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A Different Study on the Spaces of Generalized Fibonacci Difference *bs* and *cs* Spaces Sequence

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Abstract: The main topic in this article is to define and examine new sequence spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r)))$, where $\hat{F}(s,r)$ is generalized difference Fibonacci matrix in which $s,r \in \mathbb{R}\setminus\{0\}$. Some algebric properties including some inclusion relations, linearly isomorphism and norms defined over them are given. In addition, it is shown that they are Banach spaces. Finally, the α-, β- and γ-duals of the spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ are appointed and some matrix transformations of them are given.

Keywords: Fibonacci numbers; Fibonacci double band matrix; sequence spaces; difference matrix; matrix transformations; α, β, γ-duals

1. Introduction

Italian mathematician Leonardo Fibonacci found the Fibonacci number sequence. The Fibonacci sequence actually originated from a rabbit problem in his first book "Liber Abaci". This sequence is used in many fields. The Fibonacci sequence is as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots.$$

The Fibonacci sequence, which is denoted by (f_n), is defined as the linear reccurence relation

$$f_n = f_{n-1} + f_{n-2}.$$

$f_0 = 1, f_1 = 1$ and $n \geq 2$. The golden ratio is

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \varphi \text{ (Golden Ratio).}$$

The Golden Ratio, which is also known outside the academic community, is used in many fields of science.

Let w be the set of all real valued sequences. Any subspace of w is called the sequence space. c, c_0 and ℓ_∞ are called as sequences space convergent, convergent to zero and bounded, respectively. In addition to these representations, ℓ_1, bs and cs are sequence spaces, which are called absolutely convergent, bounded and convergent series, respectively.

Let us take a two-indexed real valued infinite matrix $A = (a_{nk})$, where a_{nk} is real number and $k, n \in \mathbb{N}$. A is called a matrix transformation from X to Y if, for every $x = (x_k) \in X$, sequence $Ax = \{A_n(x)\}$ is A transform of x and in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \tag{1}$$

and Equation (1) converges for each $n \in \mathbb{N}$.

Let λ be a sequence space and K be an infinite matrix. Then, the matrix domain λ_K is introduced by

$$\lambda_K = \{t = (t_k) \in w : \ Kt \in \lambda\}. \tag{2}$$

Here, it can be seen that λ_K is a sequence space.

For calculation of any matrix domain of a sequence, a triangle infinite matrix is used by many authors. So many sequence spaces have been recently defined in this way. For more details, see [1–22].

Kara [23] recently introduced the $\hat{F}$ which is derived from the Fibonacci sequence (f_n) and defined the new sequence spaces $\ell_p(\hat{F})$ and $\ell_\infty(\hat{F})$ by using sequence spaces ℓ_p and ℓ_∞, respectively, where $1 \leq p < \infty$. The sequence space $\ell_p(\hat{F})$ has been defined as:

$$\ell_p(\hat{F}) = \left\{x \in w : \ \hat{F}x \in \ell_p\right\}, \ (1 \leq p < \infty),$$

where $\hat{F} = (f_{nk})$ defined by the sequence (f_n) as follows:

$$f_{nk} := \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. In addition, Kara et al. [24] have characterized some class of compact operators on the spaces $\ell_p(\hat{F})$ and $\ell_\infty(\hat{F})$, where $1 \leq p < \infty$.

Candan [25] introduced $c(\hat{F}(s,r))$ and $c_0(\hat{F}(s,r))$. Later, Candan and Kara [15] have investigated the sequence spaces $\ell_p(\hat{F}(s,r))$ in which $1 \leq p \leq \infty$.

The α-, β- and γ-duals P^α, P^β and P^γ of a sequence spaces P are defined, respectively, as

$$\begin{aligned} P^\alpha &= \{a = (a_k) \in w : \ at = (a_k t_k) \in \ell_1 \text{ for all } t \in P\}, \\ P^\beta &= \{a = (a_k) \in w : \ at = (a_k t_k) \in cs \text{ for all } t \in P\}, \\ P^\gamma &= \{a = (a_k) \in w : \ at = (a_k t_k) \in bs \text{ for all } t \in P\}, \end{aligned}$$

respectively.

In Section 2, sequence space $bs(\hat{F})$ and $cs(\hat{F})$ are defined and some algebric properties of them are investigated. In the last section, the α-, β- and γ-duals of the spaces $bs(\hat{F})$ and $cs(\hat{F})$ are found and some matrix tranformations of them are given.

2. Generalized Fibonacci Difference Spaces of *bs* and *cs* Sequences

In this section, spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ of generalized Fibonacci difference of sequences, which constitutes bounded and convergence series, respectively, will be defined. In addition, some algebraic properties of them are investigated.

Now, we introduce the sets $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ as the sets of all sequences whose $\hat{F}(s,r) = \{f_{nk}(s,r)\}$ transforms are in the sequence space bs and cs,

$$\begin{aligned} bs(\hat{F}(s,r)) &= \left\{x = (x_k) \in w : \ \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}} x_k + r\frac{f_{k+1}}{f_k} x_{k-1} \right) \right| < \infty \right\}, \\ cs(\hat{F}(s,r)) &= \left\{x = (x_k) \in w : \ \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}} x_k + r\frac{f_{k+1}}{f_k} x_{k-1} \right) \in c \right\}, \end{aligned}$$

where $\hat{F}(s,r) = \{f_{nk}(s,r)\}$ is

$$f_{nk}(s,r) := \begin{cases} r\frac{f_{n+1}}{f_n}, & k = n-1, \\ s\frac{f_n}{f_{n+1}}, & k = n, \\ 0, & k < n-1 \text{ or } 0 \leq k > n, \end{cases} \tag{3}$$

for all $k, n \in \mathbb{N}$ where $s, r \in \mathbb{R} \setminus \{0\}$. Actually, by using Equation (2), we can get

$$bs(\hat{F}(s,r)) = (bs)_{\hat{F}(s,r)} \text{ and } cs(\hat{F}(s,r)) = (cs)_{\hat{F}(s,r)}.$$

With a basic calculation, we can find the inverse matrix of $\hat{F}(s,r) = \{f_{nk}(s,r)\}$. The inverse matrix of $\hat{F}(s,r) = \{f_{nk}(s,r)\}$ is $\hat{F}^{-1}(s,r) = \left(f_{nk}^{-1}(s,r)\right)$ such that

$$f_{nk}^{-1}(s,r) = \begin{cases} \frac{1}{s}(-\frac{r}{s})^{n-k}\frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \leq k < n, \\ 0, & k > n, \end{cases} \tag{4}$$

for all $k, n \in \mathbb{N}$. If $y = (y_n)$ is $\hat{F}(s,r)$-transform of a sequence $x = (x_n)$, then the below equality is justified:

$$y_n = (\hat{F}(s,r)x)_n = \begin{cases} sx_0, & n = 0, \\ s\frac{f_n}{f_{n+1}}x_n + r\frac{f_{n+1}}{f_n}x_{n-1}, & n \geq 1, \end{cases} \tag{5}$$

for all $n \in \mathbb{N}$. In this situation, we see that $x_n = \hat{F}^{-1}(s,r)y$, i.e.,

$$x_n = \sum_{k=0}^{n} \frac{1}{s}(-\frac{r}{s})^{n-k}\frac{f_{n+1}^2}{f_k f_{k+1}}y_k \tag{6}$$

for all $n \in \mathbb{N}$.

Theorem 1. *$bs(\hat{F}(s,r))$ is the linear space with the co-ordinatewise addition and scalar multiplation.*

Proof. We omit the proof because it is clear and easy. □

Theorem 2. *$cs(\hat{F}(s,r))$ is the linear space with the co-ordinatewise addition and scalar multiplation.*

Proof. We omit the proof because it is clear and easy. □

Theorem 3. *The space $bs(\hat{F}(s,r))$ is a normed space with*

$$\|x\|_{bs(\hat{F}(s,r))} = \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}}x_k + r\frac{f_{k+1}}{f_k}x_{k-1} \right) \right|. \tag{7}$$

Proof. It is clear that space $bs(\hat{F}(s,r))$ ensures normed space conditions. □

Theorem 4. *The space $cs(\hat{F}(s,r))$ is a normed space with norm Equation (7).*

Proof. It is clear that normed space conditions are ensured by space $cs(\hat{F}(s,r))$. □

Theorem 5. *$bs(\hat{F}(s,r))$ is linearly isomorphic as isometric to the space bs, that is, $bs(\hat{F}(s,r)) \cong bs$.*

Proof. For proof, we must demonstrate that bijection and linearly transformation T exist between the space $bs(\hat{F}(s,r))$ and bs. Let us take the transformation $T : bs(\hat{F}(s,r)) \to bs$ mentioned above with the help of Equation (5) by $Tx = \hat{F}(s,r)x$. We omit the details that T is both linear and injective because the demonstration is clear. □

Let us prove that transformation T is surjective. For this, we get $y = (y_n) \in bs$.
In this case, by using Equations (6) and (7), we find

$$\begin{aligned}
\|x\|_{bs(\hat{F}(s,r))} &= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}} x_k + r\frac{f_{k+1}}{f_k} x_{k-1} \right) \right| \\
&= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left[s\frac{f_k}{f_{k+1}} \left(\sum_{i=0}^{k} -\frac{1}{s}(-\frac{r}{s})^{k-i} \frac{f_{k+1}^2}{f_i f_{i+1}} y_i \right) \right.\right. \\
&\quad \left.\left. + r\frac{f_{k+1}}{f_k} \left(\sum_{i=0}^{k-1} -\frac{1}{s}(-\frac{r}{s})^{k-i-1} \frac{f_k^2}{f_i f_{i+1}} y_i \right) \right] \right| \\
&= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} y_k \right| = \|y\|_{bs}.
\end{aligned}$$

This result shows that $x \in bs(\hat{F}(s,r))$. That is, T is surjective. At the same time, this result also indicates that T is preserving the norm. Therefore, the sequence spaces $bs(\hat{F}(s,r))$ and bs are linearly isomorphic as isometric.

Theorem 6. *The sequence space $cs(\hat{F}(s,r))$ is linearly isomorphic as isometric to the space cs, that is, $cs(\hat{F}(s,r)) \cong cs$.*

Proof. If we write cs instead of bs and $cs(\hat{F}(s,r))$ instead of $bs(\hat{F}(s,r))$ in Theorem 5, the proof will be demonstrated. □

Theorem 7. *The space $bs(\hat{F}(s,r))$ is a Banach space with the norm, which is given in Equation (7).*

Proof. We can easily see that norm conditions are ensured. Let us take that $x^i = (x_k^i)$ is a Cauchy sequence in $bs(\hat{F}(s,r))$ for all $i \in \mathbb{N}$. By using Equation (5), we have

$$y_k^i = s\frac{f_k}{f_{k+1}} x_k^i + r\frac{f_{k+1}}{f_k} x_{k-1}^i$$

for all $i,k \in \mathbb{N}$. Since $x^i = (x_k^i)$ is a Cauchy sequence, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\begin{aligned}
\left\| x^i - x^m \right\|_{bs(\hat{F}(s,r))} &= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}} (x_k^i - x_k^m) + r\frac{f_{k+1}}{f_k} (x_{k-1}^i - x_{k-1}^m) \right) \right| \\
&= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left(y_k^i - y_k^m \right) \right| = \left\| y^i - y^m \right\|_{bs} < \varepsilon
\end{aligned}$$

for all $i, m \geq n_0$. Since bs is complete, $y^i \to y$ $(i \to \infty)$ such that $y \in bs$ exist and since the sequence spaces $bs(\hat{F}(s,r))$ and bs are linearly isomorphic as isometric $bs(\hat{F}(s,r))$ is complete. Consequently, $bs(\hat{F}(s,r))$ is a Banach space. □

Theorem 8. *The space $cs(\hat{F}(s,r))$ is a Banach space with the norm, which is given in Equation (7).*

Proof. We can easily see that norm conditions are ensured. Let us take that $x^i = (x_k^i)$ is a Cauchy sequence in $cs(\hat{F}(s,r))$ for all $i \in \mathbb{N}$. By using Equation (5), we have

$$y_k^i = s\frac{f_k}{f_{k+1}} x_k^i + r\frac{f_{k+1}}{f_k} x_{k-1}^i$$

for all $i, k \in \mathbb{N}$. Since $x^i = (x_k^i)$ is a Cauchy sequence, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\begin{aligned}\left\|x^i - x^m\right\|_{cs(\hat{F}(s,r))} &= \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\left(s\frac{f_k}{f_{k+1}}(x_k^i - x_k^m) + r\frac{f_{k+1}}{f_k}(x_{k-1}^i - x_{k-1}^m)\right)\right| \\ &= \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\left(y_k^i - y_k^m\right)\right| = \left\|y^i - y^m\right\|_{cs} < \varepsilon\end{aligned}$$

for all $i, m \geq n_0$. Since cs is complete, $y^i \to y$ $(i \to \infty)$ such that $y \in cs$ exists and since the sequence spaces $cs(\hat{F}(s,r))$ and cs are linearly isomorphic as isometric $cs(\hat{F}(s,r))$ is complete. Consequently, $cs(\hat{F}(s,r))$ is a Banach space. □

Now, let $A = (a_{nk})$ be an arbitrary infinite matrix and list the following:

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}| < \infty, \tag{8}$$

$$\lim_k a_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{9}$$

$$\sup_m \sum_k \left|\sum_{n=0}^{m}(a_{nk} - a_{n,k+1})\right| < \infty, \tag{10}$$

$$\lim_n \sum_k a_{nk} = \alpha \text{ for each } k \in \mathbb{N},\ \alpha \in \mathbb{C}, \tag{11}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty, \tag{12}$$

$$\lim_n a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N},\ \alpha_k \in \mathbb{C}, \tag{13}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(a_{nk} - a_{n,k+1})\right| < \infty, \tag{14}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(a_{nk} - a_{n,k-1})\right| < \infty, \tag{15}$$

$$\lim_n (a_{nk} - a_{n,k+1}) = \alpha \text{ for each } k \in \mathbb{N},\ \alpha \in \mathbb{C}, \tag{16}$$

$$\lim_{n\to\infty}\sum_k |a_{nk} - a_{n,k+1}| = \sum_k \left|\lim_{n\to\infty}(a_{nk} - a_{n,k+1})\right|, \tag{17}$$

$$\sup_n \left|\lim_k a_{nk}\right| < \infty, \tag{18}$$

$$\lim_{n\to\infty}\sum_k |a_{nk} - a_{n,k+1}| = 0 \text{ uniformly in } n, \tag{19}$$

$$\lim_m \sum_k \left|\sum_{n=0}^{m}(a_{nk} - a_{n,k+1})\right| = 0, \tag{20}$$

$$\lim_m \sum_k \left|\sum_{n=0}^{m}(a_{nk} - a_{n,k+1})\right| = \sum_k \left|\sum_n (a_{nk} - a_{n,k+1})\right|, \tag{21}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}[(a_{nk} - a_{n,k+1}) - (a_{n-1,k} - a_{n-1,k+1})]\right| < \infty, \tag{22}$$

$$\sup_{m\in\mathbb{N}}\left|\lim_k \sum_{n=0}^{m} a_{nk}\right| < \infty, \tag{23}$$

$$\exists \alpha_k \in \mathbb{C} \ni \sum_n a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{24}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} [(a_{nk} - a_{n-1,k}) - (a_{n,k-1} - a_{n-1,k-1})] \right| < \infty, \tag{25}$$

where $\mathcal{F}$ denote the collection of all finite subsets of $\mathbb{N}$.

Now, we can give some matrix transformations in the following Lemma for the next step that we will need in the inclusion Theorems.

Lemma 1. *Let $A = (a_{nk})$ be an arbitrary infinite matrix. Then,*

(1) $A = (a_{nk}) \in (bs, \ell_\infty)$ iff Equations (9) and (12) hold (Stieglitz and Tietz [26]),
(2) $A = (a_{nk}) \in (cs, c)$ iff Equations (12) and (13) hold (Wilansky [27]),
(3) $A = (a_{nk}) \in (bs, \ell_1)$ iff Equations (9) and(14) hold (K.-G. Grosse-Erdman [28]).
(4) $A = (a_{nk}) \in (cs, \ell_1)$ iff Equation (15) holds (Stieglitz and Tietz [26]).
(5) $A = (a_{nk}) \in (bs, c)$ iff Equations (9), (16) and (17) hold (K.-G. Grosse-Erdman [28]).
(6) $A = (a_{nk}) \in (cs, \ell_\infty)$ iff Equations (12) and (18) hold (Stieglitz and Tietz [26]).
(7) $A = (a_{nk}) \in (bs, c_0)$ iff Equations (9) and (19) hold (Stieglitz and Tietz [26]).
(8) $A = (a_{nk}) \in (bs, cs_0)$ iff Equations (9) and (20) hold (Zeller [29]).
(9) $A = (a_{nk}) \in (bs, cs)$ iff Equations (9) and (21) hold (Zeller [29]).
(10) $A = (a_{nk}) \in (bs, bv)$ iff Equations (9) and (22) hold (Zeller [29]).
(11) $A = (a_{nk}) \in (bs, bs)$ iff Equations (9) and (10) hold (Zeller [29]).
(12) $A = (a_{nk}) \in (cs, cs)$ iff Equations (10) and (11) hold (Hill, [30]).
(13) $A = (a_{nk}) \in (bs, bv_0)$ iff Equations (12), (19) and (22) hold (Stieglitz and Tietz [26]).
(14) $A = (a_{nk}) \in (cs, c_0)$ iff Equation (12) holds and Equation (13) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ (Dienes [31]).
(15) $A = (a_{nk}) \in (cs, bs)$ iff Equations (10) and (23) hold (Zeller [29]).
(16) $A = (a_{nk}) \in (cs, cs_0)$ iff Equation (10) holds and Equation (24) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ (Zeller [29]).
(17) $A = (a_{nk}) \in (cs, bv)$ iff Equation (25) holds (Zeller [29]).
(18) $A = (a_{nk}) \in (cs, bv_0)$ iff Equation (25) holds and Equation (13) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ (Stieglitz and Tietz [26]).

Theorem 9. *The inclusion $bs \subset bs(\hat{F}(s,r))$ is valid.*

Proof. Let $x \in bs$. We must demonstrate that $x \in bs(\hat{F}(s,r))$. It means that $\hat{F}(s,r) \in (bs, bs)$. For $\hat{F}(s,r) \in (bs, bs)$, $\hat{F}(s,r)$ must ensure to the conditions of (11) of Lemma 1. We see that

$$\lim_k f_{nk}(s,r) = 0 \text{ for each } n \in \mathbb{N}.$$

The other condition also holds as follows:

$$\begin{aligned} \sup_m \sum_k \left| \sum_{n=0}^{m} (f_{nk}(s,r) - f_{n,k+1}(s,r)) \right| &= \sup_m \lim_p \left(\frac{|s+r|}{f_1.f_2} + \frac{|s+r|}{f_2.f_3} + \ldots + \frac{|s+r|}{f_{p+1}.f_{p+2}} \right) \\ &= \frac{17}{10} |s+r| < \infty. \end{aligned}$$

Consequently, the conditions of (11) of Lemma 1 hold. The proof is complete. □

Theorem 10. *If $|r/s| < 1/4$, then $bs(\hat{F}(s,r)) \subset \ell_\infty$ is valid.*

Proof. Let $x \in bs(\hat{F}(s,r))$. Then, $y = \hat{F}(s,r)x \in bs$. We must demonstrate that $x = \hat{F}^{-1}(s,r)y \in \ell_\infty$. That is, $\hat{F}^{-1}(s,r) \in (bs, \ell_\infty)$. For $\hat{F}^{-1}(s,r) \in (bs, \ell_\infty)$, $\hat{F}^{-1}(s,r)$ must satisfy the conditions of (1) of Lemma 1. It is clear that

$$\lim_k f^{-1}_{nk}(s,r) = 0 \text{ for each } n \in \mathbb{N}.$$

The other condition is also holds as follows:

$$\begin{aligned}\sup_n\sum_k\left|(f^{-1}_{nk}(s,r)-f^{-1}_{n,k+1}(s,r))\right| &\leq 2\sup_n\sum_k\left|(f^{-1}_{nk}(s,r)\right|-\left|\frac{r}{s}\right| \\ &\leq \frac{4}{s}\sum_k\left(\frac{4r}{s}\right)^k<\infty.\end{aligned} \tag{26}$$

Consequently, the conditions of (1) of Lemma 1 hold. The proof is complete. □

Theorem 11. *The inclusion $cs\subset cs(\hat{F}(s,r))$ is valid.*

Proof. Let $x\in cs$. We must demonstrate that $x\in cs(\hat{F}(s,r))$. It means that $\hat{F}(s,r)\in(cs,cs)$. For $\hat{F}(s,r)\in(cs,cs)$, $\hat{F}(s,r)$ must satisfy the conditions of (12) of Lemma 1. Equation (10) has been satisfied in Theorem 9. Now, we must demonstrate Equation (11). For every $k\in\mathbb{N}$,

$$\lim_{n\to\infty}\sum_k f_{nk}(s,r)=\lim_n(s\frac{f_n}{f_{n+1}}+r\frac{f_{n+1}}{f_n})=\frac{s}{\varphi}+r\varphi=\ell$$

such that $\ell\in\mathbb{C}$ exist. Consequently, the conditions of (12) of Lemma 1 hold. The proof is complete. □

Theorem 12. *If $|r/s|<1/4$, then $cs(\hat{F}(s,r))\subset c$ is valid.*

Proof. Let $x\in cs(\hat{F}(s,r))$. Then, $y=\hat{F}(s,r)x\in cs$. We must demonstrate that $x=\hat{F}^{-1}(s,r)y\in c$. That is, $\hat{F}^{-1}(s,r)\in(cs,c)$. For $\hat{F}^{-1}(s,r)\in(cs,c)$, $\hat{F}^{-1}(s,r)$ must satisfy the conditions of (2) of Lemma 1. Equation (12) has been satisfied in Theorem 10. Now, we must demonstrate Equation (13). For each $k\in\mathbb{N}$,

$$\begin{aligned}\lim_n f^{-1}_{nk}(s,r) &\leq \lim_n\left|f^{-1}_{nk}(s,r)\right|=\lim_n\left|\frac{f_{n+1}}{sf_n}\left(-\frac{r}{s}\right)^{n-k}\frac{\frac{f_{n+1}}{f_{k+1}}}{\frac{f_k}{f_n}}\right|=\lim_n\left|\frac{f_{n+1}}{sf_n}\prod_{i=k}^{n-1}\frac{r\frac{f_{i+2}}{f_{i+1}}}{s\frac{f_i}{f_{i+1}}}\right| \\ &\leq \lim_n\frac{f_{n+1}}{|s|\,f_n}\prod_{i=k}^{n-1}\left|\frac{\sup_{i\in\mathbb{N}}r\frac{f_{i+2}}{f_{i+1}}}{\inf_{i\in\mathbb{N}}s\frac{f_i}{f_{i+1}}}\right|\leq\lim_n\frac{f_{n+1}}{|s|\,f_n}\left(\frac{4r}{s}\right)^{n-k}=\frac{\varphi}{|s|}.0=0.\end{aligned}$$

Thus, Equation (13) is also satisfied. □

Theorem 13. *The inclusion $cs(\hat{F}(s,r))\subset bs(\hat{F}(s,r))$ is valid.*

Proof. Let $x\in cs(\hat{F}(s,r))$. Then, $y=\hat{F}(s,r)x\in cs$. Hence, $\sum_k\hat{F}(s,r)x\in c$. $c\subset\ell_\infty$, so it becomes $\sum_k\hat{F}(s,r)x\in\ell_\infty$. That is, $\hat{F}(s,r)x\in bs$. Hence, $x\in bs(\hat{F}(s,r))$. Consequently, $cs(\hat{F}(s,r))\subset bs(\hat{F}(s,r))$.

Before giving the corollary about the Schauder basis for the space $cs(\hat{F}(r,s))$, let us define the Schauder basis which was introduced by J. Schauder in 1927. Let $(X,\|.\|)$ be normed space and be a sequence $(a_k)\in X$. There exists a unique sequence (λ_k) of scalars such that $x=\sum\limits_{k=0}^{\infty}\lambda_k a_k$, and

$$\lim_{n\to\infty}\left\|x-\sum_{k=0}^{n}\lambda_k a_k\right\|=0.$$

Then, (a_k) is called a Schauder basis for X. □

Now, we can give the corollary about Schauder basis.

Corollary 1. *Let us sequence* $b^{(k)} = \left\{ b_n^{(k)} \right\}_{n\in\mathbb{N}}$ *defined in the* $cs(\hat{F}(s,r))$ *such that*

$$b_n^{(k)} = \begin{cases} \frac{1}{s}(-\frac{r}{s})^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}}, & n > k, \\ \frac{1}{s} \frac{f_{k+1}}{f_k}, & n = k, \\ 0, & n < k. \end{cases}$$

Then, sequence $\left\{ b^{(k)} \right\}_{n\in\mathbb{N}}$ *is a basis of* $cs(\hat{F}(s,r))$ *and every sequence* $x \in cs(\hat{F}(s,r))$ *has a unique representation* $x = \sum_k y_k b^k$, *where* $y_k = (\hat{F}(s,r)x)_k$.

3. The α-, β- and γ-Duals of the Spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ and Some Matrix Transformations

In this section, the alpha-, beta-, gamma-duals of the spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ are determined and characterized the classes of infinite matrices from the space $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ to some other sequence spaces.

Now, we give the two lemmas to prove the theorems that will be given in the next stage.

Lemma 2. *Suppose that* $a = (a_n) \in w$ *and the infinite matrix* $B = (b_{nk})$ *is defined by* $B_n = a_n(\hat{F}^{-1}(s,r))_n$, *that is,*

$$b_{nk} = \begin{cases} a_n f_{nk}^{-1}(s,r), & 0 \le k < n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$, $\delta \in \{bs, cs\}$. *Then,* $a \in \left\{ \delta(\hat{F}(s,r)) \right\}^{\alpha}$ *iff* $B \in (\delta, \ell_1)$.

Proof. Let $a = (a_n)$ and $x = (x_n)$ be an arbitrary subset of w. $y = (y_n)$ such that $y = \hat{F}(s,r)x$, which is defined by Equation (5). Then,

$$a_n x_n = a_n(\hat{F}^{-1}(s,r)y)_n = (By)_n \tag{27}$$

for all $n \in \mathbb{N}$. Hence, we obtain by Equation (5) that $ax = (a_n x_n) \in \ell_1$ with $x = (x_n) \in \delta(\hat{F}(s,r))$ iff $By \in \ell_1$ with $y \in \delta$. That is, $B \in (\delta, \ell_1)$. □

Lemma 3. *Let [32]* $C = (c_{nk})$ *be defined via a sequence* $a = (a_k) \in w$ *and the inverse matrix* $V = (v_{nk})$ *of the triangle matrix* $U = (u_{nk})$ *by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \le k < n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. *Then, for any sequence space* λ,

$$\begin{aligned} \lambda_U^{\gamma} &= \{a = (a_k) \in w : C \in (\lambda, \ell_\infty)\}, \\ \lambda_U^{\beta} &= \{a = (a_k) \in w : C \in (\lambda, c)\}. \end{aligned}$$

If we consider Lemmas 1–3 together, the following is obtained.

Corallary 1. *Let* $B = (b_{nk})$ *and* $C = (c_{nk})$ *such that*

$$b_{nk} = \begin{cases} a_n f_{nk}^{-1}(s,r), & 0 \le k < n \\ 0, & k > n \end{cases} \quad \text{and } c_{nk} = \sum_{j=k}^{n} \frac{1}{s}(-\frac{r}{s})^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j.$$

If we take $t_1, t_2, t_3, t_4, t_5, t_6, t_7$ and t_8 as follows:

$$
\begin{aligned}
t_1 &= \left\{ a = (a_k) \in w : \sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (b_{nk} - b_{n,k+1}) \right| < \infty \right\}, \\
t_2 &= \left\{ a = (a_k) \in w : \sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (b_{nk} - b_{n,k-1}) \right| < \infty \right\}, \\
t_3 &= \left\{ a = (a_k) \in w : \lim_{k \to \infty} c_{nk} = 0 \right\}, \\
t_4 &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} (c_{nk} - c_{n,k+1}) = \alpha \right\}, \\
t_5 &= \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k |c_{nk} - c_{n,k+1}| = \sum_k \left| \lim_{n \to \infty} (c_{nk} - c_{n,k+1}) \right| \right\}, \\
t_6 &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{C} \lim_{n \to \infty} c_{nk} = \alpha, \text{ for all } k \in \mathbb{N} \right\}, \\
t_7 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_k |c_{nk} - c_{n,k+1}| < \infty \right\}, \\
t_8 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \left| \lim_k c_{nk} \right| < \infty \right\}.
\end{aligned}
$$

Then, the following statements hold:

(1) $\{bs(\hat{F}(s,r))\}^{\alpha} = t_1$,
(2) $\{cs(\hat{F}(s,r))\}^{\alpha} = t_2$,
(3) $\{bs(\hat{F}(s,r))\}^{\beta} = t_3 \cap t_4 \cap t_5$,
(4) $\{cs(\hat{F}(s,r))\}^{\beta} = t_6 \cap t_7$,
(5) $\{bs(\hat{F}(s,r))\}^{\gamma} = t_3 \cap t_7$,
(6) $\{cs(\hat{F}(s,r))\}^{\gamma} = t_7 \cap t_8$.

Theorem 14. *Let $\lambda \in \{bs, cs\}$ and $\mu \subset w$. Then, $A = (a_{nk}) \in (\lambda(\hat{F}(s,r)), \mu)$ iff*

$$D^m = (d_{nk}^{(m)}) \in (\lambda, c) \text{ for all } n \in \mathbb{N}, \tag{28}$$

$$D = (d_{nk}) \in (\lambda, \mu), \tag{29}$$

where

$$d_{nk}^{(m)} = \begin{cases} \sum_{j=k}^{m} \frac{1}{s}(-\frac{r}{s})^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}, & 0 \le k < m, \\ 0, & k > m, \end{cases} \tag{30}$$

and

$$d_{nk} = \sum_{j=k}^{\infty} \frac{1}{s}(-\frac{r}{s})^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \tag{31}$$

for all $k, m, n \in \mathbb{N}$.

Proof. To prove the necessary part of the theorem, let us suppuse that $A = (a_{nk}) \in (\lambda(\hat{F}(s,r), \mu)$ and $x = (x_k) \in \lambda(\hat{F}(s,r))$. By using Equation (6), we find

$$
\begin{aligned}
\sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \sum_{j=o}^{k} \frac{1}{s}(-\frac{r}{s})^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \\
&= \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{1}{s}(-\frac{r}{s})^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k = \sum_{k=0}^{m} d_{nk}^{(m)} y_k = D_n^{(m)}(y)
\end{aligned} \tag{32}
$$

for all $m, n \in \mathbb{N}$. For each $m \in \mathbb{N}$ and $x = (x_k) \in \lambda(\hat{F}(s,r))$, $A_m(x)$ exists and also lies in c. Then, $D_n^{(m)}$also lies in c for each $m \in \mathbb{N}$. Hence, $D^{(m)} \in (\lambda, c)$. Now, from Equation (32), we consider for $m \to \infty$, and then $Ax = Dy$. Consequently, we obtain $D = (d_{nk}) \in (\lambda, \mu)$.

If we want to prove the sufficient part of the theorem, then let us assume that Equations (28) and (29) are satisfied and $x = (x_k) \in \lambda(\hat{F}(s,r))$. By using Corollary 1 and Equations (28) and (32), we obtain $y = \hat{F}(s,r)x \in \lambda$ and $D_n^{(m)}(y) = \sum_{k=0}^{m} d_{nk}^{(m)} y_k = \sum_{k=0}^{m} a_{nk}x_k = A_n^{(m)}(x) \in c$. Hence, $A = (a_{nk})_{k\in\mathbb{N}}$ exists. In addition, in Equation (32), if we consider $m \to \infty$. Then, $Ax = Dy$. Consequently, we obtain $A = (a_{nk}) \in (\lambda(\hat{F}(s,r)), \mu)$.

In Theorem 14, we take $\lambda(\hat{F}(s,r))$ instead of μ and μ instead of $\lambda(\hat{F}(s,r))$, and then we get the following theorem. □

Theorem 15. *Let $\lambda \in \{bs, cs\}$ and μ be an arbitrary subset of w and $A = (a_{nk})$ and $B = (b_{nk})$ be infinite matrices. If we take*

$$b_{nk} := r\frac{f_{n+1}}{f_n}a_{n-1,k} + s\frac{f_n}{f_{n+1}}a_{nk} \tag{33}$$

for all $k, n \in \mathbb{N}$, then $A \in (\mu, \lambda(\hat{F}(s,r)))$ iff $B \in (\mu, \lambda)$.

Proof. Let us suppose that $A \in (\mu, \lambda(\hat{F}(s,r)))$ and Equation (33) exist. For $z = (z_k) \in \mu$, we obtain $Az \in \lambda(\hat{F}(s,r))$ from $A \in (\mu, \lambda(\hat{F}(s,r)))$. Hence, $\hat{F}(s,r)(Az) \in \lambda$. On the other hand, we have

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{k=0}^{m}\left(r\frac{f_{n+1}}{f_n}a_{n-1,k} + s\frac{f_n}{f_{n+1}}a_{nk}\right)z_k \tag{34}$$

for all $m, n \in \mathbb{N}$. If we carry out $m \to \infty$ to Equation (34), we obtain that

$$(Bz)_n = \left((\hat{F}(s,r)A)z\right)_n = \left(\hat{F}(s,r)(Az)\right)_n. \tag{35}$$

Since $\hat{F}(s,r)(Az) \in \lambda$, we find $Bz = (Bz)_n \in \lambda$ for $z = (z_k) \in \mu$ from Equation (35). Hence, we obtain that $B \in (\mu, \lambda)$.This is the desired result. □

At this stage, let us consider almost convergent sequences spaces, which were given by Lorentz [33]. This is because they will help in calculating some of the results of Theorems 14 and 15. Let a sequence $x = (x_k) \in \ell_\infty$. x is said to be almost convergent to the generalized limit ℓ iff $\lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \ell$ uniformly in n and is denoted by $f - \lim x = \ell$. By f and f_0, we indicate the space of all almost convergent and almost null sequences, respectively. However, in this article, we use $\hat{c}$ and $\hat{c}_0$ instead of f and f_0, respectively, in order to avoid any confusion. This is because the Fibonacci sequence is also denoted by f. In addition, by $\hat{c}s$, we indicate the space of sequences, which is composed of all almost convergent series. The sequences spaces $\hat{c}$ and $\hat{c}_0$ are

$$\begin{aligned}
\hat{c}_0 &= \left\{x = (x_k) \in \ell_\infty : \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n\right\}, \\
\hat{c} &= \left\{x = (x_k) \in \ell_\infty : \exists \ell \in \mathbb{C} \ni \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1} = \ell \text{ uniformly in } n\right\}.
\end{aligned}$$

Now, let $A = (a_{nk})$ be an arbitrary infinite matrix and list the following conditions:

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{36}$$

$$\lim_q \sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q} \triangle\left[\sum_{j=0}^{n+i}(a_{jk} - \alpha_k)\right]\right| = 0 \text{ uniformly in } n, \tag{37}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\triangle\left[\sum_{j=0}^{n}a_{jk}\right]\right|<\infty, \tag{38}$$

$$\exists\alpha_k\in\mathbb{C}\ni f-\lim\sum_{j=0}^{n}a_{jk}=\alpha_k \text{ for each } k\in\mathbb{N}, \tag{39}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n}a_{jk}\right|<\infty, \tag{40}$$

$$\exists\alpha_k\in\mathbb{C}\ni\sum_n\sum_k a_{nk}=\alpha_k \text{ for all } k\in\mathbb{N}, \tag{41}$$

$$\lim_n\sum_k\left|\triangle\left[\sum_{j=0}^{n}(a_{jk}-\alpha_k)\right]\right|=0\,, \tag{42}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n}a_{jk}\right|^q<\infty,\ q=\frac{p}{p-1}, \tag{43}$$

$$\sup_{m,n\in\mathbb{N}}\left|\sum_{n=0}^{m}a_{nk}\right|<\infty, \tag{44}$$

$$\sup_{m,l\in\mathbb{N}}\left|\sum_{n=0}^{m}\sum_{k=l}^{\infty}a_{nk}\right|<\infty, \tag{45}$$

$$\sup_{m,l\in\mathbb{N}}\left|\sum_{n=0}^{m}\sum_{k=0}^{l}a_{nk}\right|<\infty, \tag{46}$$

$$\lim_m\sum_k\left|\sum_{n=m}^{\infty}a_{nk}\right|=0, \tag{47}$$

$$\sum_n\sum_k a_{nk},\ \text{convergent}, \tag{48}$$

$$\lim_{m\to\infty}\sum_{n=0}^{m}(a_{nk}-a_{n,k+1})=\alpha, \text{ for each } k\in\mathbb{N}, \alpha\in\mathbb{C}. \tag{49}$$

Let us give some matrix transformations in the following Lemma for use in the next step.

Lemma 4. *Let $A=(a_{nk})$ be an infinite matrix for all $k,n\in\mathbb{N}$. Then,*

(1) $A=(a_{nk})\in(\hat{c},cs)$ iff Equations (24) and (40)–(42) hold (Başar [34]).
(2) $A=(a_{nk})\in(cs,\hat{c})$ iff Equations (12) and (36) hold (Başar and Çolak [35]).
(3) $A=(a_{nk})\in(bs,\hat{c})$ iff Equations (9), (12), (36) and (37) hold (Başar and Solak [36]).
(4) $A=(a_{nk})\in(bs,\hat{c}s)$ iff Equations (9) and (37)–(39) hold (Başar and Solak [36]).
(5) $A=(a_{nk})\in(cs,\hat{c}s)$ iff Equations (38) and (39) hold (Başar and Çolak [35]).
(6) $A=(a_{nk})\in(\ell_\infty,bs)=(c,bs)=(c_0,bs)$ iff Equation (40) holds (Zeller [29]).
(7) $A=(a_{nk})\in(\ell_p,bs)$ iff Equation (43) holds (Jakimovski and Russell [37]).
(8) $A=(a_{nk})\in(\ell,bs)$ iff Equation (44) holds (Zeller [29]).
(9) $A=(a_{nk})\in(bv,bs)$ iff Equation (45) holds (Zeller [29]).
(10) $A=(a_{nk})\in(bv_0,bs)$ iff Equation (46) holds (Jakimovski and Russell [37]).
(11) $A=(a_{nk})\in(\ell_\infty,cs)$ iff Equation (47) holds (Zeller [29]).
(12) $A=(a_{nk})\in(c,cs)$ iff Equations (11), (40) and (48) hold (Zeller [29]).
(13) $A=(a_{nk})\in(cs_0,cs)$ iff Equations (10) and (49) hold (Zeller [29]).
(14) $A=(a_{nk})\in(\ell_p,cs)$ iff Equations (11) and (43) hold (Jakimovski and Russell [37]).
(15) $A=(a_{nk})\in(\ell,cs)$ iff Equations (11) and (44) hold (Jakimovski and Russell [37]).
(16) $A=(a_{nk})\in(bv,cs)$ iff Equations (11), (44) and (46) hold (Zeller [29]).

(17) $A = (a_{nk}) \in (bv_0, cs)$ *iff Equations (11) and (46) hold (Jakimovski and Russell [37]).*

Now, let us list the following condition, where d_{nk} and $d_{nk}^{(m)}$ are taken as in Equations (30) and (31):

$$\lim_k d_{nk}^{(m)} = 0 \text{ for all } n \in \mathbb{N}, \tag{50}$$

$$\exists d_{nk} \in \mathbb{C} \ni \lim_{n\to\infty} (d_{nk}^{(m)} - d_{n,k+1}^{(m)}) = d_{nk} \text{ for all } k, n \in \mathbb{N}, \tag{51}$$

$$\lim_{n\to\infty} \sum_k \left| d_{nk}^{(m)} - d_{n,k+1}^{(m)} \right| < \infty \text{ uniformly in } n, \tag{52}$$

$$\lim_k d_{nk} = 0 \text{ for all } n \in \mathbb{N}, \tag{53}$$

$$\sup_n \sum_k |d_{nk} - d_{n,k+1}| < \infty, \tag{54}$$

$$\exists d_k \in \mathbb{C} \ni \lim_{n\to\infty} (d_{nk} - d_{n,k+1}) = d_k \text{ for all } k, n \in \mathbb{N}, \tag{55}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k |d_{nk} - d_{n,k+1}| = \alpha \text{ uniformly in } n, \tag{56}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| < \infty, \tag{57}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| = \sum_k \left| \sum_n (d_{nk} - d_{n,k+1}) \right|, \tag{58}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| = 0, \tag{59}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} (d_{nk} - d_{n,k+1}) \right| < \infty, \tag{60}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} (d_{nk} - d_{n,k+1}) - (d_{n-1,k} - d_{n-1,k+1}) \right| < \infty, \tag{61}$$

$$\sup_n \sum_k \left| d_{nk}^{(m)} - d_{n,k+1}^{(m)} \right| < \infty, \tag{62}$$

$$\exists d_k \in \mathbb{C} \ni \lim_n d_{nk}^{(m)} = d_k \text{ for all } k, n \in \mathbb{N}, \tag{63}$$

$$\sup_{n\in\mathbb{N}} \left| \lim_k d_{nk} \right| < \infty, \tag{64}$$

$$\exists d_k \in \mathbb{C} \ni \lim_n d_{nk} = d_k \text{ for all } k, n \in \mathbb{N}, \tag{65}$$

$$\sup_{m\in\mathbb{N}} \left| \lim_k \sum_{n=0}^{m} d_{nk} \right| < \infty, \tag{66}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k-1}) \right| < \infty, \tag{67}$$

$$\exists d_k \in \mathbb{C} \ni \sum_n d_{nk} = d_k \text{ for each } k \in \mathbb{N}, \tag{68}$$

$$\sup_{N,K\in\mathcal{F}} \sum_{n\in N} \left| \sum_{k\in K} (d_{nk} - d_{n,k-1}) \right| < \infty, \tag{69}$$

$$\exists d_k \in \mathbb{C} \ni f - \lim d_{nk} = d_k \text{ for each } k \in \mathbb{N}, \tag{70}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N}\sum_{k\in K} (d_{nk} - d_{n-1,k}) - (d_{n,k-1} - d_{n-1,k-1}) \right| < \infty, \tag{71}$$

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \triangle \left[\sum_{j=0}^{n+i} (d_{jk} - \alpha_k) \right] \right| = 0 \text{ uniformly in } n, \tag{72}$$

$$\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=0}^{n} d_{jk} \right| < \infty, \tag{73}$$

$$\exists d_k \in \mathbb{C} \ni \sum_n \sum_k d_{nk} = d_k \text{ for all } k \in \mathbb{N}, \tag{74}$$

$$\lim_n \sum_k \left| \triangle \left[\sum_{j=0}^{n} (d_{jk} - \alpha_k) \right] \right| = 0\,, \tag{75}$$

$$\sup_{n\in\mathbb{N}} \sum_k \left| \triangle \left[\sum_{j=0}^{n} d_{jk} \right] \right| < \infty, \tag{76}$$

$$\exists d_k \in \mathbb{C} \ni f - \lim \sum_{j=0}^{n} d_{jk} = d_k \text{ for each } k \in \mathbb{N}, \tag{77}$$

Now, we can give several conclusions of Theorems 14 and 15, and Lemmas 1 and 4.

Corallary 2. *Let $A = (a_{nk})$ be an infinite matrix for all $k, n \in \mathbb{N}$. Then,*

(1) $A = (a_{nk}) \in (bs(\hat{F}(s,r), c_0)$ *iff Equations (50)–(53) hold and Equation (56) also holds with $\alpha = 0$.*
(2) $A = (a_{nk}) \in (bs(\hat{F}(s,r), cs_0)$ *iff Equations (50)–(53) and (59) hold.*
(3) $A = (a_{nk}) \in (bs(\hat{F}(s,r), c)$ *iff Equations (50)–(53), (55) and (56) hold.*
(4) $A = (a_{nk}) \in (bs(\hat{F}(s,r), cs)$ *iff Equations (50)–(53) and (58) hold.*
(5) $A = (a_{nk}) \in (bs(\hat{F}(s,r), \ell_\infty)$ *iff Equations (50)–(54) hold.*
(6) $A = (a_{nk}) \in (bs(\hat{F}(s,r), bs)$ *iff Equations (50)–(53) and (57) hold.*
(7) $A = (a_{nk}) \in (bs(\hat{F}(s,r), \ell_1)$ *iff Equations (50)–(53) and (60) hold.*
(8) $A = (a_{nk}) \in (bs(\hat{F}(s,r), bv)$ *iff Equations (50)–(53) and (61) hold.*
(9) $A = (a_{nk}) \in (bs(\hat{F}(s,r), bv_0)$ *iff Equations (50)–(52), (54) and (61) hold and Equation (56) also holds with $\alpha = 0$.*

Corallary 3. *Let $A = (a_{nk})$ be an infinite matrix for all $k, n \in \mathbb{N}$. Then,*

(1) $A = (a_{nk}) \in (cs(\hat{F}(s,r), c_0)$ *iff Equations (54), (62) and (63) hold and Equation (65) also holds with $d_k = 0$ for all $k \in \mathbb{N}$.*
(2) $A = (a_{nk}) \in (cs(\hat{F}(s,r), cs_0)$ *iff Equations (57), (62) and (63) hold and Equation (68) also holds with $d_k = 0$ for all $k \in \mathbb{N}$.*
(3) $A = (a_{nk}) \in (cs(\hat{F}(s,r), c)$ *iff Equations (54), (62), (63) and (65) hold.*
(4) $A = (a_{nk}) \in (cs(\hat{F}(s,r), cs)$ *iff Equations (62), (63), (67) and (68) hold.*
(5) $A = (a_{nk}) \in (cs(\hat{F}(s,r), \ell_\infty)$ *iff Equations (54) and (62)–(64) hold.*
(6) $A = (a_{nk}) \in (cs(\hat{F}(s,r), bs)$ *iff Equations (57), (62), (63) and (66) hold.*
(7) $A = (a_{nk}) \in (cs(\hat{F}(s,r), \ell_1)$ *iff Equations (62), (63) and (69) hold.*
(8) $A = (a_{nk}) \in (cs(\hat{F}(s,r), bv)$ *iff Equations (62), (63) and (71) hold.*
(9) $A = (a_{nk}) \in (cs(\hat{F}(s,r), bv_0)$ *iff Equations (62), (63) and (65) hold and Equation (71) also holds with $d_k = 0$ for all $k \in \mathbb{N}$.*

Corallary 4. *Let $A = (a_{nk})$ be an infinite matrix for all $k, n \in \mathbb{N}$. Then,*

(1) $A = (a_{nk}) \in (bs(\hat{F}(s,r), \hat{c})$ *iff Equations (50)–(54), (70) and (72) hold.*

(2) $A = (a_{nk}) \in (bs(\hat{F}(s,r), \hat{c}_0)$ *iff Equations (50)–(54) hold and (70) and Equation (72) also hold with* $\alpha_k = 0$ *in Equation (70) and* $d_k = 0$ *in (72).*
(3) $A = (a_{nk}) \in (cs(\hat{F}(s,r), \hat{c})$ *iff Equations (54), (62), (63) and (70) hold.*
(4) $A = (a_{nk}) \in (cs(\hat{F}(s,r), \hat{c}_0)$ *iff Equations (62), (63) and (54) hold and Equation (70) also holds with* $\alpha_k = 0$.
(5) $A = (a_{nk}) \in (\hat{c}, cs(\hat{F}(s,r))$ *iff Equations (68) and (73)–(75) hold with* b_{nk} *instead of* d_{nk}*, where* b_{nk} *is defined by Equation (33).*
(6) $A = (a_{nk}) \in (bs(\hat{F}(s,r), \hat{c}s)$ *iff Equations (50)–(53), (72), (76) and (77) hold.*
(7) $A = (a_{nk}) \in (cs(\hat{F}(s,r), \hat{c}s)$ *iff Equations (62), (63), (76) and (77) hold.*

Corallary 5. *Let* $A = (a_{nk})$ *be an infinite matrix for all* $k, n \in \mathbb{N}$*. Then,*

(1) $A = (a_{nk}) \in (\ell_\infty, bs(\hat{F}(s,r)) = (c, bs) = (c_0, bs)$ *iff Equation (40) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by (33).*
(2) $A = (a_{nk}) \in (\ell_p, bs(\hat{F}(s,r))$ *iff Equation (43) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by (33).*
(3) $A = (a_{nk}) \in (\ell, bs(\hat{F}(s,r))$ *iff Equation (44) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(4) $A = (a_{nk}) \in (bv, bs(\hat{F}(s,r))$ *iff Equation (45) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(5) $A = (a_{nk}) \in (bv_0, bs(\hat{F}(s,r))$ *iff Equation (46) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(6) $A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}(s,r))$ *iff Equation (47) holds with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(7) $A = (a_{nk}) \in (c, cs(\hat{F}(s,r))$ *iff Equations (11), (40) and (48) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(8) $A = (a_{nk}) \in (cs_0, cs(\hat{F}(s,r))$ *iff Equations (10) and (49) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(9) $A = (a_{nk}) \in (\ell_p, cs(\hat{F}(s,r))$ *iff Equations (11) and (43) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(10) $A = (a_{nk}) \in (\ell, cs(\hat{F}(s,r))$ *iff Equations (11) and (44) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(11) $A = (a_{nk}) \in (bv, cs(\hat{F}(s,r))$ *iff Equations (11), (44) and (46) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*
(12) $A = (a_{nk}) \in (bv_0, cs(\hat{F}(s,r))$ *iff Equations (11) and (46) hold with* b_{nk} *instead of* a_{nk}*, where* b_{nk} *is defined by Equation (33).*

4. Discussion

The difference sequence operator was introduced for the first time in the literature by Kızmaz [38]. Kirişçi and Başar [4] have characterized and investigated generalized difference sequence spaces. The Fibonacci difference matrix $\hat{F}$, which is derived from the Fibonacci sequence (f_n), was recently introduced by Kara [23] in 2013 and defined the new sequence spaces $\ell_p(\hat{F})$ and $\ell_\infty(\hat{F})$, which are derived by the matrix domain of $\hat{F}$ from the sequence spaces ℓ_p and ℓ_∞, respectively, where $1 \leq p < \infty$. Candan [25] in 2015 introduced the sequence spaces $c(\hat{F}(s,r))$ and $c_0(\hat{F}(s,r))$. Later, Candan and Kara [15] studied the sequence spaces $\ell_p(\hat{F}(s,r))$ in which $1 \leq p \leq \infty$. In addition, Kara et al. [24] have characterized some class of compact operators in the spaces $\ell_p(\hat{F})$ and $\ell_\infty(\hat{F})$, where $1 \leq p < \infty$.

The study is concerned with matrix domain on a sequences space of a triangle infinite matrix. In this article, we defined spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ of Generalized Fibonacci difference of sequences, which constituted bounded and convergence series, respectively. We have demonstrated the sets of $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$, which are the linear spaces, and both spaces have the same norm

$$\|x\| = \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \left(s\frac{f_k}{f_{k+1}} x_k + r\frac{f_{k+1}}{f_k} x_{k-1} \right) \right|,$$

where $x \in bs(\hat{F}(s,r))$ or $x \in cs(\hat{F}(s,r))$. In addition, it was shown that they are normed space and Banach spaces. It was found that $bs(\hat{F}(s,r))$ and bs are linearly isomorphic as isometric. At the same time, $cs(\hat{F}(s,r))$ and cs are linearly isomorphic as isometric. Some inclusions' theorems were given with respect to $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$. According to this, inclusions $bs \subset bs(\hat{F}(s,r))$, $cs \subset cs(\hat{F}(s,r))$ are valid. In addition, if $|r/s| < 1/4$, then $bs(\hat{F}(s,r)) \subset \ell_\infty$ and $cs(\hat{F}(s,r)) \subset c$ are valid. It was concluded that $cs(\hat{F}(s,r))$ has a Schauder basis.

Finally, the α-, β- and γ-duals of the both spaces are calculated and some matrix transformations of them were given.

5. Conclusions

In this article, we have defined spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ of Generalized Fibonacci difference of sequences, which constituted bounded and convergence series, respectively. We have demonstrated that the sets of $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ are the linear spaces and both spaces have the same norm. In addition, it was shown that they are Banach spaces. Some inclusions theorems were given with respect to $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$. It was concluded that $cs(\hat{F}(s,r))$ has a Schauder basis. Finally, the α-, β- and γ-duals of the both spaces were calculated and some matrix transformations of them were given.

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Abbreviations

The following abbreviations are used in this manuscript:

iff   if and only if

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