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The Complexity of Some Classes of Pyramid Graphs Created from a Gear Graph

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Abstract: The methods of measuring the complexity (spanning trees) in a finite graph, a problem related to various areas of mathematics and physics, have been inspected by many mathematicians and physicists. In this work, we defined some classes of pyramid graphs created by a gear graph then we developed the Kirchhoff's matrix tree theorem method to produce explicit formulas for the complexity of these graphs, using linear algebra, matrix analysis techniques, and employing knowledge of Chebyshev polynomials. Finally, we gave some numerical results for the number of spanning trees of the studied graphs.

Keywords: complexity; Chebyshev polynomials; gear graph; pyramid graphs

MSC: 05C05, 05C50

1. Introduction

The graph theory is a theory that combines computer science and mathematics, which can solve considerable problems in several fields (telecom, social network, molecules, computer network, genetics, etc.) by designing graphs and facilitating them through idealistic cases such as the spanning trees, see [1–10].

A spanning tree of a finite connected graph G is a maximal subset of the edges that contains no cycle, or equivalently a minimal subset of the edges that connects all the vertices. The history of enumerating the number of spanning trees $\tau(G)$ of a graph G dates back to 1842 when the physicist Kirchhoff [11] offered the matrix tree theorem established on the determinants of a certain matrix gained from the Laplacian matrix L defined by the difference between the degree matrix D and adjacency matrix A, where D is a diagonal matrix, $D = dig(d_1, d_2, \ldots, d_n)$ corresponding to a graph G with n vertices that has the vertex degree of d_i in the ith position of a graph G and A is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (u_i, u_j) according to whether u_i and u_j are adjacent or not. That is

$$L_{ij} = \begin{cases} a_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j \\ 0 & \text{otherwise} \end{cases},$$

where a_i denotes the degree of the vertex i.

This method allows beneficial results for a graph comprising a small number of vertices, but is not feasible for large graphs. There is one more method for calculating $\tau(G)$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k = 0$

denote the eigenvalues of the matrix L of a graph G with n vertices. "Kelmans" and "Chelnokov" [12] have derived that

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1} \lambda_k.$$

One of the favorite methods of calculating the complexity is the contraction–deletion theorem. For any graph G, the complexity $\tau(G)$ of G is equal to $\tau(G) = \tau(G-e) + \tau(G/e)$, where e is any edge of G, and where $G-e$ is the deletion of e from G, and G/e is the contraction of e in G. This gives a recursive method to calculate the complexity of a graph [13,14].

Another important method is using electrically equivalent transformations of networks. Yilun Shang [15] derived a closed-form formula for the enumeration of spanning trees the subdivided-line graph of a simple connected graph using the theory of electrical networks.

Many works have conceived techniques to derive the number of spanning trees of a graph, some of which can be found at [16–18].

Now, we give the following Lemma:

Lemma 1 [19]. *$\tau(G) = \frac{1}{k^2}\det(kI - D^c + A^c)$ where A^c and D^c are the adjacency and degree matrices of G^c, the complement of G, respectively, and I is the $k \times k$ identity matrix.*

The characteristic of this formula is to express $\tau(G)$ straightway as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. Chebyshev Polynomial

In this part we insert some relations regarding Chebyshev polynomials of the first and second types which we use in our calculations.

We start from their definitions, see Yuanping, et al. [20].

Let $A_n(x)$ be $n \times n$ matrix such that

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix}.$$

Furthermore, we render that the Chebyshev polynomials of the first type are defined by

$$T_n(x) = \cos(n \ \cos^{-1} x) \tag{1}$$

The Chebyshev polynomials of the second type are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \ \cos^{-1} x)}{\sin(\cos^{-1} x)} \tag{2}$$

It is easily confirmed that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{3}$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one obtains

$$U_n(x) = \det(A_n(x)),\ n \geq 1 \tag{4}$$

Moreover, by solving the recursion (3), one gets the straightforward formula

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}, \;\; n \geq 1, \tag{5}$$

where the conformity is valid for all complex x (except at $x = \pm 1$, where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be confirmed that

$$U_{n-1}(x) = 2^{n-1} \prod_{j=1}^{n-1} (x - \cos\frac{j\pi}{n}) \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1} U_{n-1}(x) \tag{7}$$

From Equations (6) and (7), we have:

$$U_{n-1}{}^2(x) = 4^{n-1} \prod_{j=1}^{n-1} (x^2 - \cos^2\frac{j\pi}{n}) \tag{8}$$

Finally, straightforward manipulation of the above formula produces the following formula (9), which is highly beneficial to us later:

$$U_{n-1}{}^2(\sqrt{\frac{x+2}{4}}) = \prod_{j=1}^{n-1} (x - 2\cos\frac{2j\pi}{n}) \tag{9}$$

Moreover, one can see that

$$U_{n-1}{}^2(x) = \frac{1 - T_{2n}(x)}{2(1-x^2)} = \frac{1 - T_n(2x^2-1)}{2(1-x^2)} \tag{10}$$

$$T_n(x) = \frac{1}{2}[(x+\sqrt{x^2-1}\,)^n + ((x-\sqrt{x^2-1}\,)^n] \tag{11}$$

Now we introduce the following important two Lemmas.

Lemma 2 [21]. *Let $B_n(x)$ be $n \times n$ Circulant matrix such that*

$$B_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & x \end{pmatrix}.$$

Then for $n \geq 3,\;\; x \geq 4$, one has

$$\det(B_n(x)) = \frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2}) - 1].$$

Lemma 3 [22]. *If $A \in F^{n\times n}$, $B \in F^{n\times m}$, $C \in F^{m\times n}$ and $D \in F^{m\times m}$. Suppose that A and D are nonsingular matrices, then*:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det D = \det A \det(D - CA^{-1}B).$$

This Lemma gives a type of symmetry for some matrices which simplify our calculations of the complexity of graphs studied in this paper.

3. Main Results

Definition 1. *The pyramid graph* $A_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0;\ u_1, u_2, \ldots, u_m\,;\ w_1, w_2, \ldots, w_m\}$ and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, v_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1},where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Figure 1.

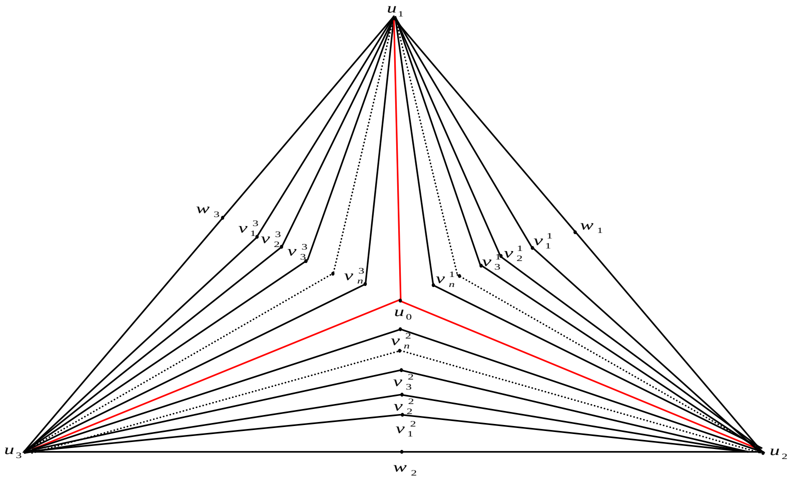

Figure 1. The pyramid graph $A_n^{(3)}$.

Theorem 1. *For* $n \geq 0,\ m \geq 3,\ \tau(A_n^{(m)}) = 2^{mn}[(n+2+\sqrt{2n+3}\,)^m + (n+2-\sqrt{2n+3}\,)^m - 2(n+1)^m]$.

Proof. Using Lemma 1, we have

$$\tau(A_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)\,I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det \left(\begin{array}{cccccccccccccccccccccccccccccc}
(m+1) & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & 2(n+2) & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 1 & \cdots & \cdots & \cdots & 1 & 2(n+2) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{array}\right)$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n be the $n \times n$ matrix with all one. Set $a = 2n + 4$ and $b = mn + 2m + 1$. Then we obtain:

$$\tau\left(A_n^{(m)}\right) = \frac{1}{b^2} \times \det \begin{pmatrix}
m+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
0 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

$$= \frac{1}{b^2} \times \det \begin{pmatrix}
b & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

$$= \frac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

$$= \tfrac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 1 & \cdots & \cdots & \cdots & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & 1 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}$$

$$= \tfrac{1}{b} \times \det \begin{pmatrix}
a & 1 & \cdots & \cdots & \cdots & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & \cdots & 1 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}$$

Using Lemma 3, yields

$$\tau\,(A_n^{(m)}) = \tfrac{1}{b} \times \det \begin{pmatrix} A & B \\ C & 2I_{mn} \end{pmatrix} = \tfrac{1}{b} \times \det(A - B\tfrac{1}{2I_{mn}}\ C) \times 2^{mn}$$

$$= \tfrac{1}{b} 2^{mn} \times 2^{-2m} \times \det \begin{pmatrix}
2a & n+2 & 2(n+1) & \cdots & 2(n+1) & n+2 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
n+2 & 2a & n+2 & 2(n+1) & \cdots & 2(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
2(n+1) & n+2 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2(n+1) & \ddots & \ddots & \ddots & \ddots & n+2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
n+2 & 2(n+1) & \cdots & 2(n+1) & n+2 & 2a & 0 & \cdots & \cdots & 0 & -2 & -2 \\
0 & 0 & 2 & \cdots & \cdots & 2 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}$$

Using Lemma 3 again, yields

$$\tau\,(A_n^{(m)}) = \frac{2^{mn-2m}}{b} \times \det \begin{pmatrix} D & E \\ F & 4I_m \end{pmatrix} = \frac{2^{mn}}{b} \times \det(D - E\frac{1}{4I_m}\ F)$$

$$\tau\,(A_n^{(m)}) = \frac{2^{mn}}{b} \times \det \begin{pmatrix} 2a & (n+3) & 2(n+2) & \cdots & 2(n+2) & (n+3) \\ (n+3) & 2a & (n+3) & \ddots & \cdots & 2(n+2) \\ 2(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2(n+2) \\ 2(n+2) & \ddots & \ddots & \ddots & \ddots & (n+3) \\ (n+3) & 2(n+2) & \cdots & 2(n+2) & (n+3) & 2a \end{pmatrix}$$

Straightforward inducement using the properties of determinants, one can obtain

$$\tau\,(A_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{2b}{m\,n+m+2} \times \det \begin{pmatrix} (2a-n-3) & 0 & (n+1) & \cdots & (n+1) & 0 \\ 0 & (2a-n-3) & 0 & \ddots & \cdots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+1) & \cdots & (n+1) & 0 & (2a-n-3) \end{pmatrix}$$

$$= \frac{2^{m\,n+1}\,(n+1)^m}{m\,n+m+2} \times \det \begin{pmatrix} \frac{(2a-n-3)}{(n+1)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-3)}{(n+1)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-3)}{(n+1)} \end{pmatrix}$$

Using Lemma 2, yields

$$\begin{aligned} \tau\,(A_n^{(m)}) &= 2^{m\,n+1} \times \frac{(n+1)^m}{mn+m+2} \times \frac{2(\frac{2a-n-3}{n+1}+m-3)}{\frac{2a-n-3}{n+1}-3} \times [T_m(\frac{\frac{2a-n-3}{n+1}-1}{2}) - 1] \\ &= 2^{mn+1} \times (n+1)^m \times [T_m(\tfrac{n+2}{n+1}) - 1]. \end{aligned}$$

Using Equation (11), yields the result. □

Definition 2. *The pyramid graph $B_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0\,;\ u_1, u_2, \ldots, u_m\,;\ w_1, w_2, \ldots, w_m\}$ with double internal edges and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, v_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Figure 2.*

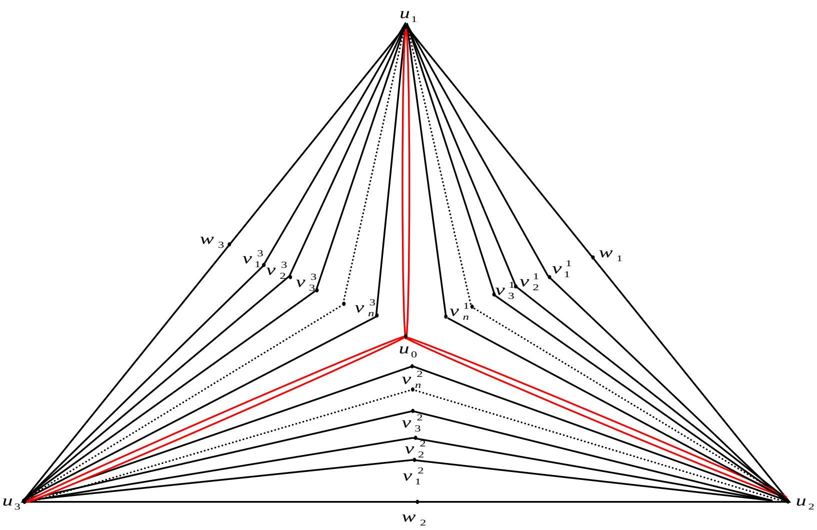

Figure 2. The pyramid graph $B_n^{(3)}$.

Theorem 2. *For $n \geq 0$, $m \geq 3$,* $\tau(B_n^{(m)}) = 2^{mn}\left[\left(n+3+2\sqrt{n+2}\,\right)^m + \left(n+3-2\sqrt{n+2}\,\right)^m - 2(n+1)^m\right].$

Proof. Using Lemma 1, we get:

$$\tau(B_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)\, I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det \left(\begin{array}{*{30}{c}}
(2m+1) & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & (2n+5) & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
-1 & 1 & \cdots & \cdots & \cdots & 1 & (2n+5) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{array}\right)$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n be the $n \times n$ matrix with all one. Set $a = 2n+5$ and $b = mn+2m+1$. Then we get:

$$\tau\left(B_n^{(m)}\right) = \tfrac{1}{b^2} \times \det \left(\begin{array}{*{19}{c}}
2m+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{array}\right)$$

$$
= \frac{1}{b^2} \times \det \begin{pmatrix}
b & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (a+1) & 2 & \cdots & \cdots & \cdots & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & \cdots & 2 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

$$= \frac{1}{b} \times \det \begin{pmatrix}
(a+1) & 2 & \cdots & \cdots & \cdots & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \cdots & \cdots & \cdots & 2 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & 2I_{mn} & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}$$

Using Lemma 3, yields

$$\tau\,(B_n^{(m)}) = \frac{1}{b} \times \det \begin{pmatrix} A & B \\ C & 2I_{mn} \end{pmatrix} = \frac{1}{b} \times \det(A - B\frac{1}{2I_{mn}}\ C) \times 2^{mn}$$

$$= \frac{1}{b} 2^{mn} \times 2^{-2m} \times \det \begin{pmatrix}
(2a+2n+2) & 3n+4 & 4(n+1) & \cdots & 4(n+1) & 3n+4 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
3n+4 & (2a+2n+2) & 3n+4 & 4(n+1) & \cdots & 4(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
4(n+1) & 3n+4 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4(n+1) & \ddots & \ddots & \ddots & \ddots & 3n+4 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
3n+4 & 4(n+1) & \cdots & 4(n+1) & 3n+4 & (2a+2n+2) & 0 & \cdots & \cdots & 0 & -2 & -2 \\
2 & 2 & 4 & \cdots & \cdots & 4 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
4 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4 & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & 4 & \cdots & \cdots & 4 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}$$

Using Lemma 3 again, yields

$$\tau\,(B_n^{(m)}) = \frac{2^{mn-2m}}{b} \times \det \begin{pmatrix} D & E \\ F & 4I_m \end{pmatrix} = \frac{2^{mn}}{b} \times \det(D - E\frac{1}{4I_m}\ F)$$

$$\tau\,(B_n^{(m)}) = \frac{2^{mn}}{b} \times \det \begin{pmatrix}
(2a+2n+4) & (3n+7) & 4(n+2) & \cdots & 4(n+2) & (3n+7) \\
(3n+7) & (2a+2n+4) & (3n+7) & \ddots & \cdots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & (3n+7) \\
(3n+7) & 4(n+2) & \cdots & 4(n+2) & (3n+7) & (2a+2n+4)
\end{pmatrix}$$

With a straightforward inducement using properties of determinants, we obtain

$$\tau\,(B_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{4b}{mn+m+4} \times \det \begin{pmatrix}
(2a-n-3) & 0 & (n+1) & \cdots & (n+1) & 0 \\
0 & (2a-n-3) & 0 & \ddots & \cdots & (n+1) \\
(n+1) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (n+1) \\
(n+1) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (n+1) & \cdots & (n+1) & 0 & (2a-n-3)
\end{pmatrix}$$

$$= \frac{2^{mn+2} \times (n+1)^m}{mn+m+4} \times \det \begin{pmatrix} \frac{(2a-n-3)}{(n+1)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-3)}{(n+1)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-3)}{(n+1)} \end{pmatrix}$$

Using Lemma 2, yields

$$\begin{aligned} \tau\,(B_n^{(m)}) &= 2^{m\,n+2} \times \frac{(n+1)^m}{mn+m+4} \times \frac{2(\frac{2a-n-3}{n+1}+m-3)}{\frac{2a-n-3}{n+1}-3} \times [T_m(\frac{\frac{2a-n-3}{n+1}-1}{2}) - 1] \\ &= 2^{mn+1} \times (n+1)^m \times [T_m(\tfrac{n+3}{n+1}) - 1]. \end{aligned}$$

Using Equation (11), yields the result. □

Definition 3. *The pyramid graph* $C_n^{(m)}$ *is the graph created from the gear graph* G_{m+1} *with vertices* $\{u_0;\ u_1, u_2, \ldots, u_m;\ w_1, w_2, \ldots, w_m\}$ *with double external edges and* m *sets of vertices, say,* $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, u_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, *such that for all* $i = 1, 2, \ldots, n$ *the vertex* v_i^j *is adjacent to* u_j *and* u_{j+1}, *where* $j = 1, 2, \ldots, m-1$, *and* v_i^m *is adjacent to* u_1 *and* u_m. *See Figure 3.*

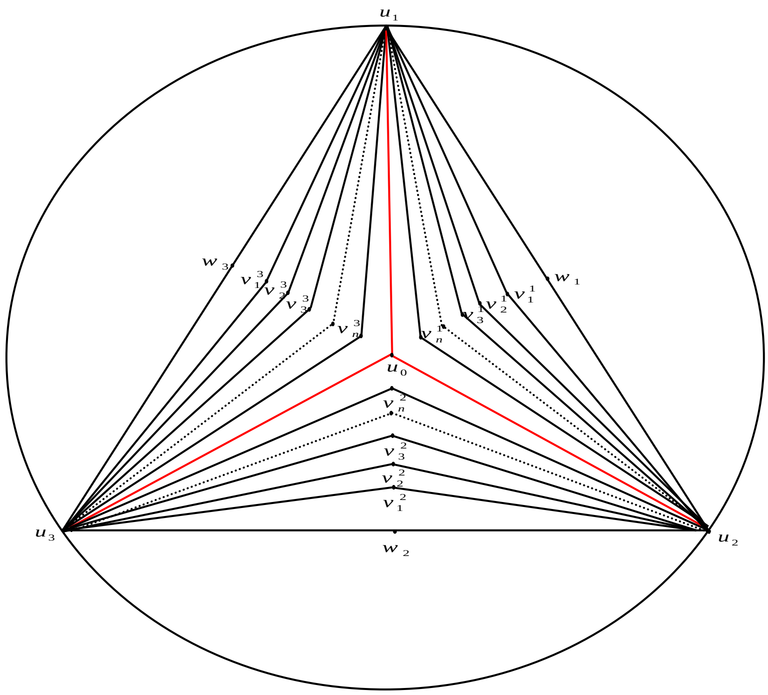

Figure 3. The pyramid graph $C_n^{(3)}$.

Theorem 3. *For* $n \geq 0$, $m \geq 3$, $\tau(C_n^{(m)}) = 2^{mn}[(n+4+\sqrt{2n+7}\,)^m + (n+4-\sqrt{2n+7}\,)^m - 2(n+3)^m]$.

Proof. Using Lemma 1, we have:

$$\tau(C_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)\,I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$
\det\begin{pmatrix}
(m+1) & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & 2(n+3) & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 1 & \cdots & 1 & 0 & 2(n+3) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}
$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n be the $n \times n$ matrix with all one. Set $a = 2n+6$ and $b = mn + 2m + 1$. Then we have:

$$
\tau\left(C_n^{(m)}\right) = \frac{1}{b^2} \times \det\begin{pmatrix}
m+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
0 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b^2} \times \det \begin{pmatrix}
b & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \multicolumn{6}{c}{2I_{mn}+J_{mn}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \multicolumn{6}{c}{2I_{mn}+J_{mn}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 0 & 1 & \cdots & 1 & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 1 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 1 & \cdots & 1 & 0 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \multicolumn{6}{c}{2I_{mn}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
a & 0 & 1 & \cdots & 1 & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
0 & \ddots & \ddots & \ddots & \ddots & 1 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & \cdots & 1 & 0 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

Using Lemma 3, yields

$$
\tau\,(C_n^{(m)}) = \frac{1}{b} \times \det \begin{pmatrix} A & B \\ C & 2I_{mn} \end{pmatrix} = \frac{1}{b} \times \det(A - B\frac{1}{2I_{mn}}\ C) \times 2^{mn}
$$

$$
= \frac{1}{b} \times 2^{mn} \times 2^{-2m} \times \det \begin{pmatrix}
2a & n & 2(n+1) & \cdots & 2(n+1) & n & -2 & 0 & \cdots & \cdots & 0 & -2 \\
n & 2a & n+2 & 2(n+1) & \cdots & 2(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
2(n+1) & n & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2(n+1) & \ddots & \ddots & \ddots & \ddots & n & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
n & 2(n+1) & \cdots & 2(n+1) & n & 2a & 0 & \cdots & \cdots & 0 & -2 & -2 \\
0 & 0 & 2 & \cdots & \cdots & 2 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
$$

Using Lemma 3 again, yields

$$
\tau\,(C_n^{(m)}) = \frac{2^{mn-2m}}{b} \times \det \begin{pmatrix} D & E \\ F & 4I_m \end{pmatrix} = \frac{2^{mn}}{b} \times \det(D - E\frac{1}{4I_m}\ F)
$$

$$
\tau\,(C_n^{(m)}) = \frac{2^{mn}}{b} \times \det \begin{pmatrix}
2a & (n+1) & 2(n+2) & \cdots & 2(n+2) & (n+1) \\
(n+1) & 2a & (n+3) & \ddots & \cdots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & (n+1) \\
(n+1) & 2(n+2) & \cdots & 2(n+2) & (n+1) & 2a
\end{pmatrix}
$$

Using properties of determinants, we have:

$$\tau\,(C_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{2b}{m\,n+3m+2} \times \det \begin{pmatrix} (2a-n-1) & 0 & (n+3) & \cdots & (n+3) & 0 \\ 0 & (2a-n-1) & 0 & \ddots & \cdots & (n+3) \\ (n+3) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+3) \\ (n+3) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+3) & \cdots & (n+3) & 0 & (2a-n-1) \end{pmatrix}$$

$$= \frac{2^{m\,n+1}\,(n+3)^m}{m\,n+3m+2} \times \det \begin{pmatrix} \frac{(2a-n-1)}{(n+3)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-1)}{(n+3)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-1)}{(n+3)} \end{pmatrix}$$

Using Lemma 2, yields:

$$\begin{aligned} \tau\,(C_n^{(m)}) &= 2^{m\,n+1} \times \frac{(n+3)^m}{mn+3m+2} \times \frac{2(\frac{2a-n-1}{n+3}+m-3)}{\frac{2a-n-1}{n+3}-3} \times [T_m(\frac{\frac{2a-n-1}{n+3}-1}{2}) - 1] \\ &= 2^{mn+1} \times (n+3)^m \times [T_m(\tfrac{n+4}{n+3}) - 1]. \end{aligned}$$

Using Equation (11), yields the result. □

Definition 4. *The pyramid graph $D_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0\,;\ u_1, u_2, \ldots, u_m\,;\ w_1, w_2, \ldots, w_m\}$ with double internal and external edges and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, v_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Figure 4.*

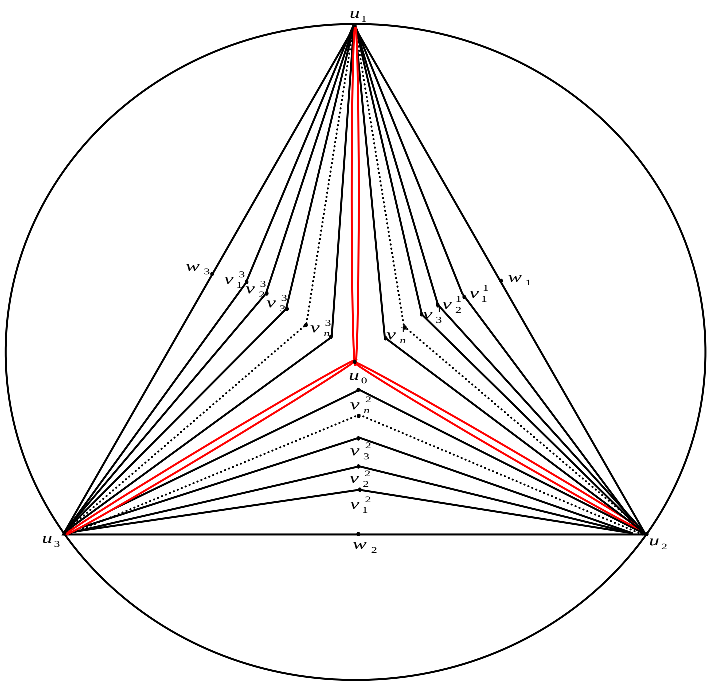

Figure 4. The pyramid graph $D_n^{(3)}$.

Theorem 4. *For* $n \geq 0$, $m \geq 3$, $\tau(D_n^{(m)}) = 2^{mn}[(n+5+2\sqrt{n+4}\,)^m + (n+5-2\sqrt{n+4}\,)^m - 2(n+3)^m]$.

Proof. Applying Lemma 1, we have:

$$\tau(D_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)\, I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det \begin{pmatrix}
(2m+1) & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & (2n+7) & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
-1 & 0 & 1 & \cdots & 1 & 0 & (2n+7) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
1 & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n the $n \times n$ matrix with all one. Set $a = 2n+7$ and $b = mn+2m+1$. Then we have:

$$\tau\left(D_n^{(m)}\right) = \frac{1}{b^2} \det \begin{pmatrix}
2m+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

$$
= \frac{1}{b^2}\det\begin{pmatrix}
b & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\ddots\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (a+1) & 1 & 2 & \cdots & 2 & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 2 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & 2 & \cdots & 2 & 1 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b}\det\begin{pmatrix}
(a+1) & 1 & 2 & \cdots & 2 & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & \ddots & \ddots & \ddots & \ddots & 2 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 2 & \cdots & 2 & 1 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

Using Lemma 3, yields

$$
\tau\,(D_n^{(m)}) = \frac{1}{b}\times\det\begin{pmatrix} A & B \\ C & 2I_{mn} \end{pmatrix} = \frac{1}{b}\times\det(A - B\frac{1}{2I_{mn}}\ C)\times 2^{mn}
$$

$$
= \frac{1}{b}2^{mn}\times 2^{-2m}\times\det\begin{pmatrix}
(2a+2n+2) & 3n+2 & 4(n+1) & \cdots & 4(n+1) & 3n+2 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
3n+2 & (2a+2n+2) & 3n+2 & 4(n+1) & \cdots & 4(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
4(n+1) & 3n+4 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4(n+1) & \ddots & \ddots & \ddots & \ddots & 3n+2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
3n+2 & 4(n+1) & \cdots & 4(n+1) & 3n+2 & (2a+2n+2) & 0 & \cdots & \cdots & 0 & -2 & -2 \\
2 & 2 & 4 & \cdots & \cdots & 4 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
4 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4 & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & 4 & \cdots & \cdots & 4 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
$$

Using Lemma 3, yields

$$
\tau\,(D_n^{(m)}) = \frac{2^{mn-2m}}{b}\times\det\begin{pmatrix} A & B \\ C & 4I_m \end{pmatrix} = \frac{2^{mn}}{b}\times\det(A - B\frac{1}{4I_m}\ C)
$$

$$
\tau\,(D_n^{(m)}) = \frac{2^{mn}}{b}\times\det\begin{pmatrix}
(2a+2n+4) & (3n+5) & 4(n+2) & \cdots & 4(n+2) & (3n+5) \\
(3n+5) & (2a+2n+4) & (3n+5) & \ddots & \cdots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & (3n+5) \\
(3n+5) & 4(n+2) & \cdots & 4(n+2) & (3n+5) & (2a+2n+4)
\end{pmatrix}
$$

Straightforward inducement using properties of determinants, we get:

$$
\tau\,(D_n^{(m)}) = \frac{2^{mn}}{b}\times\frac{4b}{mn+3m+4}\times\det\begin{pmatrix}
(2a-n-1) & 0 & (n+3) & \cdots & (n+3) & 0 \\
0 & (2a-n-1) & 0 & \ddots & \cdots & (n+3) \\
(n+3) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (n+3) \\
(n+3) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (n+3) & \cdots & (n+3) & 0 & (2a-n-1)
\end{pmatrix}
$$

$$
= \frac{2^{m\,n+2}\,(n+3)^m}{m\,n+3m+4} \times \det \begin{pmatrix} \frac{(2a-n-1)}{(n+3)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-1)}{(n+3)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-1)}{(n+3)} \end{pmatrix}
$$

Using Lemma 2, yields:

$$
\tau\,(D_n^{(m)}) = 2^{m\,n+2} \times \frac{(n+3)^m}{mn+3m+4} \times \frac{2(\frac{2a-n-1}{n+3}+m-3)}{\frac{2a-n-1}{n+3}-3} \times [T_m(\frac{\frac{2a-n-1}{n+3}-1}{2}) - 1] = 2^{mn+1} \times (n+3)^m \times [T_m(\frac{n+5}{n+3}) - 1].
$$

Using Equation (11), yields the result. □

4. Numerical Results

The following Table 1 illustrates some values of the number of spanning trees of studied pyramid graphs.

Table 1. Some values of the number of spanning trees of studied pyramid graphs.

m	n	$\tau\,(P_n^{(m)})$	$\tau\,(A_n^{(m)})$	$\tau\,(B_n^{(m)})$	$\tau\,(C_n^{(m)})$
3	0	50	196	242	676
3	1	1024	3200	3136	8192
3	2	15,488	43,264	36,992	92,416
3	3	200,704	524,288	409,600	991,232
3	4	2,367,488	5,914,624	4,333,568	10,240,000
3	5	26,214,400	63,438,848	44,302,336	102,760,448
4	0	192	1152	1792	6400
4	1	11,520	49,152	57,600	184,320
4	2	458,752	1,638,400	1,622,016	4,816,896
4	3	14,745,600	47,185,920	41,746,432	117,440,512
4	4	415,236,096	1,233,125,376	1,006,632,960	2,717,908,992
4	5	10,687,086,592	30,064,771,072	23,102,226,432	60,397,977,600
5	0	722	6724	12,482	58,564
5	1	123,904	739,328	984,064	3,964,928
5	2	12,781,568	59,969,536	65,619,968	237,899,776
5	3	1,007,681,536	4,060,086,272	3,901,751,296	13,088,325,632
5	4	67,194,847,232	243,609,370,624	213,408,284,672	674,448,277,504
5	5	3,995,393,327,104	243,609,370,624	10,953,240,346,624	33,019,708,571,648

5. Conclusions

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The computation of this number is not only beneficial from a mathematical (computational) standpoint, but it is also an important measure of the reliability of a network and electrical circuit layout. Some computationally laborious problems such as the traveling salesman problem can be resolved approximately by using spanning trees. In this paper, we define some classes of pyramid graphs created by a gear graph and we have studied the problem of computing the number of spanning trees of these graphs.

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References

1. Applegate, D.L.; Bixby, R.E.; Chvátal, V.; Cook, W.J. *The Traveling Salesman Problem: A Computational Study*; Princeton University Press: Princeton, NJ, USA, 2006.
2. Cvetkovič, D.; Doob, M.; Sachs, H. *Spectra of Graphs: Theory and Applications*, 3rd ed.; Johann Ambrosius Barth: Heidelberg, Germany, 1995.
3. Kirby, E.C.; Klein, D.J.; Mallion, R.B.; Pollak, P.; Sachs, H. A theorem for counting spanning trees in general chemical graphs and its particular application to toroidal fullerenes. *Croat. Chem. Acta* **2004**, *77*, 263–278.
4. Boesch, F.T.; Satyanarayana, A.; Suffel, C.L. A survey of some network reliability analysis and synthesis results. *Networks* **2009**, *54*, 99–107. [CrossRef]
5. Boesch, F.T. On unreliability polynomials and graph connectivity in reliable network synthesis. *J. Graph Theory* **1986**, *10*, 339–352. [CrossRef]
6. Wu, F.Y. Number of spanning trees on a Lattice. *J. Phys. A* **1977**, *10*, 113–115. [CrossRef]
7. Zhang, F.; Yong, X. Asymptotic enumeration theorems for the number of spanning trees and Eulerian trail in circulant digraphs & graphs. *Sci. China Ser. A* **1999**, *43*, 264–271.
8. Chen, G.; Wu, B.; Zhang, Z. Properties and applications of Laplacian spectra for Koch networks. *J. Phys. A Math. Theor.* **2012**, *45*, 025102.
9. Atajan, T.; Inaba, H. Network reliability analysis by counting the number of spanning trees. In Proceedings of the IEEE International Symposium on Communications and Information Technology, ISCIT 2004, Sapporo, Japan, 26–29 October 2004; pp. 601–604.
10. Brown, T.J.N.; Mallion, R.B.; Pollak, P.; Roth, A. Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes. *Discret. Appl. Math.* **1996**, *67*, 51–66. [CrossRef]
11. Kirchhoff, G.G. Uber die Auflosung der Gleichungen, auf welche man be ider Untersuchung der Linearen Verteilung galvanischer Storme gefuhrt wird. *Ann. Phys. Chem.* **1847**, *72*, 497–508. [CrossRef]
12. Kelmans, A.K.; Chelnokov, V.M. A certain polynomials of a graph and graphs with an extermal number of trees. *J. Comb. Theory B* **1974**, *16*, 197–214. [CrossRef]
13. Biggs, N.L. *Algebraic Graph Theory*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1993; p. 205.
14. Daoud, S.N. The deletion-contraction method for counting the number of spanning trees of graphs. *Eur. Phys. J. Plus* **2015**, *130*, 217. [CrossRef]
15. Shang, Y. On the number of spanning trees, the Laplacian eigenvalues, and the Laplacian Estrada index of subdivided-line graphs. *Open Math.* **2016**, *14*, 641–648. [CrossRef]
16. Bozkurt, Ş.B.; Bozkurt, D. On the Number of Spanning Trees of Graphs. *Sci. World J.* **2014**, *2014*, 294038. [CrossRef] [PubMed]
17. Daoud, S.N. Number of Spanning Trees in Different Product of Complete and Complete Tripartite Graphs. *ARS Comb.* **2018**, *139*, 85–103.
18. Daoud, S.N. Complexity of Graphs Generated by Wheel Graph and Their Asymptotic Limits. *J. Egypt. Math. Soc.* **2017**, *25*, 424–433. [CrossRef]
19. Daoud, S.N. Chebyshev polynomials and spanning tree formulas. *Int. J. Math. Comb.* **2012**, *4*, 68–79.
20. Zhang, Y.; Yong, X.; Golin, M.J. Chebyshev polynomials and spanning trees formulas for circulant and related graphs. *Discret. Math.* **2005**, *298*, 334–364. [CrossRef]

21. Daoud, S.N. On a class of some pyramid graphs and Chebyshev polynomials. *J. Math. Probl. Eng. Hindawi Publ. Corp.* **2013**, *2013*, 820549.
22. Marcus, M. *A Survey of Matrix Theory and Matrix Inequalities*; University Allyn and Bacon. Inc.: Boston, MA, USA, 1964.