symmetry

MDPI

Article

Symmetry Reduction and Numerical Solution of Von Kármán Swirling Viscous Flow

XiaoMin Wang and SuDao Bilige *

Department of Mathematics, Inner Mongolia University of Technology, Hohhot 010051, China; wxmmath@sina.cn
* Correspondence: inmathematica@126.com

Received: 28 February 2018; Accepted: 8 April 2018; Published: 19 April 2018

Abstract: In this paper, the numerical solutions of von Kármán swirling viscous flow are obtained based on the effective combination of the symmetry method and the Runge-Kutta method. Firstly, the multi-parameter symmetry of von Kármán swirling viscous flow is determined based on the differential characteristic set algorithm. Secondly, we used the symmetry to reduce von Kármán swirling viscous flow to an initial value problem of the original differential equations. Finally, we numerically solve the initial value problem of the original differential equations by using the Runge-Kutta method.

Keywords: von Kármán swirling viscous flow; symmetry; differential characteristic set algorithm; Runge-Kutta method

1. Introduction

The symmetry group method is very important in the analysis of partial differential equations (PDEs), and this method is applied widely in many field [1–3]. The applications of Lie's continuous symmetry groups include such diverse fields as differential geometry, bifurcation theory, mechanics, hydrodynamics, relativity, diffusion and wave phenomena, astrophysics, plasma and so on [3–7]. There have been several studies about the symmetry method, such as symmetry classification [8], potential symmetry [9], approximate symmetry [10], etc. Based on the symmetries of a PDE, many important properties of the equation such as Lie algebras [11,12], conservation laws [13–18], and exact solutions [16–22] can be considered successively. Recently, some researchers focus on the applications of the symmetry method for solving boundary value problems (BVP) of a PDE [2,23–25].

As it is well known, the similarity transformation is used frequently in solving nonlinear PDEs problems. The Lie transformation group of PDEs can yield a more general form of similarity transformation, and these transformations have more significance for mathematics and physics. So the symmetry method has a higher superiority than the similarity transformation in BVP of nonlinear PDEs. At present, combining the symmetry method with other methods to solve BVP of the nonlinear PDEs are the new research subjects. Recently, we have studied this topic based on the differential characteristic set algorithm [26–28].

It is an new research to the application of the Lie symmetry method in the BVP for nonlinear PDEs in fluid mechanics. We will study the symmetry reduction and the numerical solutions of von Kármán swirling viscous flow based on the effective combination of the Lie symmetry method and the Runge-Kutta method. This investigation will widen the application of Lie symmetry. The rest of the paper is organized as follows. In Section 2, we give the formulation of invariance for a BVP of PDEs. In Section 3, we obtain the symmetry of von Kármán swirling viscous flow based on the differential characteristic set algorithm and then we reduce it. In Section 4, we give numerical solutions of von

Kármán swirling viscous flow by applying the Runge-Kutta method. In Section 5, we give a discussion and conclusion remarks.

2. Formulation of Invariance for a BVP of PDEs

Consider a BVP for kth order scalar PDEs ($k \geq 2$)

$$F^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u) = u_{i_1,i_2,\cdots,i_l} - f^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u) = 0, \tag{1}$$

(where $f^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)$ does not depend on $u_{i_1,i_2,\cdots,i_l}$) where $x = (x_1,x_2,\cdots,x_n)$ are n independent variables, $u = (u^1,u^2,\cdots,u^m)$ are m dependent variables and defined on a domain Ω_x in x-space with boundary conditions

$$B^{\nu}_{\alpha}(x,u,\partial u,\partial^2 u,\ldots,\partial^{k-1} u) = 0, \quad \nu = 1,2,\cdots,m \tag{2}$$

prescribed on boundary surfaces

$$\omega_{\alpha}(x) = 0, \;\; \alpha = 1,2,\cdots,s. \tag{3}$$

Assume that BVP (1)–(3) has a unique solution. Consider an infinitesimal generator of the form

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{\nu}(x,u)\frac{\partial}{\partial u_{\nu}}, \quad i = 1,2,\cdots,n; \; \nu = 1,2,\cdots,m. \tag{4}$$

which defines a one-parameter Lie group of transformations in x-space as well as in (x,u)-space.

Definition 1. *X is admitted by BVP (1)–(3) if and only if [2]*

$$X^{(k)}F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u) = 0, \; when \; F(x,u,\partial u,\partial^2 u,\ldots,\partial^k u) = 0; \tag{5}$$

$$X\omega_{\alpha}(x) = 0, \; when \; \omega_{\alpha}(x) = 0, \alpha = 1,2,\cdots,s; \tag{6}$$

$$X^{(k-1)}B^{\nu}_{\alpha}(x,u,\partial u,\partial^2 u,\ldots,\partial^{k-1} u) = 0, \; when \; B^{\nu}_{\alpha}(x,u,\partial u,\partial^2 u,\ldots,\partial^{k-1} u) = 0, \omega_{\alpha}(x) = 0, \\ \alpha = 1,2,\cdots,s \tag{7}$$

where $X^{(k)}$ is kth $(k \geq 1)$ extended infinitesimal generator of X given by

$$X^{(k)} = \xi_i(x,u)\tfrac{\partial}{\partial x_i} + \eta^{\nu}(x,u)\tfrac{\partial}{\partial u^{\nu}} + \eta^{(1)\nu}_i(x,u,\partial u)\tfrac{\partial}{\partial u^{\nu}_i} + \cdots \\ + \eta^{(k)\nu}_{i_1 i_2\cdots i_k}(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)\tfrac{\partial}{\partial u^{\nu}_{i_1 i_2\cdots i_k}}, \tag{8}$$

$$u^{\nu}_i = \frac{\partial u^{\nu}}{\partial x_i}, \;\; \eta^{(1)\nu}_i = D_i\eta^{\nu} - (D_i\xi_j)u^{\nu}_j, \; \eta^{(t)\nu}_{i_1 i_2\cdots i_t} = D_{i_t}\eta^{(t-1)\nu}_{i_1 i_2\cdots i_{t-1}} - (D_{i_t}\xi_j)^{\nu}_{i_1 i_2\cdots i_{t-1}j}, \tag{9}$$

$$D_i = \frac{\partial}{\partial x_i} + u^{\nu}_i\frac{\partial}{\partial u^{\nu}} + u^{\nu}_{ij}\frac{\partial}{\partial u^{\nu}_j} + \cdots + u^{\nu}_{ii_1 i_2\cdots i_n}\frac{\partial}{\partial u^{\nu}_{i_1 i_2\cdots i_n}} + \cdots, \tag{10}$$

and $i = 1,2,\cdots,n$; $\nu = 1,2,\cdots,m$; $i_{\ell} = 1,2,\cdots,n$; $\ell = 1,2,\cdots,t$, $t \geq 2$.

3. The Symmetry and Symmetry Reduction of von Kármán sWirling Viscous Flow

Let us consider von Kármán swirling viscous flow which is a famous classical problem in fluid mechanics. The governing equations are as follows:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}=0, \tag{11}$$
$$V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}-\frac{V_\theta^2}{r}=\nu\left(\frac{\partial^2 V_r}{\partial r^2}+\frac{1}{r}\frac{\partial V_r}{\partial r}+\frac{\partial^2 V_r}{\partial z^2}-\frac{V_r}{r^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{12}$$
$$V_r\frac{\partial V_\theta}{\partial r}+V_z\frac{\partial V_\theta}{\partial z}+\frac{V_rV_\theta}{r}=\nu\left(\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}+\frac{\partial^2 V_\theta}{\partial z^2}-\frac{V_\theta}{r^2}\right), \tag{13}$$
$$V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=\nu\left(\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}+\frac{\partial^2 V_z}{\partial z^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{14}$$

where V_r, V_θ, V_z, p are all functions of r, θ, z. ρ is the fluid density, ν is the kinematic viscosity, p is the pressure. The boundary conditions of Equations (11)–(14) are

$$V_r(r,\theta,0)=B_1(r,\theta), V_{rz}(r,\theta,0)=B_2(r,\theta), V(r,\theta,+\infty)=0, \tag{15}$$
$$V_\theta(r,\theta,0)=B_3(r,\theta), V_{\theta z}(r,\theta,0)=B_4(r,\theta), V(r,\theta,+\infty)=0, \tag{16}$$
$$V_z(r,\theta,0)=B_5(r,\theta), V_{zz}(r,\theta,0)=B_6(r,\theta), \tag{17}$$
$$p(r,\theta,0)=B_7(r,\theta), p_z(r,\theta,0)=B_8(r,\theta), \tag{18}$$

where $U_{rz}=\partial U_r/\partial z, U_{\theta z}=\partial U_\theta/\partial z, U_{zz}=\partial U_z/\partial z$, and $B_i(r,\theta)(i=1,\cdots,8)$ are the functions of (r,θ) to be determined later.

3.1. First Symmetry Reduction

The symmetry group of Equations (11)–(14) will be generated by the vector field of the form

$$X_1=\xi_1\frac{\partial}{\partial r}+\xi_2\frac{\partial}{\partial\theta}+\xi_3\frac{\partial}{\partial z}+\eta_1\frac{\partial}{\partial V_r}+\eta_2\frac{\partial}{\partial V_\theta}+\eta_3\frac{\partial}{\partial V_z}+\eta_4\frac{\partial}{\partial p}, \tag{19}$$

where $\xi_i=\xi_i(r,\theta,z,V_r,V_\theta,V_z,p), \eta_j=\eta_j(r,\theta,z,V_r,V_\theta,V_z,p)$ are the infinitesimal functions of the symmetry.

We obtain the determining equations of symmetry (19) by using the Lie algorithm, but it is too difficult to get their solutions. However, we use the differential characteristic set algorithm to obtain the following equivalent system of the determining equations [29].

$$\begin{aligned}&\xi_{1\theta}=\xi_{1z}=\xi_{1V_r}=\xi_{1V_\theta}=\xi_{1V_z}=\xi_{1p}=0,\xi_{2r}=\xi_{2\theta}=\xi_{2z}=\xi_{2V_r}=\xi_{2V_\theta}=\xi_{2V_z}=\xi_{2p}=0,\\&\xi_{3r}=\xi_{3\theta}=\xi_{3V_r}=\xi_{3V_\theta}=\xi_{3V_z}=\xi_{3p}=0,\eta_{4r}=\eta_{4z}=\eta_{4V_r}=\eta_{4V_\theta}=\eta_{4V_z}=0,\\&\xi_1-r\xi_{1r}=0,\xi_1-r\xi_{3z}=0,r\eta_1+V_r\xi_1=0,r\eta_2+V_\theta\xi_1=0,r\eta_3+V_z\xi_1=0,r\eta_4+2\xi_1=0.\end{aligned} \tag{20}$$

By solving the above PDEs, we get

$$\xi_1=a_1r,\ \xi_2=a_2,\ \xi_3=a_1z+a_3,\ \eta_1=-a_1V_r,\ \eta_2=-a_1V_\theta,\ \eta_3=-a_1V_z,\ \eta_4=f(\theta)-2a_1p. \tag{21}$$

where a_1, a_2, a_3 are arbitrary symmetry parameters, and $f(\theta)$ is an arbitrary function. Then the corresponding infinitesimal vector has the following form

$$X_1=a_1r\frac{\partial}{\partial r}+a_2\frac{\partial}{\partial\theta}+(a_1z+a_3)\frac{\partial}{\partial z}-a_1V_r\frac{\partial}{\partial V_r}-a_1V_\theta\frac{\partial}{\partial V_\theta}-a_1V_z\frac{\partial}{\partial V_z}+[f(\theta)-2a_1p]\frac{\partial}{\partial p}. \tag{22}$$

The characteristic equations for the symmetry X_1 are as follows

$$\frac{dr}{a_1r}=\frac{d\theta}{a_2}=\frac{dz}{a_1z+a_3}=\frac{dV_r}{-a_1V_r}=\frac{dV_\theta}{-a_1V_\theta}=\frac{dV_z}{-a_1V_z}=\frac{dp}{f(\theta)-2a_1p}. \tag{23}$$

By solving $\frac{dr}{a_1 r} = \frac{d\theta}{a_2}$ and $\frac{d\theta}{a_2} = \frac{dz}{a_1 z + a_3}$, we obtain two invariants as follows

$$\varsigma = r e^{-\frac{a_1}{a_2}\theta}, \tau = \frac{a_1 z + a_3}{a_1} e^{-\frac{a_1}{a_2}\theta}. \tag{24}$$

By using the invariant form method, we get the solutions of Equations (23)

$$V_r = \frac{U(\varsigma,\tau)}{r},\ V_\theta = \frac{V(\varsigma,\tau)}{r},\ V_z = \frac{W(\varsigma,\tau)}{r},\ p = e^{-\frac{2a_1}{a_2}\theta}\left[P(\varsigma,\tau) + \frac{1}{a_2}\int_1^\theta f(t) e^{\frac{2a_1 t}{a_2}} dt\right]. \tag{25}$$

By substituting (25) into Equations (11)–(14), we obtain PDEs as follows

$$a_1 \tau \frac{\partial V}{\partial \tau} - \varsigma\left(a_2 \frac{\partial W}{\partial \tau} + a_2 \frac{\partial U}{\partial \varsigma} - a_1 \frac{\partial V}{\partial \varsigma}\right) = 0, \tag{26}$$

$$U^2 + V^2 - \frac{\varsigma^3}{\rho}\frac{\partial P}{\partial \varsigma} - \varsigma\left(\nu \frac{\partial U}{\partial \varsigma} + U \frac{\partial U}{\partial \varsigma} + W \frac{\partial U}{\partial \tau}\right) + \nu \varsigma^2 \left(\frac{\partial^2 U}{\partial \varsigma^2} + \frac{\partial^2 U}{\partial \tau^2}\right) = 0, \tag{27}$$

$$\nu \frac{\partial V}{\partial \varsigma} + U \frac{\partial V}{\partial \varsigma} + W \frac{\partial V}{\partial \tau} - \nu \varsigma \left(\frac{\partial^2 V}{\partial \varsigma^2} + \frac{\partial^2 V}{\partial \tau^2}\right) = 0, \tag{28}$$

$$W\left(\nu + U - \varsigma \frac{\partial W}{\partial \tau}\right) - \frac{\varsigma^3}{\rho}\frac{\partial P}{\partial \tau} - \varsigma(\nu + U)\frac{\partial W}{\partial \varsigma} + \nu \varsigma^2 \left(\frac{\partial^2 W}{\partial \varsigma^2} + \frac{\partial^2 W}{\partial \tau^2}\right) = 0. \tag{29}$$

According to invariance for a BVP of the PDEs, the symmetry X_1 leaves the boundary conditions (15)–(18) invariant, namely

$$X_1[V_r(r,\theta,z) - B_1(r,\theta)] = 0, \text{ when } V_r(r,\theta,0) = B_1(r,\theta), \tag{30}$$

$$X_1^{(1)}[V_{rz}(r,\theta,z) - B_2(r,\theta)] = 0, \text{ when } V_{rz}(r,\theta,0) = B_2(r,\theta), \tag{31}$$

$$X_1[V_\theta(r,\theta,z) - B_3(r,\theta)] = 0, \text{ when } V_\theta(r,\theta,0) = B_3(r,\theta), \tag{32}$$

$$X_1^{(1)}[V_{\theta z}(r,\theta,z) - B_4(r,\theta)] = 0, \text{ when } V_{\theta z}(r,\theta,0) = B_4(r,\theta), \tag{33}$$

$$X_1[V_z(r,\theta,z) - B_5(r,\theta)] = 0, \text{ when } V_z(r,\theta,0) = B_5(r,\theta), \tag{34}$$

$$X_1^{(1)}[V_{zz}(r,\theta,z) - B_6(r,\theta)] = 0, \text{ when } V_{zz}(r,\theta,0) = B_6(r,\theta), \tag{35}$$

$$X_1[p(r,\theta,z) - B_7(r,\theta)] = 0, \text{ when } p(r,\theta,0) = B_7(r,\theta), \tag{36}$$

$$X_1^{(1)}[p_z(r,\theta,z) - B_8(r,\theta)] = 0, \text{ when } p_z(r,\theta,0) = B_8(r,\theta), \tag{37}$$

$$X_1[V(r,\theta,+\infty)] = 0, \text{ when } V(r,\theta,+\infty) = 0, \tag{38}$$

$$X_1[V(r,\theta,+\infty)] = 0, \text{ when } V(r,\theta,+\infty) = 0, \tag{39}$$

where $X_1^{(1)}$ is the 1st extended infinitesimal generator of X_1 as follow

$$\begin{aligned} X_1^{(1)} = {} & X_1 - 2a_1 V_{rr}\frac{\partial}{\partial V_{rr}} - a_1 V_{r\theta}\frac{\partial}{\partial V_{r\theta}} - 2a_1 V_{rz}\frac{\partial}{\partial V_{rz}} - 2a_1 V_{\theta r}\frac{\partial}{\partial V_{\theta r}} - a_1 V_{\theta\theta}\frac{\partial}{\partial V_{\theta\theta}} - 2a_1 V_{\theta z}\frac{\partial}{\partial V_{\theta z}} \\ & - 2a_1 V_{zr}\frac{\partial}{\partial V_{zr}} - a_1 V_{z\theta}\frac{\partial}{\partial V_{z\theta}} - 2a_1 V_{zz}\frac{\partial}{\partial V_{zz}} - 3a_1 p_r \frac{\partial}{\partial p_r} - [f'(\theta) - 2a_1 p_\theta]\frac{\partial}{\partial p_\theta} - 3a_1 p_z \frac{\partial}{\partial p_z}. \end{aligned} \tag{40}$$

We can determine the functions $B_1(r,\theta),\cdots,B_8(r,\theta)$ by solving (37)–(59), namely

$$B_1(r,\theta)=\frac{1}{r}\tilde{B}_1(\theta-\frac{a_2}{a_1}\ln r),\ B_2(r,\theta)=\frac{1}{r^2}\tilde{B}_2(\theta-\frac{a_2}{a_1}\ln r), \tag{41}$$

$$B_3(r,\theta)=\frac{1}{r}\tilde{B}_3(\theta-\frac{a_2}{a_1}\ln r),\ B_4(r,\theta)=\frac{1}{r^2}\tilde{B}_4(\theta-\frac{a_2}{a_1}\ln r), \tag{42}$$

$$B_5(r,\theta)=\frac{1}{r}\tilde{B}_5(\theta-\frac{a_2}{a_1}\ln r),\ B_6(r,\theta)=\frac{1}{r^2}\tilde{B}_6(\theta-\frac{a_2}{a_1}\ln r), \tag{43}$$

$$B_7(r,\theta)=\frac{1}{r^2}\tilde{B}_7(\theta-\frac{a_2}{a_1}\ln r),\ B_8(r,\theta)=\frac{1}{r^3}\tilde{B}_8(\theta-\frac{a_2}{a_1}\ln r), \tag{44}$$

where c_i $(i=1,\cdots,8)$ are arbitrary constants. Because the arbitrary function $f(\theta)$ does not affect the reduced results (26)–(29), we let $f(\theta)=0$ in the process of calculation (44).

Let $a_3=0$, then we have

$$\tau=0,\ \text{when}\ z=0;\quad \tau\longrightarrow+\infty,\ \text{when}\ z\longrightarrow+\infty, \tag{45}$$

according to (15)–(18) and (25), we get boundary conditions of Equations (26)–(29) as follows

$$U(\varsigma,0)=\tilde{B}_1(-\frac{a_2}{a_1}\ln\varsigma),\ U_\tau(\varsigma,0)=\frac{1}{\varsigma}\tilde{B}_2(-\frac{a_2}{a_1}\ln\varsigma),\ U(\varsigma,+\infty)=0, \tag{46}$$

$$V(\varsigma,0)=\tilde{B}_3(-\frac{a_2}{a_1}\ln\varsigma),\ V_\tau(\varsigma,0)=\frac{1}{\varsigma}\tilde{B}_4(-\frac{a_2}{a_1}\ln\varsigma),\ V(\varsigma,+\infty)=0, \tag{47}$$

$$W(\varsigma,0)=\tilde{B}_5(-\frac{a_2}{a_1}\ln\varsigma),\ W_\tau(\varsigma,0)=\frac{1}{\varsigma}\tilde{B}_6(-\frac{a_2}{a_1}\ln\varsigma), \tag{48}$$

$$P(\varsigma,0)=\frac{1}{\varsigma^2}\tilde{B}_7(-\frac{a_2}{a_1}\ln\varsigma),\ P_\tau(\varsigma,0)=\frac{1}{\varsigma^3}\tilde{B}_8(-\frac{a_2}{a_1}\ln\varsigma), \tag{49}$$

where $\tilde{B}_i(i=1,\cdots,8)$ are the functions of ς.

3.2. Second Symmetry Reduction

By the same manner, we get the following infinitesimal vector of symmetry

$$X_2=b_1\varsigma\frac{\partial}{\partial\varsigma}+b_1\tau\frac{\partial}{\partial\tau}+(b_2-2b_1P)\frac{\partial}{\partial P}, \tag{50}$$

for the system (26)–(29), where b_1,b_2 are arbitrary symmetry parameters.

In the following, the BVP for PDEs (26)–(49) will be reduced to the initial value problem of the ordinary differential equations (ODEs) by using the invariant form method [2].

The characteristic equations for the symmetry X_2 are as follows

$$\frac{d\varsigma}{b_1\varsigma}=\frac{d\tau}{b_1\tau}=\frac{dU}{0}=\frac{dV}{0}=\frac{dW}{0}=\frac{dP}{b_2-2b_1P}. \tag{51}$$

By solving $\frac{d\varsigma}{\varsigma}=\frac{d\tau}{\tau}$, we obtain the invariant as follows

$$\zeta=\frac{\tau}{\varsigma}. \tag{52}$$

By using the invariant form method, we get the solutions of Equations (51) as follows

$$U=u(\zeta),\ V=v(\zeta),\ W=w(\zeta),\ P=\frac{b_2}{2b_1}+\frac{1}{\varsigma^2}g(\zeta). \tag{53}$$

By substituting (52) and (53) into Equations (26)–(29), we obtain ODEs as follows

$$\zeta u' - w' = 0, \tag{54}$$
$$u^2 + v^2 + \frac{2g + \zeta g'}{\rho} + (3\nu\zeta + \zeta u - w)u' + \nu(1+\zeta^2)u'' = 0, \tag{55}$$
$$(3\nu\zeta + \zeta u - w)v' + \nu(1+\zeta^2)v'' = 0, \tag{56}$$
$$(\nu + u - w')w + \zeta(3\nu + u)w' + \nu(1+\zeta^2)w'' - \frac{g'}{\rho} = 0. \tag{57}$$

The symmetry X_2 leaves the boundary conditions (46)–(49) invariant, namely

$$X_2[U(\varsigma,\tau) - \tilde{B}_1(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } U(\varsigma,0) = \tilde{B}_1(-\frac{a_2}{a_1}\ln\varsigma); \tag{58}$$
$$X_2^{(1)}[U_\tau(\varsigma,\tau) - \frac{1}{\varsigma}\tilde{B}_2(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } U_\tau(\varsigma,0) = \frac{1}{\varsigma}\tilde{B}_2(-\frac{a_2}{a_1}\ln\varsigma), \tag{59}$$
$$X_2[V(\varsigma,\tau) - \tilde{B}_3(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } V(\varsigma,0) = \tilde{B}_3(-\frac{a_2}{a_1}\ln\varsigma); \tag{60}$$
$$X_2^{(1)}[V_\tau(\varsigma,\tau) - \frac{1}{\varsigma}\tilde{B}_4(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } V_\tau(\varsigma,0) = \frac{1}{\varsigma}\tilde{B}_4(-\frac{a_2}{a_1}\ln\varsigma), \tag{61}$$
$$X_2[U(\varsigma,\tau) - \tilde{B}_5(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } W(\varsigma,0) = \tilde{B}_5(-\frac{a_2}{a_1}\ln\varsigma); \tag{62}$$
$$X_2^{(1)}[W_\tau(\varsigma,\tau) - \frac{1}{\varsigma}\tilde{B}_6(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } W_\tau(\varsigma,0) = \frac{1}{\varsigma}\tilde{B}_6(-\frac{a_2}{a_1}\ln\varsigma), \tag{63}$$
$$X_2[P(\varsigma,\tau) - \frac{1}{\varsigma^2}\tilde{B}_7(-\frac{a_2}{a_1}\ln\varsigma) = 0, \text{ when } P(\varsigma,0) = \frac{1}{\varsigma^2}\tilde{B}_7(-\frac{a_2}{a_1}\ln\varsigma); \tag{64}$$
$$X_2^{(1)}[P_\tau(\varsigma,\tau) - \frac{1}{\varsigma^3}\tilde{B}_8(-\frac{a_2}{a_1}\ln\varsigma)] = 0, \text{ when } P_\tau(\varsigma,0) = \frac{1}{\varsigma^3}\tilde{B}_8(-\frac{a_2}{a_1}\ln\varsigma), \tag{65}$$
$$X_2[U(\varsigma,\tau)] = 0, \text{ when } U(\varsigma,+\infty) = 0, \tag{66}$$
$$X_2[V(\varsigma,\tau)] = 0, \text{ when } V(\varsigma,+\infty) = 0, \tag{67}$$

where $X_2^{(1)}$ is the 1st extended infinitesimal generator of X_2 as follow

$$X_2^{(1)} = X_2 - b_1 U_\varsigma \frac{\partial}{\partial U_\varsigma} - b_1 U_\tau \frac{\partial}{\partial U_\tau} - b_1 V_\varsigma \frac{\partial}{\partial V_\varsigma} - b_1 V_\tau \frac{\partial}{\partial V_\tau} - b_1 W_\varsigma \frac{\partial}{\partial W_\varsigma} - b_1 W_\tau \frac{\partial}{\partial W_\tau}$$
$$- 3b_1 P_\varsigma \frac{\partial}{\partial P_\varsigma} - 3b_1 P_\tau \frac{\partial}{\partial P_\tau}. \tag{68}$$

We can determine the functions $\tilde{B}_1(\varsigma), \cdots, \tilde{B}_8(\varsigma)$ by solving (58)–(65), namely

$$\tilde{B}_1(\varsigma) = c_1,\ \tilde{B}_2(\varsigma) = c_2,\ \tilde{B}_3(\varsigma) = c_3,\ \tilde{B}_4(\varsigma) = c_4, \tag{69}$$
$$\tilde{B}_5(\varsigma) = c_5,\ \tilde{B}_6(\varsigma) = c_6,\ \tilde{B}_7(\varsigma) = \frac{b_2}{2b_1}\varsigma^2 + c_7,\ \tilde{B}_8(\varsigma) = c_8, \tag{70}$$

where c_i $(i = 1, \cdots, 8)$ are arbitrary constants.

Because of

$$\zeta = 0, \text{ when } \tau = 0; \quad \zeta \longrightarrow +\infty, \text{ when } \tau \longrightarrow +\infty, \tag{71}$$

according to the boundary conditions (46)–(49) and the expression (53), we obtain the initial conditions as follows

$$u(0) = c_1, u'(0) = c_2, v(0) = c_3, v'(0) = c_4, w(0) = c_5, w'(0) = c_6, g(0) = c_7, g'(0) = c_8, \quad (72)$$

$$u(+\infty) = 0, v(+\infty) = 0. \quad (73)$$

4. Numerical Solutions

According to (54) one has

$$w' = \zeta u'. \quad (74)$$

Substitute (74) into (57), one has

$$(\nu + u - \zeta u')w + \zeta^2(3\nu + u)u' + \nu(1+\zeta^2)w'' - \frac{g'}{\rho} = 0. \quad (75)$$

In order to solve the numerical solutions of the initial value problems in ODEs (55), (56),(75) and (72) by using the Runge-Kutta method, we considers $g(\zeta) = c_7 + c_8\zeta$. Firstly, we change (55), (56), (75) and (72) into first order initial value problems in ODEs. Let

$$y_1 = u, y_2 = u', y_3 = v, y_4 = v', y_5 = w, y_6 = w', \quad (76)$$

then Equations (55), (56) and (75) are changed into the following form

$$y_1' = y_2,\ y_2' = -\frac{1}{\nu(1+\zeta^2)}[y_1^2 + y_3^2 + \frac{1}{\rho}(2c_7 + 3c_8\zeta) + (3\nu\zeta + \zeta y_1 - y_5)y_2], \quad (77)$$

$$y_3' = y_4,\ y_4' = -\frac{1}{\nu(1+\zeta^2)}(3\nu\zeta + \zeta y_1 - y_5)y_4, \quad (78)$$

$$y_5' = y_6,\ y_6' = -\frac{1}{\nu(1+\zeta^2)}[y_5(\nu + y_1 - y_6) + \zeta(3\nu + y_1)y_6 - \frac{c_8}{\rho}]. \quad (79)$$

The corresponding initial conditions have the following form

$$y_1(0) = c_1, y_2(0) = c_2, y_3(0) = c_3, y_4(0) = c_4, y_5(0) = c_5, y_6(0) = c_6. \quad (80)$$

We let

$$f_1 = y_2,\ f_2 = -\frac{1}{\nu(1+\zeta^2)}[y_1^2 + y_3^2 + \frac{1}{\rho}(2c_7 + 3c_8\zeta) + (3\nu\zeta + \zeta y_1 - y_5)y_2], \quad (81)$$

$$f_3 = y_4,\ f_4 = -\frac{1}{\nu(1+\zeta^2)}(3\nu\zeta + \zeta y_1 - y_5)y_4, \quad (82)$$

$$f_5 = y_6,\ f_6 = -\frac{1}{\nu(1+\zeta^2)}[y_5(\nu + y_1 - y_6) + \zeta(3\nu + y_1)y_6 - \frac{c_8}{\rho}], \quad (83)$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are the functions of $\zeta, y_1, y_2, y_3, y_4, y_5, y_6$, then we give the following Runge-Kutta formula

$$\begin{aligned}
y_{i,n+1} &= y_{in} + \frac{h}{6}(K_{i1} + 2K_{i2} + 2K_{i3} + K_{i4}), \quad i = 1, \cdots, 6. \\
K_{i1} &= f_i(\zeta_n, y_{1n}, y_{2n}, y_{3n}, y_{4n}, y_{5n}, y_{6n}) \\
K_{i2} &= f_i(\zeta_n + \frac{h}{2}, y_{1n} + \frac{h}{2}K_{11}, y_{2n} + \frac{h}{2}K_{21}, y_{3n} + \frac{h}{2}K_{31}, y_{4n} + \frac{h}{2}K_{41}, y_{5n} + \frac{h}{2}K_{51}, y_{6n} + \frac{h}{2}K_{61}) \quad (84) \\
K_{i3} &= f_i(\zeta_n + \frac{h}{2}, y_{1n} + \frac{h}{2}K_{12}, y_{2n} + \frac{h}{2}K_{22}, y_{3n} + \frac{h}{2}K_{32}, y_{4n} + \frac{h}{2}K_{42}, y_{5n} + \frac{h}{2}K_{52}, y_{6n} + \frac{h}{2}K_{62}) \\
K_{i4} &= f_i(\zeta_n + h, y_{1n} + hK_{13}, y_{2n} + hK_{23}, y_{3n} + hK_{33}, y_{4n} + hK_{43}, y_{5n} + hK_{53}, y_{6n} + hK_{63})
\end{aligned}$$

where $\zeta_0 = 0,\ \zeta_n = \zeta_0 + nh$, and h is the step size.

Figures 1 and 2 show the numerical solutions of $u(\zeta), v(\zeta), w(\zeta)$ in [0,2] when the parameters have the following proper values:

$$c_1 = 2, c_2 = 4, c_3 = 0, c_4 = 1, c_5 = 0.3, c_6 = 3, c_7 = 0.1, c_8 = 0.2, \rho = 100, \nu = 0.6, h = 0.1. \quad (85)$$

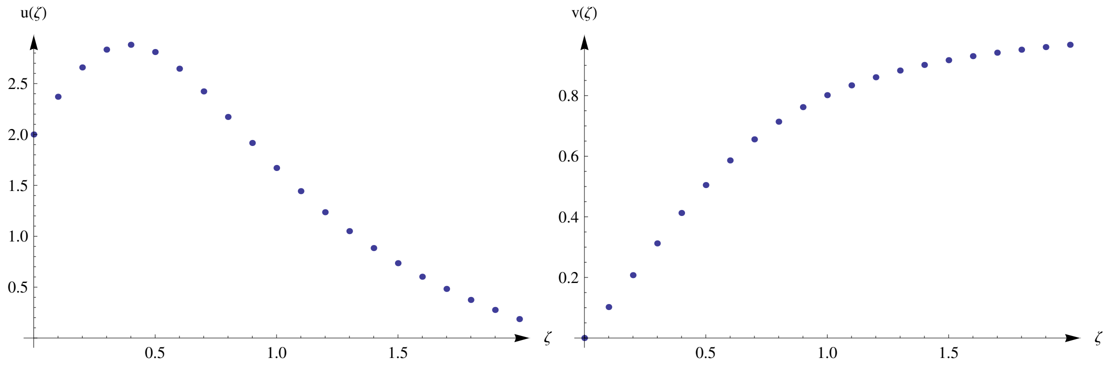

Figure 1. Numerical solutions of $u(\zeta), v(\zeta)$ in [0,2].

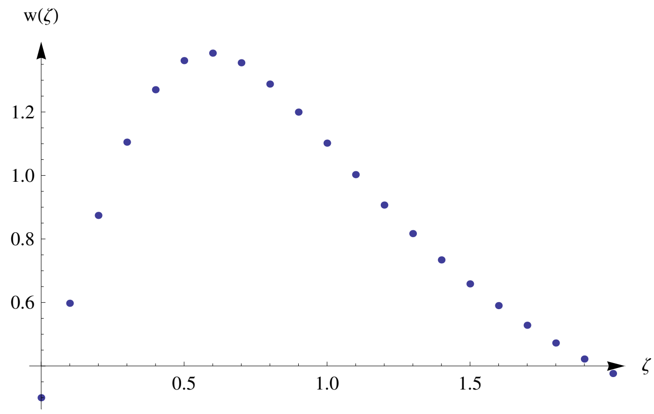

Figure 2. Numerical solutions of $w(\zeta)$ in [0,2].

5. Conclusions

In this paper, the application of the symmetry method on BVP for nonlinear PDEs is studied. Firstly, we have got the multi-parameter symmetry of von Kármán swirling viscous flow based on the differential characteristic set algorithm. Via twice symmetry reducions, BVP (11)–(18) became an initial value problem of ODEs. Secondly, we solved numerically the initial value problem of ODEs by using the Runge-Kutta method. The differential characteristic set algorithm is a key factor which influences the calculation of the symmetry of PDEs.

We considered that the boundary conditions are the arbitrary functions $B_i(r,\theta)$. However, $B_i(r,\theta)$ are determined by using the invariance of the boundary conditions under a multi-parameter Lie group of transformations. This approach is different from other research. For example, the boundary conditions are given by the following forms $V_r(r,\theta,0) = 0, V_z(r,\theta,0) = 0$ in [30]. The Lie symmetry and Runge-Kutta methods are effective methods which are applied to solving PDEs. Hence, their combination will advance the availability of solutions. At present, it is very valuable to solve nonlinear PDEs by combining the symmetry method, the differential characteristic set algorithm and other methods.

Acknowledgments: This work is supported by the National Natural Science Foundation of China (11661060); Natural Science Foundation of Inner Mongolia Autonomous Region of China (2016MS0116).

Author Contributions: Sudao Bilige conceived and designed the experiments. Xiaomin Wang performed the calculation. Xiaomin Wang, Sudao Bilige analyzed the data. Sudao Bilige, Xiaomin Wang wrote the paper. Sudao Bilige designed the software used in analysis.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Lie, S. *Vorlesungen uber Differentialgleichungen Mit Bekannten Infinitesimalen Transformationen*; BG Teubner: Leipzig, Germany, 1891.
2. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA; Berlin, Germany, 1989.
3. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Spinger: New York, NY, USA; Berlin, Germany, 1986.
4. Bluman, G.W.; Cheviakov, A.; Anco, S. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA; Berlin, Germany, 2010.
5. Ibragimov, N.H.; Ibragimov, R.N. Applications of Lie Group Analysis to Mathematical Modelling in Natural Sciences. *Math. Model. Nat. Phenom.* **2012**, *7*, 52–65. [CrossRef]
6. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations*; CRC Press: Boca Raton, FL, USA, 1994; Volume 1.
7. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations*; CRC Press: Boca Raton, FL, USA, 1995; Volume 2.
8. Tiwari Ajey, K.; Pandey, S.N.; Senthilvelan, M.; Lakshmanan, M. Lie point symmetries classification of the mixed Lienard-type equation. *Nonlinear Dyn.* **2015**, *82*, 1953–1968. [CrossRef]
9. Davison, A.H.; Kara, A.H. Potential Symmetries and Associated Conservation Laws with Application to Wave Equations. *Nonlinear Dyn.* **2003**, *33*, 369–377. [CrossRef]
10. Chaolu, T.M.; Bai, Y.S. An Algorithm for Determining Approximate Symmetries of Differential Equations Based on Wu's Method. *Chin. J. Eng. Math.* **2011**, *28*, 617–622.
11. Ma, W.X. K-symmetries and τ-symmetries of evolution equations and their Lie algebras. *J. Phys. A Math. Gen.* **1990**, *23*, 2707–2716. [CrossRef]
12. Ma, W.X. A soliton hierarchy associated with so(3,R). *Appl. Math. Comput.* **2013**, *220*, 117–122.
13. Bluman, G.W.; Chaolu, T.M. Conservation laws of nonlinear telegraph equations. *J. Math. Anal. Appl.* **2005**, *310*, 459–476. [CrossRef]
14. Anco, S.; Bluman, G.W.; Wolf, T. Invertible mappings of nonlinear PDEs to linear PDEs through admitted conservation laws. *Acta Appl. Math.* **2008**, *101*, 21–38. [CrossRef]
15. Abdulwahhab, M.A. Exact solutions and conservation laws of the system of two-dimensional viscous Burgers equations. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *39*, 283–299. [CrossRef]
16. Avdonina, E.D.; Ibragimov, N.H.; Khamitova, R. Exact solutions of gasdynamic equations obtained by the method of conservation laws. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 2359–2366. [CrossRef]
17. Ma, W.X. Conservation Laws of Discrete Evolution Equations by Symmetries and Adjoint Symmetries. *Symmetry* **2015**, *7*, 714–725. [CrossRef]
18. Wang, G.W.; Liu, X.Q.; Zhang, Y.Y. Symmetry reduction, exact solutions and conservation laws of a new fifth-order nonlinear integrable equation. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 2313–2320. [CrossRef]
19. Sahoo, S.; Saha Ray, S. Lie symmetry analysis and exact solutions of (3+1) dimensional Yu-Toda-Sasa-Fukuyama equation in mathematical physics. *Comput. Math. Appl.* **2017**, *73*, 253–260. [CrossRef]
20. Ma, W.X.; Chen, M. Direct search for exact solutions to the nonlinear Schrödinger equation. *Appl. Math. Comput.* **2009**, *215*, 2835–2842.
21. Yang, H.Z.; Liu, W.; Yang, B.Y.; He, B. Lie symmetry analysis and exact explicit solutions of three-dimensional Kudryashov-Sinelshchikov equation. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *27*, 271–280. [CrossRef]
22. Yang, S.J.; Hua, C.C. Lie symmetry reductions and exact solutions of a coupled KdV-Burgers equation. *Appl. Math. Comput.* **2014**, *234*, 579–583. [CrossRef]
23. Seshadri, R.; Na, T.Y. *Group Invariance in Engineering Boundary Value Problems*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1985.

24. Yürüsoy, M.; Pakdemirli, M.; Noyan, Ö.F. Lie group analysis of creeping flow of a second grade fluid. *Int. J. Non-Linear Mech.* **2001**, *36*, 955–960.
25. Vaneeva, O.O.; Papanicolaou, N.C.; Christou, M.A.; Sophocleous, C. Numerical solutions of boundary value problems for variable coefficient generalized KdV equations using Lie symmetries. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 3074–3085. [CrossRef]
26. Bilige, S.D.; Wang, X.M.; Morigen, W.Y. Application of the symmetry classification to the boundary value problem of nonlinear partial differential equations (in Chinese). *Acta Phys. Sin.* **2014**, *63*, 040201.
27. Gai, L.T.; Bilige, S.D.; Jie, Y.M. The exact solutions and approximate analytic solutions of (2+1)-dimensional KP equation based on symmetry method. *SpringerPlus* **2016**, *5*, 1267. [CrossRef]
28. Bilige, S.D.; Han, Y.Q. Symmetry reduction and numerical solution of a nonlinear boundary value problem in fluid mechanics. *Int. J. Numer. Methods Heat Fluid Flow* **2018**, *28*, 518–531. [CrossRef]
29. Chaolu, T.M.; Bai, Y.S. A new algorithmic theory for determining and classifying classical and non-classical symmetries of partial differential equations (in Chinese). *Sci. China Math.* **2010**, 40, 331–348.
30. Yang, C.; Liao, S.J. On the explicit, purely analytic solution of Von K*á*rm*á*n swirling viscous flow. *Commun. Nonlinear Sci. Numer. Simul.* **2006**, *11*, 83–93.[CrossRef]