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Second-Order Conditional Lie–Bäcklund Symmetries and Differential Constraints of Nonlinear Reaction–Diffusion Equations with Gradient-Dependent Diffusivity

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Abstract: The radially symmetric nonlinear reaction–diffusion equation with gradient-dependent diffusivity is investigated. We obtain conditions under which the equations admit second-order conditional Lie–Bäcklund symmetries and first-order Hamilton–Jacobi sign-invariants which preserve both signs (≥ 0 and ≤ 0) on the solution manifold. The corresponding reductions of the resulting equations are established due to the compatibility of the invariant surface conditions and the governing equations.

Keywords: conditional Lie–Bäcklund symmetry; differential constraint; sign-invariant; nonlinear reaction–diffusion equation; symmetry reduction

1. Introduction

The classical symmetry [1] for a general system of partial differential equations (PDEs) is originally created by Sophus Lie more than 100 years ago. A Lie group of transformation depends on continuous parameters and consists of either point transformations (point symmetries) acting on the independent and dependent variables of the system. The Lie group is completely characterized by the infinitesimal generator and the determining equation for the infinitesimal generator of such group is a linear homogeneous system of PDEs. The setting up of the determining equation for a given system of PDEs is entirely a matter of routine.

Over the years, many generalizations of the classical symmetry have been proposed. The possibility of the existence of Lie–Bäcklund symmetry (also known as generalized symmetry or higher-order symmetry) appears to have been first considered by Noether [2] and is characterized by an infinitesimal generator that acts on independent variables, dependent variables and their derivatives to some finite order. The procedure for constructing Lie–Bäcklund symmetries is about the same as that for the classical symmetry and the Lie–Bäcklund symmetries admitted by the system in question are found by solving a linear system of PDEs called the determining system. The method of conditional symmetry can be referred to [3–5] and the references therein.

Zhdanov [6], Fokas and Liu [7] independently introduced the method of conditional Lie–Bäcklund symmetry (CLBS) for scalar evolution equations. The reductions of the initial value problem for an evolution equation due to CLBS are considered by Zhdanov et al. in [8–10]. The studies of CLBS for an evolution system are discussed by Sergyeyev [11] and independently by Ji et al. in [12,13]. The CLBS can be regarded as a natural generalization of the conditional symmetry in a similar way to how the

Lie–Bäcklund symmetry is a generalization of the classical symmetry. Therefore, the procedure for computing CLBS is about the same as that for the conditional symmetry. The crucial problem is that we need to determine the form of CLBS presumably. It has been proved that CLBS with the characteristic

$$\eta = [f(u)]_{lx} + a_1(x)\,[f(u)]_{(l-1)x} + \cdots + a_{l-1}(x)\,[f(u)]_x + a_l(x)f(u)$$

is powerful to study classifications and reductions of second-order nonlinear diffusion equations [14–16]. In fact, the linear CLBS with the characteristic

$$\sigma_1 = v_{lx} + a_1(x)v_{(l-1)x} + \cdots + a_{l-1}(x)v_x + a_l(x)v$$

is responsible for the invariant subspace [17,18]

$$W_l = W\{f_1(x), f_2(x), \cdots, f_l(x)\} \equiv \left\{\sum_{j=1}^{l} C_j f_j(x), C_j \in R\right\}$$

admitted by the considered equation. It is noted that $[f(u)]_{jx} = \frac{\partial^j f(u)}{\partial x^j}$, $v_{jx} = \frac{\partial^j v}{\partial x^j}$ and $j = 1, 2, \cdots, l$. The CLBSs with the characteristic

$$\eta = u_{xx} + H(u)u_x^2 + G(x,u)u_x + F(x,u)$$

and

$$\eta = u_{xx} + H(u)u_x^2 + G(x,u)u_x^{2-m} + F(x,u)u_x^{1-m}$$

of nonlinear diffusion equations are considered in [19–23] and are related to first-order Hamilton–Jacobi sign-invariants (H–J SIs) of second-order nonlinear equations in question. In addition, the CLBS with the characteristic

$$\eta = [f(u)]_{xt}$$

and

$$\eta = [f(u, u_x)]_{xt}$$

can respectively be used to construct functionally separable solutions [24] and derivative-dependent functionally separable solutions [25] of nonlinear diffusion equations. A large number of exact solutions for different types of nonlinear diffusion equations have been constructed due to the compatibility of the invariant surface conditions $\eta = 0$ and the governing equations. The base of the reductions of CLBS is due to the compatibility of the invariant surface condition $\eta = 0$ and the governing equation, that is, the invariant surface condition $\eta = 0$ is nothing but a differential constraint (DC) of the original equation.

The key idea of the method of DC is proposed by Yanenko in [26]. The survey of this method is presented by Sidorvo, Shapeev and Yanenko in [27], where the method of DC is successfully introduced into practice on gas dynamics. The general formulation of this method requires that the original differential equation be enlarged by appending additional equations such that the over-determined system satisfies some conditions of compatibility. The procedure to determine whether or not a given differential equation is compatible with the original equation is straightforward; however, for a given equation, the determination of the most general DC that is compatible with the original equation is a very difficult, if not impossible, problem. Therefore, it is better to make oneself content with finding constraints in some classes, and these classes must be chosen using additional considerations. It is proved that DCs corresponding to the invariant surface conditions of the above CLBSs related to invariant subspace [14–16], sign-invariant [23,28] and separation of variables [24,25] are all very effective to study classifications and reductions of nonlinear diffusion equations.

It is also interesting to note that second-order CLBSs of nonlinear diffusion equations can be translated to first-order H–J SIs of the considered equations [23,28]. The property of SI is a junction of qualitative and quantitative properties of parabolic equations. In [18], the SIs for second-order parabolic equations are extensively discussed from many points of view, including constructing SIs of the considered equations, deriving partial differential inequalities with physical interpretations and discussing the geometrical and physical properties of the exact solutions obtained.

In this paper, we are mainly concerned with the second-order CLBS

$$\eta = u_{rr} + H(u)u_r^2 + G(r,u)u_r + F(r,u) \tag{1}$$

and the first-order H–J SI

$$J = u_t - A(u)u_r^{m+1} - B(r,u)u_r^m - C(r,u)u_r^{m-1} - E(r,u) \tag{2}$$

as well as the exact solutions of the radially symmetric nonlinear reaction–diffusion equation with gradient-dependent diffusivity

$$u_t = \frac{1}{r^{n-1}}[r^{n-1}D(u)u_r^m]_r + Q(r,u), \tag{3}$$

where $D(u) = u^k$ and $Q(r,u)$ are respectively the diffusion and reaction term. It is noted that m, k in (3) are constants, which may be positive or negative. This equation has been applied to describe several situations such as nonlinear heat conduction and nonlinear shear flows of nonlinear Newtonian fluids [29–31] (see more details, e.g., in [32]). The seminal work about the application of the nonlinear Equation (3) with $D = 1$ and $Q = 0$ to the turbulent gas flows can be referred to [33].

Lie symmetry of the simplest case of Equation (3) with $n = m = 1$, $D = 1$ and $Q = 0$, i.e., $u_t = u_{rr}$ are discussed in [1] and [34]. Recently, Cherniha et al. [32] described all possible Lie symmetries of Equation (3) in the case $n = 1$ and $D = 1$. For the case of $m = 1$ and $n = 1$, Equation (3) has been studied extensively from many different points of view and a huge number of results on symmetries, exact solutions and qualitative analysis including existence, uniqueness, blow up, extinction, large time behavior and geometrical properties have been obtained (see [35–38] and references therein). Lie symmetries and reductions of the heat equation with a source (sink)

$$u_t = div\Big(D(u)\nabla u\Big) \tag{4}$$

in the two-dimensional and three-dimensional cases are studied in [39]. The nonclassical symmetries and reductions of Equation (4) in two spatial dimensions can be referred to [40]. The nonclassical symmetries and reductions of Arrhenius reaction–diffusion in n dimensions are considered in [41].

It has been known that the approaches of SI [42–44] and CLBS [14–25] have been successfully applied to obtain solutions and to explore the properties of Equation (3) with $n = 1$ or $m = 1$. Galaktionov [42–44] studied the existence of H–J SIs to the $1+1$-dimensional nonlinear diffusion equations and the higher-dimensional case. He showed that Equation (3) with $n = 1$ and $m = 1$ admits the H–J SI

$$J = u_t + A(u)u_r^2 + B(u)u_r + C(u) \tag{5}$$

for some coefficient functions. He also verified that the $n+1$-dimensional nonlinear reaction–diffusion equation with gradient-dependent diffusivity

$$u_t = div(|\nabla u|^\sigma \nabla u) + f(u),\ t > 0 \tag{6}$$

admits the H–J SIs with the form

$$J = u_t - h(u). \tag{7}$$

Thus, it is of great interest to develop both methods to study Equation (3) with $n \neq 1$ and $m \neq 1$. Notably, all possible Lie symmetries of Equation (6) for the case of u possessing two space variables were found in [32].

The remainder of this paper is organized as follows. In the following section, we review some necessary notations, definitions and fundamental theorems on CLBS, DC and H–J SI. The corresponding conditional invariance condition for (3) is also presented here. In Section 3, second-order CLBS (1) and first-order H–J SI (2) of Equation (3) are identified. The corresponding reductions of the resulting Equation (3) are presented in Section 4. The last section is devoted to conclusions and remarks.

2. Basic Notations and Theorems

Let us review some basic facts on the CLBS method, the DC method and the H–J SI method of a nonlinear evolution equation.

Let

$$V = \sum_{k=0}^{\infty} D_r^k \eta(r, t, u, u_r, \cdots, u_{lr}) \frac{\partial}{\partial u_{kr}} \tag{8}$$

be an evolutionary vector field with the characteristic η and

$$u_t = E(r, t, u, u_r, \cdots, u_{nr}) \tag{9}$$

be a nonlinear evolution equation, where we use the following notations

$$D_r = \frac{\partial}{\partial r} + \sum_{k=0}^{\infty} u_{(k+1)r} \frac{\partial}{\partial u_{kr}}, \quad D_r^{j+1} = D_r(D_r^j), \quad D_r^0 = 1, \quad u_{kr} = \frac{\partial^k u}{\partial r^k}.$$

Definition 1 ([45–47]). *The evolutionary vector field (8) is said to be a Lie–Bäcklund symmetry of (9) if and only if*

$$V(u_t - E)|_L = 0,$$

where L is the set of all differential consequences of the equation, that is, $D_r^j D_t^k (u_t - E) = 0, \quad j, k = 0, 1, 2, \cdots.$

Definition 2. *[6,7] The evolutionary vector field (8) is said to be a CLBS of (9) if and only if*

$$V(u_t - E)|_{L \cap M_r} = 0,$$

where M_r denotes the set of all differential consequences of equation $\eta = 0$ with respect to r, that is, $D_r^j \eta = 0$, $j = 0, 1, 2, \cdots$.

The above conditional invariance criterion can be reduced to

$$D_t \eta|_{L \cap M_r} = 0. \tag{10}$$

The fact that Lie–Bäcklund vector field (8) is conditional invariant with respect to evolution Equation (9) yields the compatibility of the invariant surface condition

$$\eta(r, t, u, u_r, \cdots, u_{lr}) = 0 \tag{11}$$

and the governing Equation (9).

Definition 3. *[48] The differential constraint (11) and the evolution system (9) satisfy the compatibility condition if*

$$D_t\eta|_{L_r\cap M_r} = 0, \tag{12}$$

where L_r denotes the set of all differential consequences of the Equation (9) with respect to r, that is, $D_r^j(u_t - E) = 0,\ j = 0, 1, 2, \cdots$.

Olver [49] put forward the view that conditional symmetries and their further generalizations are connected with DC. For evolution Equation (9), the compatibility condition (12) is nothing but the conditional invariance criterion (10).

Let

$$J[u] = J(r,\ t,\ u,\ u_t,\ u_r) \tag{13}$$

be sufficiently smooth; then, $J[\cdot]$ is a general first-order Hamilton–Jacobi operator.

Definition 4. *[42]. (13) is said to be a H–J SI of (9) if it preserves both signs ≥ 0 and ≤ 0 on the solution manifold of (9). This means*

$$\begin{gathered} J[u] \geq 0\ (resp. \leq 0)\ \mathit{in}\ \boldsymbol{R}\ \mathit{for}\ t = 0, \\ \Rightarrow J[u] \geq 0\ (resp. \leq 0)\ \mathit{in}\ \boldsymbol{R}\ \mathit{for}\ t > 0. \end{gathered}$$

This property is understood in the sense of a formal application of the Maximum Principle of parabolic equations. It has been applied to obtain a large number of new exact solutions to nonlinear diffusion equations with source terms [42–44]. The equivalence classes of Lie–Bäcklund transformation [45] shows that the second-order CLBS (1) of Equation (3) degenerates into conditional symmetry if the corresponding first-order H–J SI (2) is of the form

$$J = u_t - R(r,u)u_r - E(r,u). \tag{14}$$

Here, we do not consider these degenerated CLBS.

Since (3) admits the CLBS with the characteristic (1), a direct computation of the conditional invariance condition (10) will yield that

$$
\begin{aligned}
D_t\eta|_{L\cap M_r} \equiv\ & \Big[D''' - (3m+1)HD'' - 3mH'D' + m(3m+2)H^2D' \\
& - mDH'' - m^2(m+1)DH^3 + m(3m+1)DHH'\Big]u_r^{m+3} \\
& + \Big\{\frac{n-1}{r}\Big[D'' - (2m-1)HD' + m(m-1)DH^2 - (m-1)DH'\Big] \\
& - (3m+2)GD'' + m\big[3(2m+1)HD'G + (3m+1)DGH' \\
& - m(3m+1)DGH^2 + (3m-1)DHG_u - 3D'G_u - DG_{uu}\big]\Big\}u_r^{m+2} \\
& - \Big\{\frac{2(n-1)}{r^2}\big[D' - (m-1)DH\big] + \frac{n-1}{r}\big[(m-1)DG_u + 2mGD' \\
& - 2m(m-1)DHG\big] - m\big[(3m+1)G^2D' - m(3m-1)DHG^2 \\
& + (3m-1)DHG_r + (3m-1)DGG_u + 2(3m+1)HFD' \\
& + (3m+1)DFH' + 3(m-1)DHF_u - (3m^2-m+2)DFH^2 \\
& - 3F_uD' - DF_{uu} - 2DG_{ru}\big] + (3m+1)D'G_r + 3(m+1)FD''\Big\}u_r^{m+1} \\
& - \Big\{-\frac{n-1}{r^2}D\Big[(2m-1)G + \frac{2}{r}\Big] + \frac{n-1}{r}\big[mDG_r - m(m-1)DG^2 \\
& - 2m(m-1)DHF + (m-1)DF_u + (2m+1)FD'\big] + (3m+1)D'F_r \\
& + m\big[2DF_{ru} + DG_{rr} - (3m-1)DGG_r - (3m-1)DFG_r \\
& + m(m-1)DG^3 - (6m+1)D'GF + 2(3m^2-3m+2)DFGH \\
& - 3(m-1)DGF_u - 3(m-1)DHF_r\big]\Big\}u_r^m - m\Big\{\frac{n-1}{r}D\big[F_r \\
& - 2(m-1)FG - \frac{2}{r}F\big] - 3(m-1)DGF_r - (3m-1)DFG_u \\
& - 3(m-1)DFF_u + DF_{rr} + (3m^2-5m+4)DF^2H - 3mF^2D' \\
& + (m-1)(3m-2)DG^2F\Big\}u_r^{m-1} - m(m-1)DF\Big[(3m-4)FG \\
& - 3F_r - \frac{n-1}{r}F\Big]u_r^{m-2} - m(m-1)(m-2)DF^3u_r^{m-3} + (H'Q \\
& + HQ_u + Q_{uu})u_r^2 + (2Q_{ru} + 2HQ_r + QG_u)u_r + Q_{rr} + QF_u \\
& - Q_uF + GQ_r,
\end{aligned}
\tag{15}
$$

where the prime denotes the derivative with respect to u and the subscripts denotes the partial derivative with respect to the indicated variables. In view of the above expression, we need to distinguish several cases corresponding to different m for further discussion.

3. Equation (3) Admitting CLBS (1) and H–J SI (2)

It is noted from the expression (15) that for $m = 2,\ -1,\ -2$, the terms in (15) may be combined. For instance, the terms containing u_r^{m+3} and u_r can be put together when $m = -2$. The over-determined system of the coefficient functions $H(u),\ G(r,u),\ F(r,u)$ and $Q(r,u)$ is different for different m. Each case of $m = 2,\ -1,\ -2$ will be respectively considered.

For the case of $m = 2$, it follows from (15) that the coefficient functions in (3) and (1) satisfy the system

$$\begin{aligned}
&D''' - 7HD'' - 6H'D' + 16H^2D' - 2DH'' - 12DH^3 + 14DHH' = 0,\\
&\frac{n-1}{r}(D'' - 3HD' + 2DH^2 - DH') - 8GD'' - 6D'G_u + 30HD'G\\
&+ 14DGH' - 28DGH^2 + 10DHG_u - 2DG_{uu} = 0,\\
&\frac{2(n-1)}{r^2}(DH - D') - \frac{n-1}{r}(DG_u + 4GD' - 4DHG) + 14G^2D'\\
&- 7D'G_r - 9FD'' - 20DHG^2 + 10DHG_r + 10DGG_u + 28HFD'\\
&+ 14DFH' + 6DHF_u - 24DFH^2 - 6F_uD' - 2DF_{uu} - 4DG_{ru} = 0,\\
&- \frac{n-1}{r}(2DG_r - 2DG^2 - 4DHF + DF_u + 5FD') + \frac{3(n-1)}{r^2}DG\\
&+ \frac{2(n-1)}{r^3}D - 7D'F_r - 4DF_{ru} - 2DG_{rr} + 10DGG_r - 4DG^3 + 6DHF_r\\
&+ 10DFG_u + 26D'GF - 32DFGH + 6DGF_u + H'Q + HQ_u + Q_{uu} = 0,\\
&\frac{4(n-1)}{r^2}DF + \frac{2(n-1)}{r}D(2FG - F_r) + 6DGF_r + 10DFG_r + 6DFF_u\\
&- 2DF_{rr} - 12DF^2H + 12F^2D' - 8DG^2F + 2Q_{ru} + 2HQ_r + QG_u = 0,\\
&\frac{2(n-1)}{r}DF^2 + 6DFF_r - 4DGF^2 + Q_{rr} + QF_u - Q_uF + GQ_r = 0
\end{aligned} \tag{16}$$

with $D = u^k$.

Setting

$$A(u) = D' - 2DH,\ B(r,u) = \frac{n-1}{r}D - 2DG,\ C(r,u) = -2DF,\ E(r,u) = Q, \tag{17}$$

system (16) reads as

$$
\begin{aligned}
&5DD'A' - 7DAA' - 2D^2A'' + 2ADD'' - 5AD'^2 + 8A^2D' - 3A^3 = 0,\\
&\frac{6(n-1)}{r}D(AD' - DA' - A^2) + 7DBA' + 5DAB_u - 11ABD' + 2BD'^2\\
&- 3DD'B_u - BDD'' + 2D^2B_{uu} + 7BA^2 = 0,\\
&\frac{3n(n-1)}{r^2}AD^2 - \frac{2(n-1)}{r}D(2DB_u - BD' + 4AB) - 3B^2D' - 8ACD'\\
&+ 5DBB_u + 3ADC_u + 7CDA_u + 5ADB_r + 2D^2C_{uu} + 4D^2B_{ru}\\
&- DD'C_u - 2DD'B_r + 6CA^2 + 5AB^2 = 0,\\
&\frac{n-1}{r^2}D^2[(n+1)B + 5C] - \frac{n-1}{r}D(3DB_r + 2DC_u + 6CA + 2B^2)\\
&+ 5CDB_r + 3BDC_u - ADQ_u + 2BCD' + AQD' + 5DBB_r + 3ADC_r\\
&- DQA' + DD'Q_u + 4D^2C_{ru} - QD'^2 + 2D^2Q_{uu} + 2D^2B_{rr} + DQD''\\
&+ 8ABC + B^3 = 0,\\
&\frac{n-1}{r^2}D^2C - \frac{n-1}{r}D(DC_r + 2BC) + 5CDB_r - 2ADQ_r - DQB_u\\
&+ QBD' + 3BDC_r + 3DCC_u + 2DD'Q_r + 2D^2C_{rr} + 4D^2Q_{ru}\\
&+ 3AC^2 + 2CB^2 = 0,\\
&\frac{n-1}{r}D^2Q_r + 3CC_r + DCQ_u - DBQ_r - DQC_u + CQD' + 2D^2Q_{rr}\\
&+ C^2B = 0.
\end{aligned}
\tag{18}
$$

Formula (2) is a H–J SI of Equation (3) for the case of $m = 2$ if the coefficient functions satisfy system (18). This result can be proved in the same manner as that for Proposition 2.1 in [43]. We omit the details here. It follows that the CLBS (1) is equivalent to the H–J SI (2) for Equation (3). In other words, the CLBS method provides a symmetry interpretation to the method of H–J SI.

Substituting $D = u^k$ into the first one of the system (16), we derive that H satisfies the second-order ordinary differential equation (ODE)

$$
\begin{aligned}
&2u^3H'' + 2(3k - 7uH)u^2H' + 12u^3H^3 - 16ku^2H^2\\
&+7k(k-1)uH - k(k-1)(k-2) = 0.
\end{aligned}
\tag{19}
$$

It seems impossible to obtain the general solution of this nonlinear differential equation, and so we restrict ourselves to considering the case of $uH = h$, where h is an arbitrary constant. Then, (19) reads as

$$(2h - k)(2h - k + 1)(3h - k + 2) = 0,$$

which has three solutions

$$h = \frac{k}{2}, \ \frac{k-1}{2}, \ \frac{k-2}{3}.$$

Thus, Equation (19) has three particular solutions listed as

$$H(u) = \frac{k}{2u}, \ \frac{k-1}{2u}, \ \frac{k-2}{3u}.$$

Substituting $D(u) = u^k$ and $H(u) = \frac{k}{2u}$ into the second one of (16), we deduce that $G(r,u)$ satisfies

$$2G_{uu} + \frac{k}{u}G_u - \frac{k}{u^2}G + \frac{k(n-1)}{2ru^2} = 0.$$

The corresponding solutions of this linear differential equation is easily derived, which is listed as follows:

(i) For $k \neq -2,\ 0,$

$$G(r,u) = g_1(r)u^{-\frac{k}{2}} + g_2(r)u + \frac{n-1}{2r}.$$

(ii) For $k = -2,$

$$G(r,u) = g_1(r)u + g_2(r)u\ln u + \frac{n-1}{2r}.$$

(iii) For $k = 0,$

$$G(r,u) = g_1(r)u + g_2(r).$$

For case (i), the third one of (16) is simplified as

$$\begin{aligned}&2F_{uu} + \frac{3k}{u}F_u + \frac{k(k-2)}{u^2}F + ku^{-k-1}g_1^2(r) - 2(2k+5)ug_2^2(r)\\&-(3k+10)u^{-\frac{k}{2}}g_1(r)g_2(r) + 2(k+2)g_2'(r) - \frac{2(k+2)(n-1)}{r}g_2(r) = 0.\end{aligned}$$

The form of $F(r,u)$ can be determined by solving this linear differential equation, which includes the undetermined coefficients $f_1(r)$ and $f_2(r)$. Similarly, the fourth one of (16) will determine the form of $Q(r,u)$ which includes the unknown coefficients $q_1(r)$ and $q_2(r)$. The last two ones of the system (16) yield that the undetermined parts $f_1(r)$, $f_2(r)$, $g_1(r)$, $g_2(r)$, $q_1(r)$ and $q_2(r)$ satisfy a system of nonlinear coupled ODEs. It is still very difficult to manipulate this nonlinear coupled system. However, we can solve the system explicitly for several special situations. The computations involved here are quite lengthy and it is impossible to solve the nonlinear system by hand. The software Maple is used to help us complete the tedious computations. The detailed procedure is omitted and we just present the corresponding results in Table 1. Similarly, we can identify the admissible functions $D(u)$, $Q(r,u)$ and the corresponding CLBS (1) of Equation (3) for other cases. The obtained results are also listed in Table 1 and so does for the results for $m = -1$ and $m = -2$. It is noteworthy that some solutions of (16) lead to the vanishing of all the coefficients of (15). Since the vanishing of all the coefficients of (15) yields the over-determined system for general m, which will be discussed in Section 4, we omit these results here. Thus, we only consider the results that ensure that there exists at least one non-zero coefficient in the polynomial (15). For case 1 of Table 1, it is noted that we can set $a = 0$ due to the transformation $u \to u + (2n+3)at$.

For general m, it follows from Equation (15) that the coefficient functions $H(u)$, $G(r,u)$, $F(r,u)$ and $Q(r,u)$ satisfy the following over-determined system:

$$
\begin{aligned}
&D''' - (3m+1)HD'' - 3mH'D' + m(3m+2)H^2D' - mDH'' \\
&- m^2(m+1)DH^3 + m(3m+1)DHH' = 0, \\
&\frac{n-1}{r}[D'' - (2m-1)HD' + m(m-1)DH^2 - (m-1)DH'] \\
&- (3m+2)GD'' + m[3(2m+1)HD'G - m(3m+1)DGH^2 \\
&+ (3m+1)DGH' + (3m-1)DHG_u - 3D'G_u - DG_{uu}] = 0, \\
&\frac{n-1}{r}[(m-1)DG_u + 2mGD' - 2m(m-1)DHG] \\
&+ \frac{2(n-1)}{r^2}[D' - (m-1)DH] + (3m+1)D'G_r + 3(m+1)FD'' \\
&- m[(3m+1)G^2D' + (3m-1)DHG_r + (3m-1)DGG_u \\
&- m(3m-1)DHG^2 + 2(3m+1)HFD' + (3m+1)DFH' \\
&+ 3(m-1)DHF_u - (3m^2 - m + 2)DFH^2 - 3F_uD' - DF_{uu} \\
&- 2DG_{ru}] = 0, \\
&\frac{n-1}{r}[mDG_r - m(m-1)DG^2 - 2m(m-1)DHF + (m-1)DF_u \\
&+ (2m+1)FD'] - \frac{n-1}{r^2}D[(2m-1)G + \frac{2}{r}] + (3m+1)D'F_r \\
&+ m[2DF_{ru} + DG_{rr} - (3m-1)DGG_r + m(m-1)DG^3 \\
&- (3m-1)DFG_u - (6m+1)D'GF + 2(3m^2 - 3m + 2)DFGH \\
&- 3(m-1)DGF_u - 3(m-1)DHF_r] = 0, \\
&\frac{n-1}{r}D[F_r - 2(m-1)FG - \frac{2}{r}F] - 3(m-1)DGF_r - (3m-1)DFG_r \\
&- 3(m-1)DFF_u + DF_{rr} + (3m^2 - 5m + 4)DF^2H - 3mF^2D' \\
&+ (m-1)(3m-2)DG^2F = 0, \\
&DF[(3m-4)FG - 3F_r - \frac{n-1}{r}F] = 0, \\
&H'Q + HQ_u + Q_{uu} = 0, \\
&2Q_{ru} + 2HQ_r + QG_u = 0, \\
&Q_{rr} + QF_u - Q_uF + GQ_r = 0, \\
&DF^3 = 0.
\end{aligned}
\tag{20}
$$

Table 1. Admissible functions $D(u),\ Q(r,u)$ and CLBS (1) of Equation (3) for $m = 2,\ -1,\ -2$.

No.	m	n	$D(u)$	$Q(r,u)$	$H(u)$	$G(r,u)$	$F(r,u)$
1	2		1	$2(n+3)a$	$-\frac{1}{2u}$	$-\frac{1}{2r}$	$\frac{ar}{u}$
2	2	3	$\frac{1}{u^3}$	$\left(-\frac{2}{9}bcr^3 + \frac{2}{3}ar^{\frac{3}{2}} + d\right) u^2 + bu - 6c$	$-\frac{2}{u}$	$-\frac{1}{2r}$	cru^2
3	2	3	$\frac{1}{u^2}$	au	$-\frac{3}{2u}$	$-\frac{2}{r}$	$\frac{(b-18)u}{r^2}$
4	2		$\frac{1}{u}$	$\left[ar^{\frac{3}{2}} - \frac{(n-6)(2n-3)^2}{27r^3} + b\right] u$	$-\frac{1}{u}$	$-\frac{1}{2r}$	$-\frac{(3-2n)u}{2r^2}$
5	2	6	$\frac{1}{u}$	0	$-\frac{1}{u}$	$\frac{1}{2r}$	$-\frac{9u-2a}{2r^2}$
6	-1		u^k	$au + (b + c\ln r)u^{\frac{k}{2}} - \frac{k}{2}u^{k-1}$	$-\frac{k}{2u}$	$\frac{1}{r}$	0
7	-1		u^2	$(a + bu^{\frac{a}{n}})u\ln r + (c + du^{\frac{a}{n}})u$	$-\frac{n+a}{nu}$	$\frac{1}{r}$	0
8	-1	2	u^2	$[-2(h+1)\ln r + a]\, u + (b + c\ln r)u^{-h}$	$\frac{h}{u}$	$\frac{1}{r}$	0
9	-2		u^k	$[b\sqrt{r} - 2(k-3)(2kn + k - 6n - 4)r + c]u^{k-2} + au$	$\frac{2-k}{u}$	$\frac{1}{2r}$	0
10	-2		u^3	$\left[\frac{nr}{(2n+1)^2} + b\right] u + (b + c\sqrt{r})u^{\frac{4n+3}{2(2n+1)}}$	$-\frac{4n+3}{2(2n+1)u}$	$\frac{1}{2r}$	0
11	-2		$u^{\frac{2(9n+5)}{3(2n+1)}}$	$\left[br^{\frac{1}{3}} + \frac{(3n+1)r}{3(2n+1)^2} + c\right] u^{\frac{2(3n+2)}{3(2n+1)}} + au$	$-\frac{2(3n+2)}{3(2n+1)u}$	$\frac{2}{3r}$	0

It is noted that hereafter $a,\ b,\ c$ and d in the above table are arbitrary constants.
The last one of (20) implies $F = 0$. It is equivalent to the system

$$
\begin{aligned}
&(2m+1)DD'A' - (3m+1)DAA' - mD^2A'' + mADD'' - (2m+1)AD'^2 \\
&+ (3m+2)A^2D' - (m+1)A^3 = 0, \\[2ex]
&\frac{2(m+1)(n-1)}{r}D(DA' - AD' + A^2) - (3m+1)DBA' + (6m-1)ABD' \\
&- (3m-1)ADB_u - 2(m-1)BD'^2 + (2m-1)DD'B_u - mD^2B_{uu} \\
&+ (m-1)BDD'' - (3m+1)A^2B = 0, \\[2ex]
&-\frac{(m+1)n(n-1)}{r^2}AD^2 - \frac{2(n-1)}{r}D[(m-1)BD' - mDB_u - 2mAB] \\
&+ 3(m-1)B^2D' - (3m-1)DBB_u - (3m-1)ADB_r + 2(m-1)DD'B_r \\
&- 2mD^2B_{ru} - (3m-1)AB^2 = 0, \\[2ex]
&\frac{(n-1)(mn-n+1)}{r^2}D^2B - \frac{n-1}{r}D[2(m-1)B^2 + (2m-1)DB_r] \\
&+ (m-1)B^3 + (3m-1)DBB_r + mD^2B_{rr} = 0, \\[2ex]
&-DQA' + AQD' - ADQ_u + DQD'' - QD'^2 + DD'Q_u + mD^2Q_{uu} = 0, \\[2ex]
&BQD' - DQB_u + 2DD'Q_r - 2ADQ_r + 2mD^2Q_{ru} = 0, \\[2ex]
&\frac{n-1}{r}D^2Q_r + DCQ_u - BDQ_r - DQC_u + CQD' + mD^2Q_{rr} = 0,
\end{aligned}
\tag{21}
$$

where $A(u) = D' - mDH,\ B(r,u) = \frac{n-1}{r} - mDG$. It turns out that Equation (3) admits the H–J SI (2) with $C(r,u) = 0$ and $E(r,u) = Q$.

On substitution of $D = u^k$ into the first one of the system (20), we deduce that H satisfies the second-order ODE

$$
\begin{aligned}
&mu^3H'' + m[3ku^2 - (3m+1)u^3H]H' + m^2(m+1)(uH)^3 \\
&-m(3m+2)k(uH)^2 + (3m+1)k(k-1)uH - k(k-1)(k-2) = 0.
\end{aligned} \tag{22}
$$

It seems impossible to obtain the general solution of Equation (22). However, it is ready to write down three special solutions, which are listed as

$$
H(u) = \frac{k}{mu}, \ H(u) = \frac{k-1}{mu}, \ H(u) = \frac{k-2}{(m+1)u}. \tag{23}
$$

Substituting $D(u)$, $H(u)$ and $F(r,u)$ into (20), we can obtain the explicit forms of $G(r,u)$ and $Q(r,u)$. The results are summarized in Table 2.

Table 2. Admissible Functions $D(u)$, $Q(r,u)$ and CLBS (1) of Equation (3) for general m with $F(r,u) = 0$.

No.	$D(u)$	$Q(r,u)$	$H(u)$	$G(r,u)$
1	u^k	$au + \left(b + cr^{\frac{m+1}{m}}\right) u^{\frac{1-k}{m}}$	$\frac{k-1}{mu}$	$-\frac{1}{mr}$
		$k \neq -m+1$		
2	u^{-m+1}	$\left(a\ln u + b + cr^{\frac{m+1}{m}}\right) u$	$-\frac{1}{u}$	$-\frac{1}{mr}$
3	$u^{-\frac{(m-1)^2(n+2)+5m-3}{mn+m-n+1}}$	$\left(b + cr^{\frac{m+1}{m-1}}\right) u^{\frac{mn+2m-n+2}{mn+m-n+1}} + au$	$-\frac{mn+2m-n+2}{(mn+m-n+1)u}$	$-\frac{2}{(m-1)r}$
4	$u^{-\frac{m^2n-mn+2m^2+2m}{mn+m+1}}$	$\left(b + cr^{\frac{m+1}{m}}\right) u^{\frac{mn+2m+2}{mn+m+1}} + au$	$-\frac{mn+2m+2}{(mn+m+1)u}$	$-\frac{1}{mr}$

Due to the relation (17), we can derive the H–J SI (2) of Equation (3) from the results listed in Tables 1 and 2.

4. Symmetry Reductions of Equation (3)

Symmetry reductions of Equation (3) can be derived by using the compatibility of the invariant surface condition $\eta = 0$ for the admitted CLBS (1) and the governing Equation (3). One first solves the ODE $\eta = 0$ to obtain u as a function of r with r-independent integration constants, and then substitutes this solution into Equation (3) to determine the time evolution of these constants. We just present several examples to illustrate the reduction procedure.

Example 1. *Equation*

$$
u_t = \frac{1}{r^2}\left(r^2 \frac{1}{u^2} u_r^2\right)_r + au
$$

admits the second-order CLBS

$$
\eta = u_{rr} - \frac{3}{2u}u_r^2 - \frac{2}{r}u_r + \frac{(b-18)u}{r^2}
$$

and the first-order H–J SI

$$
J = u_t - \frac{1}{u^3}u_r^3 - \frac{6}{ru^2}u_r^2 + \frac{2(b+8)}{r^2 u}u_r - au.
$$

The compatibility of $\eta = 0$ and the governing equation yields the following results:

- For $b > \frac{27}{2}$,

$$u(r,t) = \frac{4(2b-27)}{\left[\alpha(t)\sinh(d_1 \ln r) + \beta(t)\cosh(d_1 \ln r)\right]^2 r^3},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned} \alpha' &= \frac{3}{8}\alpha^3 + \frac{1}{4}d_1\alpha^2\beta - \frac{3}{8}\alpha\beta^2 - \frac{1}{4}d_1\beta^3 - \frac{1}{2}a\alpha, \\ \beta' &= -\frac{3}{8}\beta^3 - \frac{1}{4}d_1\alpha\beta^2 + \frac{3}{8}\alpha^2\beta + \frac{1}{4}d_1\alpha^3 - \frac{1}{2}a\beta. \end{aligned}$$

- For $b = \frac{27}{2}$,

$$u(r,t) = \frac{4}{\left[\alpha(t) + \beta(t)\ln r\right]^2 r^3},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned} \alpha' &= \frac{3}{2}\alpha\beta^2 - \frac{1}{2}a\alpha + \beta^3, \\ \beta' &= \frac{3}{2}\beta^3 - \frac{1}{2}a\beta. \end{aligned}$$

The solutions of this system of ODEs with $a = 0$ are listed below:

$$\begin{aligned} \alpha(t) &= \frac{c_2 \mp \ln(c_1 - 3t)}{3\sqrt{c_1 - 3t}}, \\ \beta(t) &= \pm\frac{1}{\sqrt{c_1 - 3t}}. \end{aligned}$$

- For $b < \frac{27}{2}$,

$$u(r,t) = \frac{4(2b-27)}{\left[\alpha(t)\sin(d_2 \ln r) + \beta(t)\cos(d_2 \ln r)\right]^2 r^3},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned} \alpha' &= -\frac{3}{8}\alpha^3 + \frac{1}{4}d_2\alpha^2\beta - \frac{3}{8}\alpha\beta^2 + \frac{1}{4}d_2\beta^3 - \frac{1}{2}a\alpha, \\ \beta' &= -\frac{3}{8}\beta^3 - \frac{1}{4}d_2\alpha\beta^2 - \frac{3}{8}\alpha^2\beta - \frac{1}{4}d_2\alpha^3 - \frac{1}{2}a\beta. \end{aligned}$$

It is noted that $d_1 = \frac{\sqrt{2b-27}}{2}$ and $d_2 = \frac{\sqrt{27-2b}}{2}$.

Example 2. *Equation*

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u^2u_r^{-1})_r + (a + bu^{\frac{a}{n}})u\ln r + (c + du^{\frac{a}{n}})u$$

admits the second-order CLBS

$$\eta = u_{rr} - \frac{n+a}{nu}u_r^2 + \frac{1}{r}u_r$$

and the first-order H–J SI

$$J = u_t - \frac{nu^2}{r}u_r^{-1} - \left[\left(a + bu^{\frac{a}{n}}\right) u \ln r + \left(c + \frac{n-a}{n} + du^{\frac{a}{n}}\right) u\right].$$

The compatibility of $\eta = 0$ and the governing equation yields the following results:

- For $a \neq 0$,

$$u(r,t) = \left[-\frac{a}{n}\left(\alpha(t)\ln r + \beta(t)\right)\right]^{-\frac{n}{a}},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = \frac{[a-(c+1)n]\,a}{n^2}\alpha + \frac{1}{n}a^2\beta + b,$$
$$\beta' = \frac{[a-(c+1)n]\,a}{n^2}\beta + \frac{1}{n}a^2\alpha^{-1}\beta^2 + d.$$

The solutions blow up along the curves $r = \exp\left[-\beta(t)/\alpha(t)\right]$ when $a > 0$ and extinguish along the curves when $a < 0$.

- For $a = 0$,

$$u(r,t) = r^{\alpha(t)}\beta(t),$$

where $\alpha(t)$ and $\beta(t)$ are given as

(i) For $b \neq 0$ and $b \neq n$,

$$\alpha(t) = bt + c_1,\ \beta(t) = c_2(bt+c_1)^{\frac{n}{b}} - \frac{c+d+1}{b-n}(bt+c_1).$$

(ii) For $b = 0$,

$$\alpha(t) = c_1,\ \beta(t) = c_2\exp\left(\frac{n}{c_1}t\right) - \frac{(c+d+1)c_1}{n}.$$

(iii) For $b = n$,

$$\alpha(t) = nt + c_1,\ \beta(t) = \left[c_2 + \frac{c+d+1}{n}\ln(nt+c_1)\right](nt+c_1).$$

Example 3. *Equation*

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u^k u_r^{-2})_r + au$$
$$+ [b\sqrt{r} - 2(k-3)(2kn+k-6n-4)r + c]u^{k-2}$$

admits the second-order CLBS

$$\eta = u_{rr} + \frac{2-k}{u}u_r^2 + \frac{1}{2r}u_r$$

and the first-order H–J SI

$$\begin{aligned}J &= u_t - (4-k)u^{k-1}u_r^{-1} - au - \frac{n}{r}u^k u_r^{-2}\\ &-[b\sqrt{r} - 2(k-3)(2kn+k-6n-4)r + c]u^{k-2}.\end{aligned}$$

The compatibility of $\eta = 0$ and the governing equation yields the following results.

- For $k \neq 3$,

$$u(r,t) = \left[(3-k)\Big(2\alpha(t)\sqrt{r} + \beta(t)\Big)\right]^{\frac{1}{3-k}},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned}\alpha' &= \frac{1}{2}(k-3)(4nk+k-12n-4)\alpha^{-1}\beta + (3-k)a\alpha + \frac{b}{2},\\ \beta' &= n(k-3)^2\alpha^{-2}\beta^2 + (3-k)a\beta + c.\end{aligned}$$

The solutions blow up along the curves $r = \left[-\beta(t)/\Big(2\alpha(t)\Big)\right]^2$ when $k > 3$ and extinguish along the curves when $k < 3$.

- For $k = 3$,

$$u(r,t) = \exp\,[\alpha(t)\sqrt{r} + \beta(t)],$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = 2\alpha^{-1} + b,\ \beta' = 4n\alpha^{-2} + a + c.$$

Example 4. *Equation*

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u^k u_r^m)_r + au + \left(b + cr^{\frac{m+1}{m}}\right)u^{\frac{1-k}{m}},\ k \neq -m+1$$

admits the second-order CLBS

$$\eta = u_{rr} + \frac{k-1}{mu}u_r^2 - \frac{1}{mr}u_r$$

and the first-order H–J SI

$$J = u_t - u^{k-1}u_r^{m+1} - \frac{n}{r}u^k u_r^m - au - \left(b + cr^{\frac{m+1}{m}}\right)u^{\frac{1-k}{m}}.$$

The compatibility of $\eta = 0$ and the governing equation gives

$$u(r,t) = \left[(k+m-1)\left(\frac{\alpha(t)}{m+1}r^{\frac{m+1}{m}} + \frac{\beta(t)}{m}\right)\right]^{\frac{m}{m+k-1}},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = \frac{(m+k-1)n+m+1}{m}\alpha^{m+1} + \frac{(m+k-1)a}{m}\alpha + \frac{(m+1)c}{m},$$
$$\beta' = \frac{(m+k-1)n}{m}\alpha^{m}\beta + \frac{(m+k-1)a}{m}\beta + b.$$

Notably, exact solutions of this form were found by Cherniha et al. [32] in the special case $n = 2$ and $k = a = b = c = 0$, which corresponds to the turbulent gas flows. The solutions generalize the instantaneous source solution of the porous medium equation. It is noted that the solutions blow up along the curves $r = \left[-\Big((m+1)\beta(t)\Big)/\Big(m\alpha(t)\Big)\right]_+^{\frac{m}{m+1}}$ when $m(m+k-1) < 0$ and extinguish along the curves when $m(m+k-1) > 0$, namely, the interfaces of the solutions are the curves $r = \left[-\Big((m+1)\beta(t)\Big)/\Big(m\alpha(t)\Big)\right]_+^{\frac{m}{m+1}}$.

Example 5. *Equation*

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u^{-m+1}u_r^m)_r + \left(a\ln u + b + cr^{\frac{m+1}{m}}\right)u$$

admits the second-order CLBS

$$\eta = u_{rr} - \frac{1}{u}u_r^2 - \frac{1}{mr}u_r$$

and the first-order H–J SI

$$J = u_t - u^{-m}u_r^{m+1} - \frac{n}{r}u^{-m+1}u_r^m - \left(a\ln u + b + cr^{\frac{m+1}{m}}\right)u.$$

The compatibility of $\eta = 0$ and the governing equation gives

$$u(r,t) = \exp\left[\frac{m}{m+1}\alpha(t)r^{\frac{m+1}{m}} + \beta(t)\right],$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = \frac{m+1}{m}\alpha^{m+1} + a\alpha + \frac{(m+1)c}{m},$$
$$\beta' = n\alpha^m\beta + a\beta + b.$$

The solutions are also generalizations of the instantaneous source solution of the porous medium equation.

Example 6. *Equation*

$$u_t = \frac{1}{r^{n-1}}\left[r^{n-1}u^{-\frac{(m-1)^2(n+2)+5m-3}{mn+m-n+1}}u_r^m\right]_r + \left(b + cr^{\frac{m+1}{m-1}}\right)u^{\frac{mn+2m-n+2}{mn+m-n+1}} + au$$

admits the second-order CLBS

$$\eta = u_{rr} - \frac{mn+2m-n+2}{(mn+m-n+1)u}u_r^2 - \frac{2}{(m-1)r}u_r$$

and the first-order H–J SI

$$J = u_t - u^{-\frac{m(mn+2m-n+2)}{mn+m-n+1}} u_r^{m+1} - \frac{mn+m-n+1}{(m-1)r} u^{-\frac{(m-1)^2(n+2)+5m-3}{mn+m-n+1}} u_r^m - \left(b + c r^{\frac{m+1}{m-1}}\right) u^{\frac{mn+2m-n+2}{mn+m-n+1}} - au.$$

The compatibility of $\eta = 0$ and the governing equation gives

$$u(r,t) = \left[-\frac{m-1}{mn+m-n+1}\alpha(t) r^{\frac{m+1}{m-1}} - \frac{m+1}{mn+m-n+1}\beta(t)\right]^{-\frac{mn+m-n+1}{m+1}},$$

where $\alpha(t)$ and $\beta(t)$ satisfy the two-dimensional dynamical system

$$\alpha' = -\frac{(m+1)^2}{(m-1)^2}\alpha^m \beta - \frac{(m+1)a}{mn+m-n+1}\alpha + \frac{(m+1)c}{m-1},$$
$$\beta' = -\frac{(m+1)a}{mn+m-n+1}\beta + b.$$

The solutions blow up along the curves $r = \left[-\left((m+1)\beta(t)\right)/\left((m-1)\alpha(t)\right)\right]_+^{\frac{m-1}{m+1}}$ when $(m+1)(mn+m-n+1) > 0$ and extinguish along the curves when $(m+1)(mn+m-n+1) < 0$.

5. Conclusions

We have described the structure of the second-order CLBSs (1) and the first-order H–J SIs (2) for second-order nonlinear reaction–diffusion Equations (3) with gradient-dependent diffusivity. Equation (3) admitting CLBSs (1) and H–J SIs (2) are identified. The corresponding reductions of the resulting equations are established, which in general reduce Equation (3) to two-dimensional dynamical systems due to the compatibility of the invariant surface condition (11) and the governing Equation (3). Those solutions extend the known ones such as instantaneous source solutions of the porous medium equation with absorption term. The phenomena of blow up and extinguish of the obtained solutions are also described. In fact, the invariant surface condition (11) is just the second-order DC of Equation (3). For Equation (3), the first-order H–J SI (2) is a natural extension of the conditional symmetry with the characteristic (14).

For the second-order nonlinear diffusion equation, we found that the second-order CLBS (1) is very effective and may yield some interesting results. Are there other choices of CLBS? Higher-order CLBS may lead to some new results. The discussion about the study of second-order CLBS and first-order H–J SI can be also extended to consider multi-dimensional nonlinear diffusion equations or other types of evolution equations. The CLBSs related to invariant subspace [14–16], sign-invariant [23,28] and separation of variables [24,25] for scalar diffusion equation can be generalized to study classifications and reductions of diffusion system. All of these problems will be involved in our future research.

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