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On the Second-Degree Exterior Derivation of Kahler Modules on $X \otimes Y$

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Abstract: This article presents a new approach to stress the properties of Kahler modules. In this paper, we construct the Kahler modules of second-degree exterior derivations and we constitute an exact sequence of $X \otimes Y$-modules. Particularly, we examine Kahler modules on $X \otimes Y$, and search for the homological size of $\Lambda^2(\Omega_1(X \otimes Y))$.

Keywords: exterior derivation; universal module; Kahler differentials; projective dimension

1. Introduction and Preliminaries

On a differentiable manifold, the exterior derivative extends the concept of the differential of a function to differential forms of higher degree. The exterior derivative was first described in its current form by Elie Cartan in 1899; it allows for a natural, metric-independent generalization of Stokes' theorem, Gauss's theorem, and Green's theorem from vector calculus. If a k-form is thought of as measuring the flux through an infinitesimal k-parallelotope, then its exterior derivative can be thought of as measuring the net flux through the boundary of a (k + 1)-parallelotope.

In order to prove the results regarding algebraic clusters and their coordinate rings, one of the methods is to study the Kahler module of differential operators. This notion of studying the Kahler module may reduce questions about algebras to questions regarding module theory.

Differential forms are ubiquitous in modern mathematical physics, and their relevance for computations has increasingly been realized. Differential forms in mathematical physics have been studied by C. von Westenholz. An example or two will give the flavor of the subject. First, let M be an n-dimensional smooth differentiable manifold that is thought of as the configuration space of a mechanical system with n degrees of freedom. Each point of M has a neighborhood with a local coordinate system $\left(q^1, q^2, \ldots, q^n\right)$. When the system is in motion, we need not only the coordinates q^i of a point of M, but also the momentum vector $\left(p_1, p_{2,\ldots,}p_n\right)$ at q. Thus, we are led to the phase space, or cotangent bundle of M, which is denoted T $*$ M. This space already has an interesting structure: the differential form of degree one with local expression:

$$w = \sum p_i dq^i$$

which is really a global quantity on T $*$ M. Its exterior derivative:

$$\Omega = \mathrm{dw} = \sum dp_i \wedge dq^i$$

is automatically a global quantity on T $*$ M, and an exterior differential form of degree two (skew-symmetric covariant two-tensor) [1].

In this study, we searched the homological features of differential operators and the Kahler modules of second-degree exterior derivations. Let X be a commutative algebra to an algebraically closed field k with characteristic 0. Let $\Omega_n(X)$ and $\delta_n : X \to \Omega_n(X)$ symbolize the Kahler module of nth-degree derivations and standard nth degree k-derivation of X, respectively. The double $(\delta_n, \Omega_n(X))$ has the universal functioning feature that for any X-module N and any high derivation $d : X \to N$ of degree $\leq n$, there is only an X-homomorphism $t : \Omega_n(X) \to N$; so $d = t \cdot \delta_n$. $\Omega_n(X)$ is produced by the set $\{\delta_n(x) : x \in X\}$. Therefore, if X is parochially produced k-algebra, in turn, $\Omega_n(X)$ will be parochially produced X-module. Let X and Y be commutative algebras onto an algebraically closed field k with the characteristic 0. Then, $X \otimes Y$ is a commutative ring with unit by describing:

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j a_j \otimes b_j\right) = \sum_i \sum_j a_i a_j \otimes b_i b_j$$

where $\boldsymbol{a_i}, \boldsymbol{a_j} \in X$ and $\boldsymbol{b_i}, \boldsymbol{b_j} \in \boldsymbol{Y}$. Let U and V be ideals of X and Y, respectively. If $\boldsymbol{X} \to \frac{\boldsymbol{X}}{\boldsymbol{U}}$ and $\boldsymbol{Y} \to \frac{\boldsymbol{Y}}{\boldsymbol{V}}$ are standard homomorphisms of k-algebras, there exists an k-algebra isomorphism:

$$\frac{\boldsymbol{X} \otimes \boldsymbol{Y}}{\boldsymbol{U} \otimes \boldsymbol{Y} \oplus \boldsymbol{X} \otimes \boldsymbol{V}} \simeq \frac{\boldsymbol{X}}{\boldsymbol{U}} \otimes \frac{\boldsymbol{Y}}{\boldsymbol{V}}$$

Northcott [2].

2. Background Material

All of the rings considered in this paper are commutative with identity, and all of the fields are of characteristic zero, unless otherwise stated. Let's say that X is a commutative algebra onto an algebraically closed field k of characteristic 0, and M is an X-module. For any non-negative integer n, by the universal nth degree differential operator on M, we denote a pair $(\boldsymbol{d_n}, \boldsymbol{J_n}(\boldsymbol{M}))$ that is composed of an R-module $\boldsymbol{J_n}(\boldsymbol{M})$ and a differential operator $\boldsymbol{d_n} : \boldsymbol{M} \to J_n(\boldsymbol{M})$ such that for any nth degree differential operator D from M to an randomly X-module N, there is only an X-module homomorphism α from $\boldsymbol{J_n}(\boldsymbol{M})$ to N, which satisfied $\boldsymbol{\alpha d}_n = \boldsymbol{D}$. The module $\boldsymbol{J_n}(\boldsymbol{M})$ is named the Kahler module of the nth degree differential operators on M [3]. For the case M = X, $\boldsymbol{J_n}(\boldsymbol{X})$ is produced by the cluster $\{\boldsymbol{d_n}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}\}$. Therefore, if X is parochially produced k-algebra, then $\boldsymbol{J_n}(\boldsymbol{X})$ will be a parochially produced X-module.

Let M and N be X-modules. A bilinear function $\gamma : \mathbf{MxM} \to \mathbf{N}$ is named and alternating if $\gamma((\mathbf{m}, \mathbf{m})) = \mathbf{0}$ for any $\boldsymbol{m} \in \boldsymbol{M}$. Let $\boldsymbol{M} \otimes \boldsymbol{M}$ be the tensor multiplication of M with itself, and let G be the submodule of $\boldsymbol{M} \otimes \boldsymbol{M}$ that is produced by the member of the form $\boldsymbol{m} \otimes \boldsymbol{m}$ where $\boldsymbol{m} \in \boldsymbol{M}$. Bear in mind the next factor module:

$$\wedge^2(\boldsymbol{M}) = \frac{\boldsymbol{M} \otimes \boldsymbol{M}}{\boldsymbol{G}}$$

The module $\wedge^2(\boldsymbol{M})$ is called to be the second exterior power of M [4].

Lemma 1. *Let T be an X-module and $\gamma : \mathbf{MxM} \to \mathbf{T}$ be a linear alternating map. Then, there exists an R-module homomorphism $\boldsymbol{f} : \wedge^2(\boldsymbol{M}) \to \boldsymbol{T}$; so, the subsequent diagram:*

$$\begin{tikzcd} \mathbf{MxM} \arrow[rr, "\gamma"] \arrow[dr, "\wedge"'] & & \boldsymbol{T} \\ & \wedge^2(\boldsymbol{M}) \arrow[ur, "\boldsymbol{f}"'] & \end{tikzcd}$$

commutes. Erdogan [4].

Proposition 1. *Let M be an X-module, T be a submodule of M, and $\boldsymbol{L_T}$ be a submodule of $\wedge^2(\boldsymbol{M})$ produced by the cluster $\{\boldsymbol{\sigma} \wedge \boldsymbol{\tau} : \boldsymbol{\sigma} \in \boldsymbol{M}, \boldsymbol{\tau} \in \boldsymbol{T}\}$. Then, there is an X-module isomorphism:*

$$\frac{\wedge^2(\boldsymbol{M})}{\boldsymbol{L_T}} \simeq \wedge^2\left(\frac{\boldsymbol{M}}{\boldsymbol{T}}\right)$$

Erdogan [4].

Proposition 2. *Assume that* $J_a(\Omega_b(X))$ *is the Kahler module of differential operators of degree a on* $\Omega_b(X)$ *with the universal differential* Δ_a. *Then, there exists only the X-module homomorphism:*

$$\theta : \Omega_{a+b}(X) \to J_a(\Omega_b(X)) \sum_i x_i d_{a+b}(y_i) \to \Delta_a\left(\sum_i x_i d_{a+b}(y_i)\right)$$

So, the next diagram:

$$\begin{array}{ccc} X & \xrightarrow{d_b} & \Omega_b(X) \\ \downarrow d_{a+b} & & \downarrow \Delta_a \\ \Omega_{a+b}(X) & \underset{\theta}{\to} & J_a(\Omega_b(X)) \end{array}$$

commutes. Erdogan [4].

Remark 1. *Let X be a k-algebra and* $\theta : \Omega_{a+b}(X) \to J_a(\Omega_b(X))$ *be an X-module homomorphism, as given in Proposition 2. In the circumstances, we have an exact sequence of X-modules as follows:*

$$0 \to ker\theta \xrightarrow{i} \Omega_{a+b}(X) \xrightarrow{\theta} J_a(\Omega_b(X)) \xrightarrow{p} coker\theta \to 0$$

where i is the inclusion map, and p is the natural surjection:

$$J_a(\Omega_b(X)) \to \frac{J_a(\Omega_b(X))}{Im\theta}$$

Erdogan [4].

Lemma 2. *Let X be a commutative k-algebra. We presume that*$\Omega_1(X)$ *is the Kahler module of derivations of X with the universal derivation* $\Delta : X \to \Omega_1(X)$. *In the present case, the function:*

$$\begin{gathered} D : \Omega_1(X) \to \wedge^2(\Omega_1(X)) \\ D\left(\sum_i a_i \Delta b_i\right) = \sum_i \Delta a_i \wedge \Delta b_i \end{gathered}$$

is a differential operator of degree 1 over $\Omega_1(X)$ *where* $a_i, b_i \in X$ *[5].*

Proposition 3. *There is a split exact sequence of X-modules:*

$$0 \to \Omega_2 \to J_1(\Omega_1) \to \wedge^2(\Omega_1) \to 0$$

Hart [6].

Under the favor of this test, the conditions $ker\theta = 0$ and $coker\theta = 0$ are found in the following result.

Theorem 1. *Take into consideration the affine k-algebras X and Y. Let U be an ideal of X, and* $\delta_n : X \to \Omega_n(X)$ *be the standard nth degree k-derivation of X. Imagine that P is a submodule of* $\Omega_n(X)$ *that is produced by all of the members of the style* $\{\delta_n(x) : x \in U\}$. *In that case, the sequence:*

$$0 \to \frac{P + U\Omega_n(X)}{U\Omega_n(X)} \to \frac{\Omega_n(X)}{U\Omega_n(X)} \to \Omega_n\left(\frac{X}{U}\right) \to 0$$

is an exact sequence of $\frac{X}{U}$-modules [7].

Proposition 4. *Let U and V be ideals of X and Y, in return. At that rate, there is an exact sequence:*

$$0 \to ker\theta \to \Omega_n(X \otimes Y) \xrightarrow{\theta} \Omega_n\left(\frac{X}{U} \otimes \frac{Y}{V}\right) \to 0$$

of $X \otimes Y$*-modules [8].*

Theorem 2. *Take into consideration the affine k-algebras X and Y. Let U and V be ideals of X and Y, respectively, and say that* $K = U \otimes Y \oplus X \otimes V$ *and P is a submodule of* $\Omega_n(X \otimes Y)$ *that is produced by all of the members of the style* $\{\delta_n(x) : x \in K\}$*, where* $\delta_n : X \otimes Y \to \Omega_n(X \otimes Y)$ *is the standard nth degree k-derivation of* $X \otimes Y$*. At that, the sequence:*

$$0 \to \frac{P + K\Omega_n(X \otimes Y)}{K\Omega_n(X \otimes Y)} \to \frac{\Omega_n(X \otimes Y)}{K\Omega_n(X \otimes Y)} \to \Omega_n\left(\frac{X \otimes Y}{K}\right) \to 0$$

is an exact sequence of $X \otimes Y$-modules [8].

Proposition 5. *Assume that* $X = k[x_1, x_2, \ldots, x_s]$ *and* $Y = k[y_1, y_2, \ldots, y_t]$ *are polynomial algebras, and let U and V be the ideals produced by members* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *and members* $\beta_1, \beta_2, \ldots, \beta_l$ *of X and Y, respectively, and* $K = U \otimes Y \oplus X \otimes V$*. Hence, K is produced by cluster:*

$$\{\alpha_i \otimes 1, 1 \otimes \beta_j : \alpha_i \in U, \beta_j \in V\}$$

Olgun and Erdogan [8].

Corollary 1. *Let* $\delta_n : X \otimes Y \to \Omega_n(X \otimes Y)$ *and* $d_n : \frac{X \otimes Y}{K} \to \Omega_n\left(\frac{X \otimes Y}{K}\right)$ *be the nth degree Kahler derivation operators. Therefore,* $\Omega_n\left(\frac{X \otimes Y}{K}\right)$ *is produced by:*

$$\left\{d_n\left(x^\alpha + y^\beta + K\right) : 0 \le |\alpha| + |\beta| \le n\right\}$$

Olgun and Erdogan [8].

Theorem 3. *Conceive affine k-algebras X and Y. Let U and V be ideals of X and Y, respectively., and say that* $K = U \otimes Y \oplus X \otimes V$ *. Given that P is a submodule of* $\Omega_n(X \otimes Y)$ *produced by all of the members of style* $\{\delta_n(x) : x \in K\}$*, where* $\delta_n : X \otimes Y \to \Omega_n(X \otimes Y)$ *is the standard nth degree k-derivation of* $X \otimes Y$*. Hence:*

(i)

$$hd\left(\frac{P + K\Omega_n(X \otimes Y)}{K\Omega_n(X \otimes Y)}\right) < \infty \Delta hd\left(\Omega_n\left(\frac{X \otimes Y}{K}\right)\right) < \infty$$

(ii)

$$hd\left(\frac{P + K\Omega_n(X \otimes Y)}{K\Omega_n(X \otimes Y)}\right) = \infty \Delta hd\left(\Omega_n\left(\frac{X \otimes Y}{K}\right)\right) = \infty$$

Olgun and Erdogan [8].

3. Main Results

Throughout this section, X and Y affine k-algebras. Let U and V be the ideals of X and Y, respectively, and let's say that $K = U \otimes Y \oplus X \otimes V$. In this section, we will be studying the second-degree exterior derivation of Kahler modules $\wedge^2(\Omega_1(X \otimes Y))$ on $X \otimes Y$. Let us firstly mention that there exists an exact sequence:

$$0 \to T \to Q \to \Omega_1\left(\frac{X \otimes Y}{K}\right) \to 0$$

of $X \otimes Y$-modules where Q is a free $X \otimes Y$-module, and T is the submodule of Q [8].

Definition 1. *Let $X \otimes Y$ be a commutative k-algebra. Let's suppose that $\Omega_1(X \otimes Y)$ is the Kahler module of derivations of $X \otimes Y$ with the universal derivation $d_1 : X \otimes Y \to \Omega_1(X \otimes Y)$. So, the function:*

$$\widetilde{D} : \Omega_1(X \otimes Y) \to \wedge^2(\Omega_1(X \otimes Y))$$
$$\widetilde{D}\left(\sum_i (x_i \otimes y_i)\Delta(z_i \otimes t_i)\right) = \sum_i \Delta(x_i \otimes y_i) \wedge \sum_i \Delta(z_i \otimes t_i)$$

is a differential operators of degree 1 over $\Omega_1(X \otimes Y)$.

Let $X \otimes Y$ be a commutative k-algebra and $\Omega_2(X \otimes Y)$ be the Kahler module of second-degree derivations of $X \otimes Y$. Also, let $J_1(\Omega_1(X \otimes Y))$ be the Kahler module of differential operators of a degree less than or equal to 1 on $\Omega_1(X \otimes Y)$ with a universal differential operator $\Delta_1 : \Omega_1(X \otimes Y) \to J_1(\Omega_1(X \otimes Y))$. There exists $\lambda : \Omega_2(X \otimes Y) \to J_1(\Omega_1(X \otimes Y))$; so, the next diagram:

$$\begin{array}{ccc} X \otimes Y & \overset{d_1}{\to} & \Omega_1(X \otimes Y) \\ \downarrow d_2 & & \downarrow \Delta_1 \\ \Omega_2(X \otimes Y) & \underset{\lambda}{\to} & J_1(\Omega_1(X \otimes Y)) \end{array}$$

commutes, and $\lambda d_2 = \Delta_1 d_1$.

Let $\widetilde{D} : \Omega_1(X \otimes Y) \to \wedge^2(\Omega_1(X \otimes Y))$. By the universal feature of $J_1(\Omega_1(X \otimes Y)$, there is a $X \otimes Y$-module homomorphism:

$$\gamma : J_1(\Omega_1(X \otimes Y)) \to \wedge^2(\Omega_1(X \otimes Y))$$

So, the next diagram:

$$\begin{array}{ccccc} \Omega_1(X \otimes Y) & & \overset{\widetilde{D}}{\to} & & \wedge^2(\Omega_1(X \otimes Y)) \\ & \searrow \Delta_1 & & \nearrow \gamma & \\ & & J_1(\Omega_1(X \otimes Y)) & & \end{array}$$

commutes, and $\gamma\Delta_1 = \widetilde{D}$.

Remark 2. *Let $X \otimes Y$ be an affine k-algebra. The function:*

$$v : \wedge^2(Q/T) \to \wedge^2(Q)/L_T$$
$$v\left(\overline{d_1(x_i \otimes y_i)} \wedge \overline{d_1(x_j \otimes y_j)} = \overline{d_1\left(x_i \otimes y_i \wedge x_j \otimes y_j\right)}\right)$$

is an isomorphism of $X \otimes Y$-modules where $1 \leq i \leq r$ and $1 \leq j \leq s$. Here, $\wedge^2(Q)$ is a free $X \otimes Y$-module with bases $\left\{d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j) : 1 \leq i \leq r, 1 \leq j \leq s\right\}$, and L_T is a submodule of $\wedge^2(Q)$.

Theorem 4. *Let $X \otimes Y$ be an affine k-algebra. At that rate:*

$$\Omega_2(X \otimes Y) \overset{\lambda}{\to} J_1(\Omega_1(X \otimes Y)) \overset{\gamma}{\to} \wedge^2(\Omega_1(X \otimes Y)) \to 0$$

is an exact sequence of $X \otimes Y$ modules.

Proof of Theorem 4. It is enough to indicate that the sequence is exact at $J_1(\Omega_1(X \otimes Y))$. Consider $Im\lambda$ is produced by $\lambda(d_2(x_i \otimes y_i)), \lambda\left(d_2\left(x_i y_i \otimes x_j y_j\right)\right)$ for $1 \leq i \leq r, 1 \leq j \leq s$ and that

$$d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j) + d_1(x_j \otimes y_j) \wedge d_1(x_i \otimes y_i) = 0$$

□

Therefore we have:

$$\begin{aligned}\gamma\lambda(d_2(x_i \otimes y_i)) &= \gamma(\Delta_1(d_1(x_i \otimes y_i))) = \widetilde{D}(d_1(x_i \otimes y_i)) = d_1(1) \wedge d_1(x_i \otimes y_i) \\ &= 0\end{aligned}$$

And:

$$\begin{gathered}\gamma\lambda(d_2\left(x_iy_i \otimes x_jy_j\right)) = \gamma\left(\Delta_1\left(d_1\left(x_iy_i \otimes x_jy_j\right)\right)\right) \\ = \gamma\left(\Delta_1\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right)\right) + \Delta_1\left(x_j \otimes y_j d_1(x_i \otimes y_i)\right) \\ = \widetilde{D}\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right) + \widetilde{D}\left(x_j \otimes y_j d_1(x_i \otimes y_i)\right) \\ = d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j) + d_1(x_j \otimes y_j) \wedge d_1(x_i \otimes y_i) \\ = 0\end{gathered}$$

This is to say that $Im\lambda$ is included in the $ker\gamma$. Hence, we have a reduced function:

$$\begin{gathered}p : \frac{J_1(\Omega_1(X \otimes Y))}{Im\lambda} \to \wedge^2(\Omega_1(X \otimes Y)) \\ p\left(\Delta_1\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right)\right) = d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j).\end{gathered}$$

Assume that $\wedge^2(Q)$ and L_T in Remark 2. The cluster:

$$\left\{\overline{\Delta_1\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right)} : 1 \le i \le r,\ 1 \le j \le s\right\}$$

Produces:

$$\frac{J_1(\Omega_1(X \otimes Y))}{Im\lambda}$$

Since $\wedge^2(Q)$ is a free $X \otimes Y$-module with bases:

$$\left\{d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j) : 1 \le i \le r,\ 1 \le j \le s\right\},$$

we can define a function:

$$\begin{gathered}q :\ \wedge^2(Q) \to \frac{J_1(\Omega_1(X \otimes Y))}{Im\lambda} \\ q\left(d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j)\right) = \Delta_1\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right).\end{gathered}$$

Therefore, if $\{f_k\}$ is a producing set for K, we have:

$$\begin{gathered}q(d_1(f_k) \wedge d_1(x_i \otimes y_i)) = q\left(\sum_i \frac{\partial f_k}{\partial(x_i \otimes y_i)} d_1(x_i \otimes y_i) \wedge d_1(x_i \otimes y_i)\right) \\ = \sum_i \frac{\partial f_k}{\partial(x_i \otimes y_i)} \Delta_1(x_i \otimes y_i d_1(x_i \otimes y_i)) \\ = \Delta_1(f_k d_1(x_i \otimes y_i) = 0.\end{gathered}$$

Hence, $q(L_T) = 0$. So, q reduces an $X \otimes Y$-module homomorphism:

$$\begin{gathered}\overline{q} : \frac{\wedge^2(Q)}{L_T} \to J_1(\Omega_1(X \otimes Y)) \\ \overline{q}\left(\overline{d_1(x_i \otimes y_i) \wedge d_1(x_j \otimes y_j)}\right) = \overline{\Delta_1\left(x_i \otimes y_i d_1(x_j \otimes y_j)\right)}.\end{gathered}$$

It is nearly explicit that $\overline{q}p$ and $p\overline{q}$ are the identities, and so, $kerp = \frac{ker\gamma}{Im\lambda} = 0$. Therefore, the sequence is exact.

Proposition 6. *Let $X \otimes Y$ be a local k-algebra of dimension 1. Then, $X \otimes Y$ is a regular ring if* $\wedge^2(\Omega_1(X \otimes Y)) = 0$.

Proof of Proposition 6. Assume that $\boldsymbol{X \otimes Y}$ be a local k-algebra of dimension 1. So, $\boldsymbol{\Omega_1(X \otimes Y)}$ is a free $\boldsymbol{X \otimes Y}$-module of rank 1, and $\wedge^2(\boldsymbol{\Omega_1(X \otimes Y)}) = \mathbf{0}$. On the other hand, suppose that $\wedge^2(\boldsymbol{\Omega_1(X \otimes Y)}) = \mathbf{0}$. Let $\boldsymbol{m_1 \otimes m_2}$ be the maximal ideal of $\boldsymbol{X \otimes Y}$, where $\boldsymbol{m_1}$ is the maximal ideal of X and $\boldsymbol{m_2}$ is the maximal ideal of Y. Then, we have:

$$\wedge^2(\boldsymbol{\Omega_1(X \otimes Y)}) \Big/ (\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)} = \mathbf{0}.$$

□

Since:

$$(\boldsymbol{\Omega_1(X \otimes Y)})/(\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)}$$

is a vectorspace onto the $\boldsymbol{X \otimes Y}/(\boldsymbol{m_1 \otimes m_2})$, it instantly follows that either:

$$(\boldsymbol{\Omega_1(X \otimes Y)})/(\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)} = \mathbf{0}$$

or:

$$\boldsymbol{dim}_{\boldsymbol{X \otimes Y}/(\boldsymbol{m_1 \otimes m_2})}\, (\boldsymbol{\Omega_1(X \otimes Y)})/(\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)} = \mathbf{1}.$$

Let's say:

$$(\boldsymbol{\Omega_1(X \otimes Y)})/(\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)} = \mathbf{0}.$$

Then, $(\boldsymbol{\Omega_1(X \otimes Y)}) = (\boldsymbol{m_1 \otimes m_2})\boldsymbol{\Omega_1(X \otimes Y)}$; so, from Nakayama's lemma, $\boldsymbol{\Omega_1(X \otimes Y)} = \mathbf{0}$, which is a discrepancy. Therefore, we get $\boldsymbol{rank\Omega_1(X \otimes Y)} = \boldsymbol{\mu}(\boldsymbol{\Omega_1(X \otimes Y)})$, and $\boldsymbol{\Omega_1(X \otimes Y)}$ is a free $\boldsymbol{X \otimes Y}$-module of rank 1 just as we want. Here, $\boldsymbol{\mu}(\boldsymbol{\Omega_1(X \otimes Y)})$, which is the number of members of a minimal producing set of $\boldsymbol{X \otimes Y}$.

Theorem 5. *Let $\boldsymbol{X \otimes Y}$ be an affine k-algebra of dimension 1. Let $\boldsymbol{\lambda} : \boldsymbol{\Omega_2(X \otimes Y)} \to \boldsymbol{J_1(\Omega_1(X \otimes Y))}$ be as over. Then, $\boldsymbol{X \otimes Y}$ is a regular k-algebra if $\boldsymbol{\lambda}$ is a surjective $\boldsymbol{X \otimes Y}$-module homomorphism.*

Proof of Theorem 5. We have seen that if $\boldsymbol{X \otimes Y}$ is regular of dimension 1, then $\boldsymbol{\lambda}$ is an isomorphism. On the other hand, assume that $\boldsymbol{\lambda}$ is surjective. Then, by the exact sequence in Proposition 6, we see that $\wedge^2(\boldsymbol{\Omega_1(X \otimes Y)}) = \mathbf{0}$. Let $\boldsymbol{m_1 \otimes m_2}$ be the maximal ideal of $\boldsymbol{X \otimes Y}$. Then:

$$\wedge^2\, (\boldsymbol{\Omega_1(X \otimes Y)})_{\boldsymbol{m_1 \otimes m_2}} = \boldsymbol{\Omega_1(X \otimes Y)} \otimes (\boldsymbol{X \otimes Y})_{\,\boldsymbol{m_1 \otimes m_2}} = \mathbf{0}$$

So, we attain that $(\boldsymbol{X \otimes Y})_{\boldsymbol{m_1 \otimes m_2}}$ is regular just as we want. □

Theorem 6. *Let $\boldsymbol{X \otimes Y}$ be an affine domain of dimension $\boldsymbol{s} \geq \mathbf{1}$. Suppose that $\wedge^2(\boldsymbol{\Omega_1(X \otimes Y)})$ is a projective $\boldsymbol{X \otimes Y}$-module. Then, $\boldsymbol{X \otimes Y}$ is regular.*

Proof of Theorem 6. We need to see that $\boldsymbol{\Omega_1(X \otimes Y)}$ is projective. Due to $\boldsymbol{rank\Omega_1(X \otimes Y)} = \boldsymbol{s}$, it follows that $\boldsymbol{\mu}(\boldsymbol{\Omega_1(X \otimes Y)}) = \boldsymbol{s}$ just as we want. □

Now, we will give an example with regard to our above results.

Example 1. *Let $\boldsymbol{X} = \boldsymbol{k}[\boldsymbol{\phi}, \boldsymbol{\omega}]$ and $\boldsymbol{Y} = \boldsymbol{k}[\boldsymbol{\varphi}, \boldsymbol{\psi}]$. Let $\boldsymbol{U} = \left(\boldsymbol{\omega}^2 - \boldsymbol{\phi}^3\right)$ and $\boldsymbol{V} = \left(\boldsymbol{\varphi}^2 - \boldsymbol{\psi}^3\right)$ be ideals of X and Y, respectively, and let's say that $\boldsymbol{K} = \boldsymbol{U \otimes Y \oplus X \otimes V}$. Let Q be the $\boldsymbol{X \otimes Y}$-module produced by:*

$$\{\boldsymbol{d_1(\phi \otimes 1), d_1(\omega \otimes 1), d_1(1 \otimes \varphi), d_1(1 \otimes \psi)}\}$$

and let T be a submodule of Q produced by:

$$\left\{\boldsymbol{d_1(\eta \otimes 1), d_1(1 \otimes \vartheta)} : \boldsymbol{\eta} = \boldsymbol{\omega}^2 - \boldsymbol{\phi}^3, \boldsymbol{\vartheta} = \boldsymbol{\varphi}^2 - \boldsymbol{\psi}^3\right\}.$$

Now, the rank of $\wedge^2(\Omega_1(X\otimes Y)) = \mathbf{6}$ with bases:

$$\left\{ \begin{array}{c} d_1(\phi\otimes 1)\wedge d_1(\omega\otimes 1), d_1(\phi\otimes 1)\wedge d_1(1\otimes\varphi), d_1(\phi\otimes 1)\wedge d_1(1\otimes\psi), \\ d_1(\omega\otimes 1)\wedge d_1(1\otimes\varphi), d_1(\omega\otimes 1)\wedge d_1(1\otimes\psi), d_1(1\otimes\varphi)\wedge d_1(1\otimes\psi) \end{array} \right\}.$$

The rank of $J_1(\Omega_1(X\otimes Y)) = \mathbf{20}$ with bases:

$$\left\{ \begin{array}{c} \Delta_1(d_1(\phi\otimes 1)), \Delta_1(d_1(\omega\otimes 1)), \Delta_1(d_1(1\otimes\varphi)), \Delta_1(d_1(1\otimes\psi)), \\ \Delta_1((\phi\otimes 1)d_1(\phi\otimes 1)), \Delta_1((\phi\otimes 1)d_1(\omega\otimes 1)), \Delta_1((\phi\otimes 1)d_1(1\otimes\varphi)), \Delta_1((\phi\otimes 1)d_1(1\otimes\psi)), \\ \Delta_1((\omega\otimes 1)d_1(\phi\otimes 1)), \Delta_1((\omega\otimes 1)d_1(\omega\otimes 1)), \Delta_1((\omega\otimes 1)d_1(1\otimes\varphi)), \Delta_1((\omega\otimes 1)d_1(1\otimes\psi)), \\ \Delta_1((1\otimes\varphi)d_1(\phi\otimes 1)), \Delta_1((1\otimes\varphi)d_1(\omega\otimes 1)), \Delta_1((1\otimes\varphi)d_1(1\otimes\varphi)), \Delta_1((1\otimes\varphi)d_1(1\otimes\psi)), \\ \Delta_1((1\otimes\psi)d_1(\phi\otimes 1)), \Delta_1((1\otimes\psi)d_1(\omega\otimes 1)), \Delta_1((1\otimes\psi)d_1(1\otimes\varphi)), \Delta_1((1\otimes\psi)d_1(1\otimes\psi)). \end{array} \right\}$$

And finally, the rank of $\Omega_2(X\otimes Y) = \mathbf{14}$ with bases:

$$\left\{ \begin{array}{c} d_2(\phi\otimes 1), d_2(\omega\otimes 1), d_2(1\otimes\varphi), d_2(1\otimes\psi), \\ d_2(\phi\otimes\varphi), d_2(\phi\otimes\psi), d_2(\omega\otimes\varphi), d_2(\omega\otimes\psi), \\ d_2(\phi^2\otimes 1), d_2(\omega^2\otimes 1), d_2(1\otimes\varphi^2), d_2(1\otimes\psi^2), \\ d_2(\phi\omega\otimes 1), d_2(1\otimes\varphi\psi). \end{array} \right\}$$

So, we have the exact sequence:

$$\Omega_2(X\otimes Y) \xrightarrow{\lambda} J_1(\Omega_1(X\otimes Y)) \xrightarrow{\gamma} \wedge^2(\Omega_1(X\otimes Y)) \to 0$$

of $X\otimes Y$-modules. Therefore, we have also seen that the projective dimension of $\wedge^2(\Omega_1(X\otimes Y))$; that is, $hd\big(\wedge^2(\Omega_1(X\otimes Y))\big) = \infty$.

4. Discussion

Many studies related to the exterior derivation of Kahler modules have been done by researchers. Especially, a large work area related to first-degree exterior derivations of Kahler modules was created. In this paper, we investigated some homological properties of second-exterior derivations of Kahler modules. We believe that the results we found are useful, particularly in future works. Therefore, now whether or not to calculate higher degrees of exterior derivation of Kahler modules, that is the third degree, the fourth degree or the higher degree of exterior derivation, of Kahler modules comes to mind. Furthermore, we think that future research ought to relate to the following questions:

(1) Can we calculate a higher degree of exterior derivation of Kahler modules?
(2) Under which conditions can we calculate higher-degree orders of exterior derivation of Kahler modules?
(3) Which Kahler modules have finite projective dimensions?

5. Conclusions

We know that exterior derivation has an important place not only in mathematical physics, but also in commutative algebra. Study on the structure of the exterior derivation of Kahler modules over $X\otimes Y$ presents interesting aspects for both mathematical physics and commutative algebra. The second exterior derivation of Kahler modules over $X\otimes Y$ has never been studied before in detail. So, in this paper, by using the concept of the first degree of exterior derivation of Kahler modules, we searched the new approach for ways to discover the homological features of the second degree of exterior derivation of Kahler modules. Finally, we investigated some interesting properties of the algebras of Kahler modules.

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