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Symmetric Properties of Carlitz's Type q-Changhee Polynomials

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Abstract: Changhee polynomials were introduced by Kim, and the generalizations of these polynomials have been characterized. In our paper, we investigate various interesting symmetric identities for Carlitz's type q-Changhee polynomials under the symmetry group of order n arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$.

Keywords: fermionic p-adic q-integral on $\mathbb{Z}_p$; q-Euler polynomials; q-Changhee polynomials; symmetry group

MSC: 33E20; 05A30; 11B65; 11S05

1. Introduction

For an odd prime number p, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$, respectively, throughout this paper.

The p-adic norm is normalized as $|p|_p = \frac{1}{p}$, and let q be an indeterminate in $\mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$. The q-analogue of number x is defined as

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1}$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \longrightarrow \mathbb{R} \text{ is continuous}\}$. Then, a fermionic p-adic q-integral of $f\left(\in C(\mathbb{Z}_p)\right)$ is defined by Kim as [1–6] :

$$\begin{aligned} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty}\frac{[2]_q}{2}\sum_{x=0}^{p^N-1} f(x)(-q)^x. \end{aligned} \tag{2}$$

On the other hand, it is well known that the Euler polynomial $E_n(x)$ is given by the Appell sequence with $g(t) = \frac{1}{2}\left(e^t + 1\right)$, giving the the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

(see [7–17]). In particular, if $x = 0$, $E_n = E_n(0)$ $(n \in \mathbb{N})$ is called the Euler number.

As a q-analogue of Euler polynomials, the Carlitz's type q-Euler polynomial $\mathcal{E}_{n,q}(x)$ is defined by

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y), \tag{3}$$

(see [2,13–17]). In particular, if $x = 0$, $\mathcal{E}_{n,q} = \mathcal{E}_{n,q}(0)$ is called the q-Euler number.

By (3), the Carlitz's type q-Euler polynomial $\mathcal{E}_{n,q}(x)$ is obtained as

$$\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y), \ (n \geq 0). \tag{4}$$

From the fermionic p-adic q-integral on $\mathbb{Z}_p$, the degenerate q-Euler polynomial $\mathcal{E}_{n,\lambda,q}(x)$ is defined as [16]:

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y). \tag{5}$$

By the binomial expansion of $(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}$, we get

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_{-q}(y)\frac{t^n}{n!}, \tag{6}$$

where $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ for $n \in \mathbb{N}$, and by (5) and (6), we have

$$\mathcal{E}_{n,\lambda,q}(x) = \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_{-q}(y), \ (n \in \mathbb{N}). \tag{7}$$

Since

$$(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1) = \sum_{l=0}^{n} S_1(n,l)\,\alpha^l, \tag{8}$$

$$\begin{aligned}\mathcal{E}_{n,\lambda,q}(x) =& \lambda^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)^l d\mu_{-q}(y) \\ =& \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)\mathcal{E}_{l,q}(x),\end{aligned}$$

where $S_1(n,m)$ is the Stirling number of the first kind (see [2,7,8,12,17,18]).

Now, we apply these polynomials to Changhee polynomials, introduced by Kim et al. [19]. The Changhee polynomial of the first kind $Ch_n(x)$ is defined by the generating function to be

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} =& \int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) \\ =& \frac{2}{2+t}(1+t)^x.\end{aligned} \tag{9}$$

(see [20,21]).

In view point of (3) and (9), Carlitz's type q-Changhee polynomial $Ch_{n,q}(x)$ is defined by

$$\sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y), \tag{10}$$

(see [18,22]).

By the binomial expansion of $(1+t)^{[x+y]_q}$,

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_q\right)_n d\mu_{-q}(y)\frac{t^n}{n!}, \end{aligned} \tag{11}$$

and so the equation (10) and (11) yield the following:

$$Ch_{n,q}(x) = \int_{\mathbb{Z}_p} \left([x+y]_q\right)_n d\mu_{-q}(y), \tag{12}$$

(see [20,21]).

In the past decade, many different generalizations of Changhee polynomials have been studied (see [19,20,22–32]), and the relationship between important combinatorial polynomials and those polynomials was found.

Symmetric identities of special polynomials are important and interesting in number theory, pure and applied mathematics. Symmetric identities of many different polynomials were investigated in [5,10,14,16,32–39]. In particular, C. Cesarano [40] presented some techniques regarding the generating functions used, and these identities can be applicable to the theory of porous materials [41].

In current paper, we construct symmetric identities for the Carlitz's type q-Changhee polynomials under the symmetry group of order n arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$, and the proof methods which was used in the Kim's previous researches are also used as good tools in this paper (see [5,10,14,16,32–39]).

2. Symmetric Identities for the Carlitz's Type q-Changhee Polynomials

Let $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, and let S_n be the symmetry group of degree n. For positive integers $w_1, w_2, \ldots, w_n$ with $w_i \equiv 1 \pmod 2$ for each $i = 1, 2, \ldots n$, we consider the following integral equation for the fermionic p-adic q-integral on $\mathbb{Z}_p$;

$$\begin{aligned} &\int_{\mathbb{Z}_p} (1+t)^{\left[\left(\prod_{i=1}^{n-1} w_i\right)y + \left(\prod_{i=1}^{n} w_i\right)x + w_n \sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1} w_j\right)k_i\right]_q} d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y) \\ &= \frac{[2]_{q^{w_1\cdots w_{n-1}}}}{2} \lim_{N\to\infty} \sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} (1+t)^{\left[\left(\prod_{i=1}^{n-1} w_i\right)(m+w_ny) + \left(\prod_{i=1}^{n} w_i\right)x + w_n \sum_{i=1}^{n}\left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} w_j\right)k_i\right]_q} \\ &\quad \times (-1)^{m+w_ny} q^{w_1w_2\cdots w_{n-1}(m+w_ny)}. \end{aligned} \tag{13}$$

From (13), we get

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_1 w_2\cdots w_{n-1}}}}\prod_{m=1}^{n-1}\sum_{k_m=0}^{w_m-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}(1+t)^{\left[(\prod_{i=1}^{n-1}w_i)y+(\prod_{i=1}^{n}w_i)x+w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\j\neq i}}^{n-1}w_j\right)k_i\right]_q}d\mu_{-q^{w_1 w_2\cdots w_{n-1}}}(y)\\
&=\lim_{N\to\infty}\prod_{m=1}^{n-1}\sum_{k_m=0}^{w_m-1}\sum_{l=0}^{w_n-1}\sum_{y=0}^{p^N-1}(1+t)^{\left[(\prod_{i=1}^{n-1}w_i)(m+w_n y)+(\prod_{i=1}^{n}w_i)x+w_n\sum_{i=1}^{n}\left(\sum_{\substack{j=1\\j\neq i}}^{n-1}w_j\right)k_i\right]_q}\\
&\times(-1)^{\sum_{i=1}^{n-1}k_i+l+y}q^{\left(\sum_{j=1}^{n-1}\right)l+\left(\prod_{j=1}^{n}w_j\right)y+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}w_i\right)k_i}.
\end{aligned}\tag{14}$$

If we put

$$\begin{aligned}
F(w_1,w_2,\ldots,w_n)=&\frac{2}{[2]_{q^{w_1 w_2\cdots w_{n-1}}}}\prod_{m=1}^{n-1}\sum_{k_m=0}^{w_m-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}(1+t)^{\left[(\prod_{i=1}^{n-1}w_i)y+(\prod_{i=1}^{n}w_i)x+w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\j\neq i}}^{n-1}w_j\right)k_i\right]_q}d\mu_{-q^{w_1 w_2\cdots w_{n-1}}}(y),
\end{aligned}\tag{15}$$

then, by (14), we know that $F(w_1,w_2,\ldots,w_n)$ is invariant for any permutation $\sigma\in S_n$.

Hence, by (14) and (15), we obtain the following theorem.

Theorem 1. *Let $w_1,w_2,\ldots,w_n$ be positive odd integers. For any $\sigma\in S_n$, $F(w_{\sigma(1)},w_{\sigma(2)},\ldots,w_{\sigma(n)})$ have the same value.*

By (1), we know that

$$\left[\prod_{i=1}^{n-1}w_i\right]_q\left[y+w_n x+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right]_{q^{w_1 w_2\cdots w_{n-1}}}=\left[\left(\prod_{i=1}^{n-1}w_i\right)y+\left(\prod_{i=1}^{n}w_i\right)x+w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\j\neq i}}^{n-1}w_j\right)k_i\right]_q.\tag{16}$$

From (5) and (16), we derive the following identities.

$$
\begin{aligned}
&\int_{\mathbb{Z}_p}(1+t)^{\left[(\prod_{i=1}^{n-1}w_i)y+(\prod_{i=1}^{n}w_i)x+w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_q}d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)\\
=&(1+t)^{[\prod_{i=1}^{n-1}w_i]_q}\int_{\mathbb{Z}_p}(1+t)^{\left[y+w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right]_{q^{w_1w_2\cdots w_{n-1}}}}d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)\\
=&\left(\sum_{l=0}^{\infty}\binom{\left[\prod_{i=1}^{n-1}w_i\right]_q}{l}t^l\right)\left(\sum_{m=0}^{\infty}Ch_{m,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right)\frac{t^m}{m!}\right)\\
=&\sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}\left(\left[\prod_{i=1}^{n-1}w_i\right]_q\right)_{m-r}\binom{m}{r}Ch_{r,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right)\right)\frac{t^m}{m!},
\end{aligned}
\tag{17}
$$

for each positive integer n. Thus, by Theorem 1 and (17), we obtain the following corollary.

Corollary 1. *Let $w_1,w_2,\ldots,w_n$ be positive integers with $w_i\equiv 1 \pmod 2$ for each $i=1,2,\ldots,n$, and let m be a nonnegative integer. Then, for any permutation $\tau\in S_n$,*

$$
\begin{aligned}
&\frac{2}{[2]_{q^{w_{\tau(1)}w_{\tau(2)}\cdots w_{\tau(n)}}}}\sum_{r=0}^{m}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\tau(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_{\tau(n)}\sum_{i=1}^{n-1}\left(\sum_{\substack{j=1\\ j\neq i}}^{n-1}w_{\tau(j)}\right)k_i}\\
&\times\left(\left[\prod_{i=1}^{n-1}w_{\tau(i)}\right]_q\right)_{m-r}\binom{m}{r}Ch_{r,q^{w_{\tau(1)}w_{\tau(2)}\cdots w_{\tau(n-1)}}}\left(w_{\tau(n)}x+w_{\tau(n)}\sum_{i=1}^{n-1}\frac{k_i}{w_{\tau(i)}}\right)
\end{aligned}
$$

have the same expressions.

Note that, by the definition of $[x]_q$,

$$
\begin{aligned}
&\left[y+w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right]_{q^{w_1w_2\cdots w_{n-1}}}\\
=&\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}+q^{w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}.
\end{aligned}
\tag{18}
$$

By (12), we get

$$
\begin{aligned}
&Ch_{m,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right)\\
=&\int_{\mathbb{Z}_p}\left(\left[y+w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right]_{q^{w_1w_2\cdots w_{n-1}}}\right)_m d\mu_{-q^{w_1w_2\cdots w_{n-1}}},
\end{aligned}
\tag{19}
$$

and by (8) and (18),

$$
\begin{aligned}
&\left(\left[y+w_n x+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right]_{q^{w_1w_2\cdots w_{n-1}}}\right)_m\\
&=\left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}+q^{w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}\right)_m\\
&=\sum_{l=0}^{m}S_1(m,l)\left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}+q^{w_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}[y+w_nx]_{q^{w_1w_2\cdots w_{n-1}}}\right)^l \qquad (20)\\
&=\sum_{l=0}^{m}S_1(m,l)\sum_{i=1}^{l}\binom{l}{i}\left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\right)^{l-i}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}^{l-i}\\
&\quad\times q^{iw_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}[y+w_nx]^i_{q_1^{w}w_2\cdots w_{n-1}}.
\end{aligned}
$$

From (4), (19) and (20), we have

$$
\begin{aligned}
&Ch_{m,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+w_n\sum_{i=1}^{n-1}\frac{k_i}{w_i}\right)\\
&=\sum_{l=0}^{m}S_1(m,l)\sum_{i=1}^{l}\binom{l}{i}\left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\right)^{l-i}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}^{l-i}\\
&\quad\times q^{iw_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}\int_{\mathbb{Z}_p}[y+w_nx]^i_{q^{w_1w_2\cdots w_{n-1}}}\,d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y) \qquad (21)\\
&=\sum_{l=0}^{m}S_1(m,l)\sum_{i=1}^{l}\binom{l}{i}\left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1}w_i\right]_q}\right)^{l-i}\left[\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i\right]_{q^{w_n}}^{l-i}\\
&\quad\times q^{iw_n\sum_{i=1}^{n-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n-1}w_j\right)k_i}Ch_{i,q^{w_1w_2\cdots w_{n-1}}}(w_nx).
\end{aligned}
$$

From (21), we have

$$
\begin{aligned}
&\frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}} \sum_{r=0}^{m} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\sum_{i=1}^{n-1} k_i} q^{w_n \sum_{i=1}^{n-1} \left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i} \\
&\times \left(\left[\prod_{i=1}^{n-1} w_i \right]_q \right)_{m-r} \binom{m}{r} Ch_{r,q^{w_1 w_2 \cdots w_{n-1}}} \left(w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right) \\
=&\frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}} \sum_{r=0}^{m} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\sum_{i=1}^{n-1} k_i} q^{w_n \sum_{i=1}^{n-1} \left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i} \left(\left[\prod_{i=1}^{n-1} w_i \right]_q \right)_{m-r} \binom{m}{r} \\
&\times \sum_{p=0}^{r} S_1(r,p) \sum_{i=1}^{l} \binom{l}{i} \left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1} w_i \right]_q} \right)^{p-i} \left[\sum_{i=1}^{n-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}}^{p-i} \\
&\times q^{i w_n \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i} Ch_{i,q^{w_1 w_2 \cdots w_{n-1}}}(w_n x) \\
=&\sum_{r=0}^{m} \sum_{l=0}^{r} \sum_{i=1}^{l} S_1(r,l) \binom{m}{r} \binom{l}{i} \left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1} w_i \right]_q} \right)^{p-i} \left(\left[\prod_{i=1}^{n-1} w_i \right]_q \right)_{m-r} Ch_{i,q^{w_1 w_2 \cdots w_{n-1}}}(w_n x) \\
&\times \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\sum_{i=1}^{n-1} k_i} q^{(1+i) w_n \sum_{i=1}^{n-1} \left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i} \left[\sum_{i=1}^{n-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}}^{p-i} \\
=&\sum_{r=0}^{m} \sum_{l=0}^{r} \sum_{i=1}^{l} S_1(r,l) \binom{m}{r} \binom{l}{i} \\
&\times \left(\frac{[w_n]_q}{\left[\prod_{i=1}^{n-1} w_i \right]_q} \right)^{p-i} \left(\left[\prod_{i=1}^{n-1} w_i \right]_q \right)_{m-r} Ch_{i,q^{w_1 w_2 \cdots w_{n-1}}}(w_n x) F_{n,q^{w_n}}(w_1, \ldots, w_{n-1} | i+1),
\end{aligned}
\tag{22}
$$

where

$$
F_{n,q}(w_1, \ldots, w_{n-1} | i) = \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\sum_{i=1}^{n-1} k_i} q^{i \sum_{i=1}^{n-1} \left(\sum_{\substack{j=1\\ j\neq i}}^{n-1} w_j \right) k_i} \left[\sum_{t=1}^{n-1} \left(\prod_{\substack{j=1\\ j\neq t}}^{n-1} w_j \right) k_t \right]_q^{p-i-1}
$$

Theorem 2. *For each nonnegative odd integers $w_1, w_2, \ldots, w_n$ and for any permutation σ in the symmetry group of degree n, the expressions*

$$
\begin{aligned}
&\sum_{r=0}^{m} \sum_{l=0}^{r} \sum_{i=1}^{l} S_1(r,l) \binom{m}{r} \binom{l}{i} \left(\frac{[w_{\sigma(n)}]_q}{\left[\prod_{i=1}^{n-1} w_{\sigma(i)} \right]_q} \right)^{p-i} \left(\left[\prod_{i=1}^{n-1} w_{\sigma(i)} \right]_q \right)_{m-r} \\
&\times Ch_{i,q^{w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}}} \left(w_{\sigma(n)} x \right) F_{n,q^{w_{\sigma(n)}}} \left(w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} | i+1 \right)
\end{aligned}
$$

have the same.

3. Conclusion

The Changhee numbers are closely related with the Euler numbers, the Stirling numbers of the first kind and second kind and the harmonic numbers, and so on. Throughout this paper, we investigate that the function $F(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n)})$ for the Carlitz's type q-Changhee polynomials is invariant under the symmetry group $\sigma \in S_n$. From the invariance of $F(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n)})$, $\sigma \in S_n$, we construct symmetric identities of the Carlitz's type q-Changhee polynomials from the fermionic p-adic q-integral on $\mathbb{Z}_p$. As Bernoulli and Euler polynomials, our properties on the Carlitz's type q-Changhee polynomials play an crucial role in finding identities for numbers in algebraic number theory.

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