)T + p' N + q' B_2 - pk_1 B_1. \tag{98}$$

By taking the inner product from both sides of (98) with T and B_1 respectively, we get

$$f' T^* = p' N + q' B_2, p = \frac{1}{k_1}, q = \frac{1}{k_1 k_3}. \tag{99}$$

Denote

$$i = \frac{p'}{f'}, j = \frac{q'}{f'}, \tag{100}$$

then (99) turns into

$$T^* = iN + jB_2, i^2 + j^2 = 1. \tag{101}$$

By differentiating (101) with respect to s and using the Frenet formula (2), we get

$$f'k_1^*N^* = i\prime N - ik_1B_1 + j'B_2 + (i - jk_3)T. \tag{102}$$

By taking the inner product from both sides of (102) with N and B_2, respectively, we get $i' = o$ and $j' = 0$ which implies that i and j are constants. From (100), we get

$$p = if + p_0 = \frac{1}{k_1}, q = jf + q_0 = \frac{1}{k_1k_3}. \tag{103}$$

Moreover, (102) turns into

$$f'k_1^*N^* = -ik_1B_1 + (i - jk_3)T. \tag{104}$$

Denote

$$r = -\frac{ik_1}{f'k_1^*}, t = \frac{(i - jk_3)}{f'k_1^*}, \tag{105}$$

then (104) turns into

$$N^* = rB_1 + tT, f'k_1^* = -r^{-1}ik_1, r^2 + t^2 = 1. \tag{106}$$

Moreover,

$$tik_1 + ri - rjk_3 = 0. \tag{107}$$

Case 1: $t \neq 0$. By differentiating (106) about s and using the Frenet Formula (2), we get

$$-f'k_1^*T^* + f'k_2^*B_1^* = (tk_1 - r)N + rk_3B_2 + r'T + t'B_1. \tag{108}$$

By taking the inner product on both sides of (108) with T and B_1, respectively, we get $r' = 0$ and $t' = 0$, which implies that r and t are constants. In this case, (108) turns into

$$f'k_2^*B_1^* = (\frac{r^2 - i^2}{r} - t)N - (tk_3 + \frac{ijk_1}{r})B_2. \tag{109}$$

Denote

$$\sigma = (f'k_2^*)^{-1}(\frac{r^2 - i^2}{r}k_1 - t), \varsigma = (f'k_2^*)^{-1}(tk_3 - \frac{ij}{r}k_1), \tag{110}$$

then (109) turns into

$$B_1^* = \sigma N + \varsigma B_2, \sigma^2 + \varsigma^2 = 1. \tag{111}$$

Since $T^* \perp B_1^*$, it follows from (101) and (111) that $\frac{\sigma}{\varsigma} = -\frac{j}{i}$, which implies that

$$ik_1 + i - jk_3 = 0. \tag{112}$$

From (107) and (112), we get

$$i - jk_3 = -ik_1, (t - r)ik_1 = 0. \tag{113}$$

Since $t \neq 0$, it follows from (113) that $t = r$. Hence, (111) turns into

$$B_1^* = -jN + iB_2, f'k_2^* = -\frac{j}{r}k_1 + \frac{t}{i}k_3. \tag{114}$$

By differentiating (114) about s using (2), we get

$$-f'k_2^*N^* + f'k_3^*B_2^* = jk_1B_1 - (j + ik_3)T. \tag{115}$$

From which we have

$$f'k_3^*B_2^* = f'k_2^*N^* + jk_1B_1 - (j + ik_3)T, -\frac{t^2}{i}k_3(-T + B_1). \tag{116}$$

It follows from (116) that

$$B_2^* = -tT + rB_1, f'k_3^* = -\frac{t}{i}k_3. \tag{117}$$

From (106), (114) and (117), we can easily acquire our desired theorem. □

Case 2: For $t = 0$, we have the following theorem.

Theorem 8. *Let $\alpha : I \to E_1^4$ be a null Cartan curve with arc-length parameter s so that $k_1 = 1$, k_2 and k_3 are not zero. If α possesses the (1,3)-evolute mate curve $\alpha^*(s) = \alpha(s) + \frac{1}{ik_1(s)}[iN(s) + jB_2(s)]$, then k_2 and k_3 satisfy $ik_2 - jk_3 = 0$, where i and j are given constants. Moreover, the three curvatures of α^* are given by*

$$k_1^* = -i/f', k_2^* = -j/f', k_3^* = i^{-1}k_3/f', f' = (\frac{1}{i}). \tag{118}$$

The associated Frenet frames are given by

$$T^* = iN + jB_2, N^* = T, B_1^* = -jN + iB_2, B_2^* = B_1. \tag{119}$$

Proof. In this case, we may suppose that

$$N^* = B_1, f'k_1^* = -ik_1, i - jk_3 = 0. \tag{120}$$

Moreover, from (112) and the third equation of (120), we get

$$p = i(f + f_0) = \frac{1}{k_1}, q = j(f + f_0) = \frac{j}{ik_1}. \tag{121}$$

By differentiating (120) about s and using (2), we get

$$f'k_1^*T^* + f'k_2^*B_1^* = k_1N. \tag{122}$$

It follows that we can choose

$$B_1^* = -jN + iB_2, f'k_2^* = -jk_1. \tag{123}$$

By differentiating (123) about s using the Frenet Formula (2) and using the third equation of (120), we get

$$B_2^* = B_1, f'k_3^* = -(j + ik_3) = j^{-1}. \tag{124}$$

From (120), (123) and (124), we can easily acquire our desired theorem. □

Remark 4. *Theorems 7 and 8 are quite different.*

6. Conclusions

This paper established new kinds of generalized evolute and involute curves in four-dimensional Minkowski space by providing the necessary and sufficient conditions for the curves possessing generalized evolute and involute curves. Furthermore, the study invoked a new type of (1,3)-evolute and (0,2)-evolute curve in four-dimensional Minkowski space. The study also provided a new kind of generalized null Cartan curve in four-dimensional Minkowski space. For this new type of curve, the study provided several theorems with necessary and sufficient conditions and obtained significant results. The understanding of evolute curves with this new type evolute curve in four-dimensional Minkowski space will be beneficial for researchers in future studies. The designing of a framework for the involutes of order k of a null Cartan curve in Minkowski spaces will be considered in future work.

Author Contributions: M.H. is a doctorate candidate under the supervision of Z.H.H. and writer of original draft. Z.H.H. write, review and edit the original draft and formal analysis by K.S.N.

Funding: The work presented in this paper is supported by National Natural Science Foundation of China under Grant No. 61473059.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Huygens, C. 1673 Horologium oscillatorium. In *Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometrical (Paris: AF Muguet)*; Iowa State University Press: Ames, IA, USA, 1986.
2. Fuchs, D. Evolutes and involutes of spatial curves. *Am. Math. Mon.* **2013**, *120*, 217–231. [CrossRef]
3. Şenyurt, S.; Kılıçoğlu, Ş. On the Differential Geometric Elements of the Involute D-Scroll in E3. *Adv. Appl. Clifford Algebras* **2015**, *25*, 977–988. [CrossRef]
4. Gere, B.H.; Zupnik, D. On the Construction of Curves of Constant Width. *Stud. Appl. Math.* **1943**, *22*, 31–36. [CrossRef]
5. Özyılmaz, E.; Yılmaz, S. Involute-evolute curve couples in the euclidean 4-Space. *Int. J. Open Probl. Compt. Math.* **2009**, *2*, 168–174.
6. Izumiya, S.; Takahashi, M. Spacelike parallels and evolutes in Minkowski pseudo-spheres. *J. Geom. Phys.* **2007**, *57*, 1569–1600. [CrossRef]
7. Sato, T. Pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz—Minkowski space. *J. Geom.* **2012**, *103*, 319–331. [CrossRef]
8. Craizer, M. Iteration of involutes of constant width curves in the Minkowski plane. *Beiträge Algebra Geometrie/Contrib. Algebra Geom.* **2014**, *55*, 479–496. [CrossRef]
9. Nolasco, B.; Pacheco, R. Evolutes of plane curves and null curves in Minkowski 3-space. *J. Geom.* **2017**, *108*, 195–214. [CrossRef]
10. Turgut, M.; Yilmaz, S. On the Frenet frame and a characterization of space-like involute-evolute curve couple in Minkowski space-time. *Int. Math. Forum* **2008**, *3*, 793–801.
11. Bükcü, B.; Karacan, M.K. On the involute and evolute curves of the timelike curve in Minkowski 3-space. *Demonstr. Math.* **2007**, *40*, 721–732.
12. Bukcu, B.; Karacan, M.K. On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-space. *Int. J. Contemp. Math. Sci.* **2007**, *2*, 221–232. [CrossRef]
13. Yoon, D.W. On the evolute offsets of ruled surfaces in Minkowski 3-space. *Turk. J. Math.* **2016**, *40*, 594–604. [CrossRef]
14. As, E.; Sarıoğlugil, A. On the Bishop curvatures of involute-evolute curve couple in E3. *Int. J. Phys. Sci.* **2014**, *9*, 140–145.
15. Yu, H.; Pei, D.; Cui, X. Evolutes of fronts on Euclidean 2-sphere. *J. Nonlinear Sci. Appl.* **2015**, *8*, 678–686. [CrossRef]
16. Fukunaga, T.; Takahashi, M. Evolutes of fronts in the Euclidean plane. *Hokkaido Univ. Prepr. Ser. Math.* **2012**, *1026*, 1–17. [CrossRef]
17. İlarslan, K.; Nešović, E. Spacelike and timelike normal curves in Minkowski space-time. *Publ. Inst. Math.* **2009**, *85*, 111–118. [CrossRef]
18. O'neill, B. *Semi-Riemannian Geometry With Applications to Relativity*; Academic Press: Cambridge, MA, USA, 1983; Volume 103.
19. Bonnor, W.B. Null curves in a Minkowski space-time. *Tensor* **1969**, *20*, 229.
20. Bonnor, W. Curves with null normals in Minkowski space-time. In *A Random Walk in Relativity and Cosmology*; 1985; pp. 33–47. Avaliable online: http://adsabs.harvard.edu/abs/1985rwrc.conf...33B (accessed on 17 June 2018).