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Methods for Multiple Attribute Group Decision Making Based on Intuitionistic Fuzzy Dombi Hamy Mean Operators

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Abstract: In this paper, we extended the Hamy mean (HM) operator, the Dombi Hamy mean (DHM) operator, the Dombi dual Hamy mean (DDHM), with the intuitionistic fuzzy numbers (IFNs) to propose the intuitionistic fuzzy Dombi Hamy mean (IFDHM) operator, intuitionistic fuzzy weighted Dombi Hamy mean (IFWDHM) operator, intuitionistic fuzzy Dombi dual Hamy mean (IFDDHM) operator, and intuitionistic fuzzy weighted Dombi dual Hamy mean (IFWDDHM) operator. Following this, the multiple attribute group decision-making (MAGDM) methods are proposed with these operators. To conclude, we utilized an applicable example for the selection of a car supplier to prove the proposed methods.

Keywords: multiple attribute group decision making (MAGDM); the intuitionistic fuzzy numbers; intuitionistic fuzzy sets (IFSs); IFDHM operator; IFWDHM operator; IFDDHM operator; IFWDDHM operator

1. Introduction

Multiple attribute decision making (MADM) is a key branch of decision theory. The definition of intuitionistic fuzzy sets (IFSs) [1,2] has been utilized to deal with uncertainty and imprecision. The introduction of intuitionistic fuzzy entropy by Burillo and Bustince [3] caught the attention of researchers. Xu [4,5] developed a number of aggregation operators with intuitionistic fuzzy numbers (IFNs.) Xu [6] defined the intuitionistic preference relations for multiple attribute group decision making (MAGDM). Li [7] proposed the Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) models for MADM. Xu [8] developed the Choquet integrals of weighted IFNs. Ye [9] gave some Cosine similarity measures for intuitionistic fuzzy sets (IFSs). Li and Ren [10] considered the amount and reliability of IFNs for MADM. Wei [11] proposed some induced geometric aggregation operators with IFNs. Wei and Zhao [12] gave some induced correlated aggregating operators with IFNs. Wei [13,14] developed the gray relational analysis method for MADM with IFNs. Zhao and Wei [15] defined some Einstein hybrid aggregation operators with IFNs. Garg [16] proposed the generalized interactive geometric interaction operators using Einstein T-norm and T-conorm with IFNs. Chu et al. [17] gave a MAGDM model that considered both the additive consistency and group consensus with IFNs. Wan et al. [18] researched a novel risk attitudinal ranking method for MADM with IFNs. Zhao et al. [19] proposed the VIKOR (VIseKriterijumska Optimizacija I KOmpromisno Resenje) method using IFSs. Liu [20] proposed MADM methods with normal intuitionistic fuzzy interaction operators. Shi [21] developed some constructive methods for intuitionistic fuzzy implication operators. Otay et al. [22] studied the multi-expert performance evaluation of healthcare institutions with intuitionistic fuzzy Analytic Hierarchy Process (AHP) and Data Envelopment Analysis (DEA) methodology. Ai and Xu [23] proposed the multiple definite integrals of intuitionistic fuzzy calculus and isomorphic mappings. Montes et al. [24] defined the entropy measures for IFNs based on divergence. Liu et al. [25] evaluated the commercial bank counterparty credit risk management with IFNs. Some similarity measures and information aggregating operators between intuitionistic fuzzy sets [26–41] and their extension [42–53] have been proposed.

Dombi [54] proposed the operations of the Dombi T-norm and T-conorm. Following this, Liu et al. [55] proposed the Dombi operations with IFNs. Chen and Ye [56] proposed the Dombi weighted arithmetic average and geometric average fuse the single-valued neutrosophic numbers (SVNNs). Wei and Wei [57] gave some dombi prioritized weighting aggregation operators with single-valued neutrosophic numbers.

Through existing studies, we can see that the combination Hamy mean (HM) operator [58,59] and Dombi operations are not extended to IFNs so far. In order to develop Hamy mean operators and Dombi operations for IFNs, the main purposes of this study are (1) to develop some Dombi Hamy mean aggregating operators for IFNs and to investigate their properties, and (2) to propose two models to solve the MADM problems based on these operators with IFNs.

To do so, the rest of this paper is organized as follows. In the next section, we introduce some basic concepts of IFSs, Dombi operations and HM operators. In section three we propose some intuition fuzzy Hamy mean operators based on Dombi T-norm and T-conorm. In section four, we have applied these operators to solve the MAGDM problems with IFNs. In section five, a practical example for the selection of a car supplier is given. In section six, we conclude the paper and give some remarks.

2. Preliminaries

In this section, we introduce the concept of IFS, HM operator, and Dombi T-conorm and T-norm.

2.1. Intuitionistic Fuzzy Sets

Definition 1. *Let X be a fixed set, with a generic in X denoted by x. An intuitionistic fuzzy set (IFS) I in X is following [1,2]:*

$$I = \{\langle x, \mu_i(x), \nu_i(x)\rangle x \in X\} \tag{1}$$

where $\mu_i(x)$ is the membership function, and $\nu_i(x)$ is the non-membership function. For each point x in X, we have $\mu_i(x), \nu_i(x) \in [0,1]$ and $0 \le \mu_i(x) + \nu_i(x) \le 1$.

For each IFS I in X, let $\pi_i(x) = 1 - \mu_i(x) - \nu_i(x)$, $\forall x \in X$, and we call $\pi_i(x)$ the indeterminacy degree of the element x to the set I. It can be easily proved that $0 \le \pi_i(x) \le 1$, $\forall x \in X$. For convenience, we call $k = (\mu_k, \nu_k)$ an IFN, where $\mu_k \in [0,1]$, $\nu_k \in [0,1]$, and $0 \le \mu_k + \nu_k \le 1$.

Definition 2. *Let $k_1 = (\mu_1, \nu_1)$ and $k_2 = (\mu_2, \nu_2)$ be two IFNs, then operational laws are defined [4,5].*

1. $k_1 \oplus k_2 = (\mu_1 + \mu_2 - \mu_1\mu_2, \nu_1\nu_2)$
2. $k_1 \otimes k_2 = (\mu_1\mu_2, \nu_1 + \nu_2 - \nu_1\nu_2)$
3. $\lambda k_1 = \left(1 - (1-\mu_1)^\lambda, \nu_1^\lambda\right),\ \lambda > 0$
4. $k_1^\lambda = \left(\mu_1^\lambda, 1 - (1-\nu_1)^\lambda\right),\ \lambda > 0$

Example 1. *Suppose that $k_1 = (0.40.5)$, $k_2 = (0.60.4)$, and $\lambda = 4$, then we have*

1. $k_1 \oplus k_2 = (0.4 + 0.6 - 0.4 \times 0.60.5 \times 0.4) = (0.76000.2000)$
2. $k_1 \otimes k_2 = (0.4 \times 0.60.5 + 0.4 - 0.5 \times 0.4) = (0.24000.7000)$
3. $\lambda k_1 = \left(1 - (1-0.4)^4, 0.5^4\right) = (0.8704, 0.0625)$
4. $k_1^\lambda = \left(0.4^4, 1 - (1-0.5)^4\right) = (0.0256, 0.9375)$

Definition 3. *Let $k = (\mu_k, \nu_k)$ be an IFN, then a score function is [60]:*

$$S(k) = \mu_k - \nu_k \tag{2}$$

where $S(k) \in [-1, 1]$, from (2), we can give the comparison method of IFNs on the basis of the above score function. For the difference $\mu_k - \nu_k$, the larger the $S(k)$ is, the greater the IFN k is.

Example 2. *Let $k_1 = (0.5\ 0.4), k_2 = (0.6\ 0.2)$ be two IFNs, we can get the scores of k_1 and k_2. $S(k_1) = 0.5 - 0.4 = 0.1$, $S(k_2) = 0.6 - 0.2 = 0.4$, since $S(k_2) > S(k_1)$, we get $k_2 > k_1$.*

Definition 4. *Let $k = (\mu_k, \nu_k)$ be an IFN, then an accuracy function H of k can be defined as follows [61]:*

$$H(k) = \mu_k + \nu_k \tag{3}$$

where $H(k) \in [0, 1]$, for the difference $\mu_k + \nu_k$, the larger the $H(k)$ is, the greater the IFN k is.

Xu and Yager [5] develop a comparison method of IFNs.

Definition 5. *Let $k_1 = (\mu_1, \nu_1)$ and $k_2 = (\mu_2, \nu_2)$ be two IFNs, $S(k_1)$ and $S(k_2)$ are the score function of k_1 and k_2 respectively, $H(k_1)$ and $H(k_2)$ are the score function of k_1 and k_2 respectively. Then,*

(1) If $S(k_1) > S(k_2)$, then $k_1 > k_2$;
(2) If $S(k_1) = S(k_2)$, then
(3) If $H(k_1) > H(k_2)$, then $k_1 > k_2$;
(4) If $H(k_1) = H(k_2)$, then $k_1 = k_2$.

Example 3. *Let $k_1 = (0.6, 0.3), k_2 = (0.4, 0.1)$ be two IFNs, we can get the scores and the accuracy of k_1 and k_2. $S(k_1) = 0.6 - 0.3 = 0.3$, $S(k_2) = 0.4 - 0.1 = 0.3$. Since $S(k_1) = S(k_2)$, we can't get the difference of k_1 and k_2, then $H(k_1) = 0.6 + 0.3 = 0.9$, $H(k_2) = 0.4 + 0.1 = 0.5$, since $H(k_1) > H(k_2)$, we can get $k_1 > k_2$.*

2.2. HM Operator

Definition 6. *The HM operator is defined as follows [58]:*

$$\mathrm{HM}^{(x)}(k_1, k_2, \cdots k_n) = \frac{\sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(\prod\limits_{j=1}^{x} k_{i_j} \right)^{\frac{1}{x}}}{C_n^x} \tag{4}$$

where x is a parameter and $x = 1, 2, \cdots, n, i_1, i_2, \cdots i_x$ are x integer values taken from the set $\{1, 2, \cdots, n\}$ of k integer values, C_n^x denotes the binomial coefficient and $C_n^x = \frac{n!}{x!(n-x)!}$. The properties of the operator are shown as follows:

(i) When $k_i = k(i = 1, 2, \cdots, n), \mathrm{HM}^{(x)}(k_1, k_2, \cdots k_n) = k$;
(ii) When $k_i \le \pi_i(i = 1, 2, \cdots, n), \mathrm{HM}^{(x)}(k_1, k_2, \cdots k_n) \le \mathrm{HM}^{(x)}(\pi_1, \pi_2, \cdots \pi_n)$;
(iii) When $\min\{k_i\} \le \mathrm{HM}^{(x)}(k_1, k_2, \cdots k_n) \le \max\{k_i\}$.

Two particular cases of the HM operator are given as follows.

(i) When $x = 1$, $\mathrm{HM}^{(1)}(a_1, a_2, \cdots a_n) = \frac{1}{n}\sum\limits_{i=1}^{n} k_i$, it becomes the arithmetic mean operator.
(ii) When $x = k$, $\mathrm{HM}^{(k)}(k_1, k_2, \cdots k_k) = \left(\prod\limits_{i=1}^{n} k_i \right)^{\frac{1}{n}}$, it becomes the geometric mean operator.

2.3. Dombi T-Conorm and T-Norm

Dombi operations involve the Dombi product and Dombi sum, which are special cases of T-norms and T-conorms, respectively.

Definition 7. *Suppose $M = \{\langle x, \mu_M(x), \nu_M(x)\rangle\}$ and $N = \{\langle x, \mu_N(x), \nu_N(x)\rangle\}$ are any two IFNs, then the generalized intersection and generalized union are proposed as follows [54]:*

$$M \cap_{T,T^*} N = \{\langle x, T(\mu_M(x), \mu_N(x)), T^*(\nu_M(x), \nu_N(x))\rangle x \in X\} \tag{5}$$

$$M \cup_{T,T^*} N = \{\langle x, T^*(\mu_M(x), \mu_N(x)), T(\nu_M(x), \nu_N(x))\rangle x \in X\} \tag{6}$$

where T denotes a T-norm and T^* denotes a T-conorm.

Dombi proposed a generator to produce Dombi T-norm and T-conorm which are shown as follows.

$$T_{D,\lambda}(x,y) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^{\lambda} + \left(\frac{1-y}{y}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \tag{7}$$

$$T^*_{D,\lambda}(x,y) = 1 - \frac{1}{1 + \left(\left(\frac{x}{1-x}\right)^{\lambda} + \left(\frac{y}{1-y}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \tag{8}$$

where $\lambda > 0$, $x, y \in [0,1]$.

Based on the Dombi T-norm and T-conorm, we can give the operational rules of IFNs as follows. Suppose $k_1 = (\mu_1, \nu_1)$ and $k_2 = (\mu_2, \nu_2)$ are any two IFNs, then operational laws of IFNs based on the Dombi T-norm and T-conorm can be defined as follows ($\lambda > 0$):

1. $k_1 \oplus k_2 = \left(1 - \frac{1}{1+\left(\left(\frac{\mu_1}{1-\mu_1}\right)^{\lambda}+\left(\frac{\mu_2}{1-\mu_2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\left(\frac{1-\nu_1}{\nu_1}\right)^{\lambda}+\left(\frac{1-\nu_2}{\nu_2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right)$
2. $k_1 \otimes k_2 = \left(\frac{1}{1+\left(\left(\frac{1-\mu_1}{\mu_1}\right)^{\lambda}+\left(\frac{1-\mu_2}{\mu_2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\left(\frac{\nu_1}{1-\nu_1}\right)^{\lambda}+\left(\frac{\nu_2}{1-\nu_2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right)$
3. $nk_1 = \left(1 - \frac{1}{1+\left(n\left(\frac{\mu_1}{1-\mu_1}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(n\left(\frac{1-\nu_1}{\nu_1}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) (n > 0)$
4. $k_1^n = \left(\frac{1}{1+\left(n\left(\frac{1-\mu_1}{\mu_1}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(n\left(\frac{\nu_1}{1-\nu_1}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) (n > 0)$

Example 4. *Suppose that $k_1 = (0.6, 0.1)$, $k_2 = (0.7, 0.3)$, and $\lambda = 2$, $n = 3$, then we have*

(1) $k_1 \oplus k_2 = \left(\begin{array}{c} 1 - \frac{1}{1+\left(\left(\frac{0.6}{1-0.6}\right)^{2}+\left(\frac{0.7}{1-0.7}\right)^{2}\right)^{\frac{1}{2}}}, \\ \frac{1}{1+\left(\left(\frac{1-0.1}{0.1}\right)^{2}+\left(\frac{1-0.3}{0.3}\right)^{2}\right)^{\frac{1}{2}}} \end{array}\right) = (0.7350, 0.0384)$

(2) $k_1 \otimes k_2 = \left(\frac{1}{1+\left(\left(\frac{1-0.6}{0.6}\right)^{2}+\left(\frac{1-0.7}{0.7}\right)^{2}\right)^{\frac{1}{2}}}, 1 - \frac{1}{1+\left(\left(\frac{0.1}{1-0.1}\right)^{2}+\left(\frac{0.3}{1-0.3}\right)^{2}\right)^{\frac{1}{2}}}\right) = (0.5579, 0.3069)$

(3) $nk_1 = \left(1 - \frac{1}{1+\left(3\times\left(\frac{0.6}{1-0.6}\right)^2\right)^{\frac{1}{2}}}, \frac{1}{1+\left(3\times\left(\frac{1-0.1}{0.1}\right)^2\right)^{\frac{1}{2}}}\right) = (0.7221, 0.0603)$

(4) $k_1^n = \left(\frac{1}{1+\left(3\times\left(\frac{1-0.6}{0.6}\right)^2\right)^{\frac{1}{2}}}, 1 - \frac{1}{1+\left(3\times\left(\frac{0.1}{1-0.1}\right)^2\right)^{\frac{1}{2}}}\right) = (0.4641, 0.1613)$

3. Intuition Fuzzy Hamy Mean Operators Based on Dombi T-Norm and T-Conorm

In this section, we propose the intuitionistic fuzzy Dombi Hamy mean (IFDHM) operator and intuitionistic fuzzy weighted Dombi Hamy mean (IFWDHM) operator.

3.1. The IFDHM Operator

Definition 8. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right) (i = 1, 2, \cdots, n)$ *be a collection of IFNs, then we can define IFDHM operator as follows:*

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots k_n) = \frac{1}{C_n^x}\left(\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}}\right) \tag{9}$$

where x *is a parameter and* $x = 1, 2, \cdots, n, i_1, i_2, \cdots i_x$, *are* x *integer values taken from the set* $\{1, 2, \cdots, n\}$ *of* n *integer values,* C_n^x *denotes the binomial coefficient and* $C_n^x = \frac{n!}{x!(n-x)!}$.

Theorem 1. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right) (i = 1, 2, \cdots, n)$ *be a collection of IFNs, then the aggregate result of Definition 8 is still an IFN, and have*

$$\begin{aligned}
&\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots k_n) = \tfrac{1}{C_n^x}\left(\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}}\right)\\
&= \left(1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)
\end{aligned} \tag{10}$$

Proof.

1. First of all, we prove (10) is kept. According to the operational laws of IFNs, we have

$$\overset{x}{\underset{j=1}{\otimes}} k_{i_j} = \left(\frac{1}{1+\left(\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{11}$$

$$\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}} = \left(\frac{1}{1+\left(\frac{1}{x}\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{x}\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \right) \tag{12}$$

Moreover,

$$\begin{aligned} &\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}} \\ &= \left(1-\frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right) \end{aligned} \tag{13}$$

Furthermore,

$$\begin{aligned} &\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots k_n) = \frac{1}{C_n^x}\left(\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}}\right) \\ &= \left(1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right) \end{aligned} \tag{14}$$

2. Next, we prove (10) is an IFN.

   Let

$$a = 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, b = \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

   Then we need to prove that the following two conditions which are satisfied, (i) $0 \le a \le 1, 0 \le b \le 1$;

   (ii) $0 \le a + b \le 1$.

(i) Since $\mu_{i_j} \in [0,1]$, we can get

$$1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}} \ge 1 \Rightarrow 1/\left(1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}\right) \in [0,1]$$

$$\Rightarrow 1-1/\left(1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}\right) \in [0,1]$$

Therefore, $0 \le a \le 1$. Similarly, $0 \le b \le 1$.

(ii) Obviously, $0 \le a+b \le 1$, then

$$1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

$$\le 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=1$$

We get $0 \le a+b \le 1$, so the aggregated result of Definition 8 is still an IFN. Next we will discuss about some of the properties of the IFDHM operator. □

Property 1 (Idempotency). *If $k_i(1,2\cdots,n)$ and k are IFNs, and $k_i = k = (\mu_i,\nu_i)$ for all $i = 1,2\cdots,n$, then we get*

$$\mathrm{IFDHM}^{(x)}(k_1,k_2,\cdots k_n) = k \tag{15}$$

Proof. Since $k = (\mu,\nu)$, based on Theorem 1, we have

$$\mathrm{IFDHM}^{(x)}(k_1,k_2,\cdots k_n) = \frac{1}{C_n^x}\left(\bigoplus_{1\le i_1<\cdots<i_x\le n}\left(\bigotimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}\right)$$

$$=\left(1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$

$$= \left(1 - \frac{1}{1 + \left(\frac{1}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\left(\frac{1-\mu_i}{\mu_i} \right)^\lambda} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{1}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\left(\frac{\nu_i}{1-\nu_i} \right)^\lambda} \right)^{\frac{1}{\lambda}}} \right)$$

$$= \left(1 - \frac{1}{1 + \left(\frac{1}{\left(\frac{1-\mu_i}{\mu_i} \right)^\lambda} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{1}{\left(\frac{\nu_i}{1-\nu_i} \right)^\lambda} \right)^{\frac{1}{\lambda}}} \right) = \left(1 - \frac{1}{1 + \frac{1}{\frac{1-\mu_i}{\mu_i}}}, \frac{1}{1 + \frac{1}{\frac{\nu_i}{1-\nu_i}}} \right)$$

$$= (\mu_i, \nu_i) = (\mu, \nu) = k$$

□

Property 2 (Monotonicity). *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right) (i = 1, 2, \cdots, n)$ *be two sets of IFNs. If* $\mu_{i_j} \ge \mu_{\theta_j}, \nu_{i_j} \le \nu_{\theta_j}$*, for all j, then*

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) \ge \mathrm{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{16}$$

Proof. Since $x \ge 1, \mu_{i_j} \ge \mu_{\theta_j} \ge 0, \nu_{\theta_j} \ge \nu_{i_j} \ge 0$, then

$$\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda \le \sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda \Rightarrow 1/\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda \ge 1/\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda$$

$$\Rightarrow \frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} 1/\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda \ge \frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} 1/\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda$$

$$\Rightarrow 1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}} \ge 1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}}} \le \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}}}$$

$$\Rightarrow 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}}} \ge 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^\lambda} \right)^{\frac{1}{\lambda}}}$$

Similarly, we have

$$\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \leq \sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \geq 1/\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}$$

$$\Rightarrow \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \geq \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}$$

$$\Rightarrow 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}} \geq 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \leq \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Let $k = \text{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n)$, $\pi = \text{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$ and $S(k), S(\pi)$ be the score values of k and π respectively. Based on the score value of IFN in (2) and the above inequality, we can imply that $S(k) \geq S(\pi)$, and then we discuss the following cases:

(1) If $S(k) > S(\pi)$, then we can get $\text{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) > \text{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$.
(2) If $S(k) = S(\pi)$, then

$$1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$
$$=1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Since $\mu_{i_j} \geq \mu_{\theta_j} \geq 0, \nu_{\theta_j} \geq \nu_{i_j} \geq 0$, we can deduce that

$$1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

and

$$\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Therefore, it follows that

$$H(k) = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$
$$= 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} = H(\pi)$$

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$$

□

Property 3 (Boundedness). *Let* $k_i = (\mu_{i_j}, \nu_{i_j}), k^+ = (\mu_{\max i_j}, \nu_{\max i_j})(i = 1, 2, \cdots, k)$ *be a set of IFNs, and* $k^- = (\mu_{\min i_j}, \nu_{\min i_j})$ *then*

$$k^- < \mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) < k^+ \tag{17}$$

Proof. Based on Properties 1 and 2, we have

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_k) \ge \mathrm{IFDHM}^{(x)}(k^-, k^-, \cdots, k^-) = k^-,$$
$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_k) \le \mathrm{IFDHM}^{(x)}(k^+, k^+, \cdots, k^+) = k^+.$$

Then we have $k^- < \mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) < k^+$. □

Property 4 (Commutativity). *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ *be two sets of IFNs. Suppose* $(\pi_1, \pi_2, \cdots, \pi_n)$ *is any permutation of* $(k_1, k_2, \cdots, k_n)$*, then*

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{18}$$

Proof. Because $(\pi_1, \pi_2, \cdots, \pi_n)$ is any permutation of $(k_1, k_2, \cdots, k_n)$, then $\frac{1}{C_n^x} \left(\bigoplus_{1 \le i_1 < \cdots < i_x \le n} \left(\bigotimes_{j=1}^{x} k_{i_j} \right)^{\frac{1}{x}} \right) = \frac{1}{C_n^x} \left(\bigoplus_{1 \le i_1 < \cdots < i_x \le n} \left(\bigotimes_{j=1}^{x} \pi_{i_j} \right)^{\frac{1}{x}} \right)$, thus

$$\mathrm{IFDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n).$$

□

Example 5. *Let* $k_1 = (0.6, 0.3), k_2 = (0.5, 0.1), k_3 = (0.7, 0.2), k_4 = (0.8, 0.1)$ *be four IFNs. Then we use the proposed IFDHM operator to aggregate four IFNs (suppose* $x = 2, \lambda = 2$*).*

Let

$$
\begin{aligned}
k &= \mathrm{IFDHM}^{(2)}(k_1, k_2, \cdots, k_n) \\
&= \left(1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{x}{C_n^x} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \right) \\
&= \left(1 - \frac{1}{1 + \left(\frac{1}{3} \times \left(\begin{array}{l} \frac{1}{\left(\frac{1-0.6}{0.6}\right)^2 + \left(\frac{1-0.5}{0.5}\right)^2} + \frac{1}{\left(\frac{1-0.6}{0.6}\right)^2 + \left(\frac{1-0.7}{0.7}\right)^2} \\ + \frac{1}{\left(\frac{1-0.6}{0.6}\right)^2 + \left(\frac{1-0.8}{0.8}\right)^2} + \frac{1}{\left(\frac{1-0.5}{0.5}\right)^2 + \left(\frac{1-0.7}{0.7}\right)^2} \\ + \frac{1}{\left(\frac{1-0.5}{0.5}\right)^2 + \left(\frac{1-0.8}{0.8}\right)^2} + \frac{1}{\left(\frac{1-0.7}{0.7}\right)^2 + \left(\frac{1-0.8}{0.8}\right)^2} \end{array} \right) \right)^{\frac{1}{2}}}, \frac{1}{1 + \left(\frac{1}{3} \times \left(\begin{array}{l} \frac{1}{\left(\frac{0.3}{1-0.3}\right)^2 + \left(\frac{0.1}{1-0.1}\right)^2} + \frac{1}{\left(\frac{0.3}{1-0.3}\right)^2 + \left(\frac{0.2}{1-0.2}\right)^2} \\ + \frac{1}{\left(\frac{0.3}{1-0.3}\right)^2 + \left(\frac{0.1}{1-0.1}\right)^2} + \frac{1}{\left(\frac{0.1}{1-0.1}\right)^2 + \left(\frac{0.2}{1-0.2}\right)^2} \\ + \frac{1}{\left(\frac{0.1}{1-0.1}\right)^2 + \left(\frac{0.1}{1-0.1}\right)^2} + \frac{1}{\left(\frac{0.2}{1-0.2}\right)^2 + \left(\frac{0.1}{1-0.1}\right)^2} \end{array} \right) \right)^{\frac{1}{2}}} \right) \\
&= (0.5831, 0.3355)
\end{aligned}
$$

At last, we get $\mathrm{IFDHM}^{(2)}(k_1, k_2, k_3, k_4) = (0.5831, 0.3355)$.

3.2. The IFWDHM Operator

The weights of attributes play an important role in practical decision making, and they can influence the decision result. Therefore, it is necessary to consider attribute weights in aggregating information. It is obvious that the IFWDHM operator fails to consider the problem of attribute weights. In order to overcome this defect, we propose the IFWDHM operator.

Definition 9. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right)(i = 1, 2, \cdots, n)$ *be a group of IFNs,* $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$ *be the weight vector for* k_i $(i = 1, 2, \cdots, n)$*, which satisfies* $\omega_i \in [0.1]$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$*, then we can define IFWDHM operator as follows:*

$$
\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \begin{cases} \frac{\bigoplus\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \left(\bigotimes\limits_{j=1}^{x} k_{i_j} \right)^{\frac{1}{x}}}{C_{n-1}^x} & (1 \le x < n) \\ \bigotimes\limits_{i=1}^{x} k_i^{\frac{1-\omega_i}{n-1}} & (x = n) \end{cases} \tag{19}
$$

Theorem 2. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right)(i = 1, 2, \cdots, n)$ *be a group of IFNs, and their weight vector meet* $\omega_i \in [0.1]$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$ *then the result from Definition 9 is still an IFN, and have*

$$\text{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \frac{\bigoplus\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right)\left(\bigotimes\limits_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) (1 \le x < n) \tag{20}$$

or

$$\text{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \bigotimes_{i=1}^{x} k_i^{\frac{1-\omega_i}{n-1}}$$

$$= \left(\frac{1}{1 + \left(\sum\limits_{i=1}^{x} \left(\frac{1-\omega_i}{n-1}\right) \left(\frac{1-\mu_i}{\mu_i}\right)^{\lambda} \right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\sum\limits_{i=1}^{x} \left(\frac{1-\omega_i}{n-1}\right) \left(\frac{\nu_i}{1-\nu_i}\right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) (x = k) \tag{21}$$

Proof. (1) First of all, we prove that (20) and (21) are kept. For the first case, when $(1 \le x < n)$, according to the operational laws of IFNs, we get

$$\bigotimes_{j=1}^{x} k_{i_j} = \left(\frac{1}{1 + \left(\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) \tag{22}$$

$$\left(\bigotimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \left(\frac{1}{1 + \left(\frac{1}{x} \sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\frac{1}{x} \sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) \tag{23}$$

Thereafter,

$$\left(1 - \sum_{j=1}^{x} \omega_{i_j}\right)\left(\bigotimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left(\left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1 + \left(\left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) \tag{24}$$

Moreover,

$$\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\left(\underset{j=1}{\overset{x}{\otimes}}k_{i_j}\right)^{\frac{1}{x}}=\left(\begin{array}{c} 1-\frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array}\right) \tag{25}$$

Therefore,

$$\begin{aligned}&\frac{1}{C_{n-1}^{x}}\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\left(\underset{j=1}{\overset{x}{\otimes}}k_{i_j}\right)^{\frac{1}{x}}\\ &=\left(\begin{array}{c} 1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array}\right)\end{aligned} \tag{26}$$

For the second case, when $(x=n)$, we get

$$k_i^{\frac{1-\omega_i}{n-1}}=\left(\frac{1}{1+\left(\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\mu_i}{\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\nu_i}{1-\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{27}$$

Then,

$$\underset{i=1}{\overset{x}{\otimes}}k_i^{\frac{1-\omega_i}{n-1}}=\left(\frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\mu_i}{\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\nu_i}{1-\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{28}$$

(2) Next, we prove the (20) and (21) are IFNs. For the first case, when $1\le x<n$, Let

$$a=1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \tag{29}$$

$$b = \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \tag{30}$$

Then we need prove the following two conditions.
(I) $0 \le a \le 1$, $0 \le b \le 1$. (II) $0 \le a + b \le 1$.
(I) Since $a \in [0,1]$, we can get

$$1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}} \ge 1 \tag{31}$$

$$\Rightarrow \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \in [0,1] \tag{32}$$

$$\Rightarrow 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \in [0,1] \tag{33}$$

Therefore, $0 \le a \le 1$. Similarly, we can get

$$\frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \in [0,1] \tag{34}$$

Therefore, $0 \le b \le 1$.
(II) Since $0 \le a + b \le 1$, we can get the following inequality.

$$\begin{aligned} & 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \\ & \le 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \\ & = 1 \end{aligned} \tag{35}$$

For the second case, when $x = n$, we can easily prove that it is kept. So the aggregation result produced by Definition 9 is still an IFN. Next, we shall deduce some desirable properties of IFWDHM operator. □

Property 5 (Idempotency). *If $k_i (i = 1, 2, \cdots, n)$ are equal, i.e., $k_i = k = (\mu, \nu)$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum_{i=1}^{k} \omega_i = 1$ then*

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = k \tag{36}$$

Proof. Since $k_i = k = (\mu, \nu)$, based on Theorem 2, we get

(1) For the first case, when $1 \le x < n$.

$$\begin{aligned}
&\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\left(\frac{1-\mu}{\mu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j}\right) \frac{1}{\left(\frac{\nu}{1-\nu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_n^x - \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(\sum\limits_{i=1}^{x} \omega_i \right) \right) \frac{1}{\left(\frac{1-\mu}{\mu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_n^x - \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(\sum\limits_{i=1}^{x} \omega_i \right) \right) \frac{1}{\left(\frac{\nu}{1-\nu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_n^x - C_{n-1}^{x-1} \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\left(\frac{1-\mu}{\mu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_n^x - C_{n-1}^{x-1} \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\left(\frac{\nu}{1-\nu} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right)
\end{aligned}$$

Since $\sum\limits_{i=1}^{k} \omega_i = 1$, we can get

$$
\begin{aligned}
&\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}}\left(C_n^x - C_{n-1}^{x-1}\right)\frac{1}{\left(\frac{1-\mu}{\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{1}{C_{n-1}^{x}}\left(C_n^x - C_{n-1}^{x-1}\right)\frac{1}{\left(\frac{\nu}{1-\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\frac{1}{\frac{(n-1)!}{x!(k-1-x)!}}\frac{(n-1)!}{x!(k-1-x)!}\frac{1}{\left(\frac{1-\mu}{\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\frac{1}{\frac{(n-1)!}{x!(k-1-x)!}}\frac{(n-1)!}{x!(k-1-x)!}\frac{1}{\left(\frac{\nu}{1-\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= \left(1 - \dfrac{1}{1+\left(\frac{1}{\left(\frac{1-\mu}{\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \dfrac{1}{1+\left(\frac{1}{\left(\frac{\nu}{1-\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right) \\
&= (\mu, \nu) \\
&= k
\end{aligned}
$$

(2) For the second case, when $x = n$,

$$
\begin{aligned}
&\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \\
&= \left(\begin{array}{c} \dfrac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\mu_i}{\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\nu_i}{1-\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right) = \left(\begin{array}{c} \dfrac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{1-\mu}{\mu}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{\nu}{1-\nu}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right) \\
&= (\mu, \nu) \\
&= k
\end{aligned}
$$

which proves the idempotency property of the IFWDHM operator. □

Property 6 (Monotonicity). *Let $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ be two sets of IFNs. If $\mu_{i_j} \geq \mu_{\theta_j}, \nu_{i_j} \leq \nu_{\theta_j}$ for all j, and weight vector meets $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{k} \omega_i = 1$, the k and π are equal, then we have*

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \geq \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{37}$$

Proof. Since $x \geq 1, \mu_{i_j} \geq \mu_{\theta_j} \geq 0, \nu_{\theta_j} \geq \nu_{i_j} \geq 0$, then

$$\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda} \leq \sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda} \geq 1/\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda} \geq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda} \geq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda} \geq \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}} \geq 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}$$
$$\Rightarrow \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \leq \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$
$$\Rightarrow 1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \geq 1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Similarly, we have

$$\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \leq \sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \geq 1/\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \geq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}.$$
$$\Rightarrow \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda} \geq \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}$$
$$\Rightarrow 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}} \geq 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}$$
$$\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \leq \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum_{j=1}^{x}\left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Let $a = \mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n)$, $\pi = \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$ and $S(k)$, $S(\pi)$ be the score values of a and π respectively. Based on the score value of IFN in (2) and the above inequality, we can imply that $S(k) \geq S(\pi)$, and then we discuss the following cases:

(1) If $S(k) > S(\pi)$, then we can get

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) > \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$$

(2) If $S(k) = S(\pi)$, then

$$1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$
$$= 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$

Since $\mu_{i_j} \ge \mu_{\theta_j} \ge 0, \nu_{\theta_j} \ge \nu_{i_j} \ge 0$, and based on the Equations (2) and (3), we can deduce that

$$1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{i_j}}{\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} = 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\mu_{\theta_j}}{\mu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$

and

$$\frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} = \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{j=1}^{x} \omega_{i_j} \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{\theta_j}}{1-\nu_{\theta_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$

Therefore, it follows that $\mathrm{H}(k) = \mathrm{H}(\pi)$, the $\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$, When $x = n$, we can prove it in a similar way. □

Property 7 (Boundedness). *Let* $k_i = (\mu_{i_j}, \nu_{i_j}), k^+ = (\mu_{\max\, i_j}, \nu_{\max\, i_j})(i = 1, 2, \cdots, n)$ *be a set of IFNs, and* $k^- = (\mu_{\min\, i_j}, \nu_{\min\, i_j})$, *and weight vector meets* $\omega_i \in [0, 1]$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$ *then*

$$k^- \le \mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le k^+ \tag{38}$$

Proof. Based on Properties 5 and 6, we have

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \ge \mathrm{IFWDHM}_{\omega}^{(x)}(k^-, k^-, \cdots, k^-) = k^-$$
$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le \mathrm{IFWDHM}_{\omega}^{(x)}(k^+, k^+, \cdots, k^+) = k^+$$

Then we have $k^- \le \mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le k^+$. □

Property 8 (Commutativity). *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ *be two sets of IFNs. Suppose* $(\pi_1, \pi_2, \cdots, \pi_n)$ *is any permutation of* $(k_1, k_2, \cdots, k_n)$, *and weight vector meets* $\omega_i \in [0, 1]$ *and* $\sum\limits_{i=1}^{k} \omega_i = 1$, *the k and* π *are equal, then we have*

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{39}$$

Proof. Because $(\pi_1, \pi_2, \cdots, \pi_n)$ is any permutation of $(k_1, k_2, \cdots, k_n)$, then

$$\begin{array}{ll} \dfrac{\underset{1\le i_1<\cdots<i_x\le k}{\oplus}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}}}{C_{k-1}^{x}} = \dfrac{\underset{1\le i_1<\cdots<i_x\le k}{\oplus}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\otimes}} \pi_{i_j}\right)^{\frac{1}{x}}}{C_{k-1}^{x}} & (1\le x<k) \\ \overset{x}{\underset{i=1}{\otimes}} k_i^{\frac{1-\omega_i}{k-1}} = \overset{x}{\underset{i=1}{\otimes}} \pi_i^{\frac{1-\omega_i}{k-1}} & (x=k) \end{array}$$

Thus, $\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$. □

Example 6. *Let* $k_1 = (0.8, 0.2), k_2 = (0.6, 0.1), k_3 = (0.7, 0.3), k_4 = (0.4, 0.2)$ *be four IFNs, the weighting vector of attributes is* $\omega = \{0.2, 0.3, 0.4, 0.1\}$. *Then we use the proposed IFWDHM operator to aggregate four IFNs (suppose* $x = 2, \lambda = 2$*).*

Let

$$\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \frac{\underset{1\le i_1<\cdots<i_x\le n}{\oplus}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\otimes}} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}}$$

$$= \left(\begin{array}{c} 1-\dfrac{1}{1+\left(\dfrac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\dfrac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\dfrac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\dfrac{1}{\sum\limits_{j=1}^{x}\left(\frac{v_{i_j}}{1-v_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array}\right)$$

$$= \left(\begin{array}{c} 1-\dfrac{1}{1+\left(\dfrac{2}{C_{4-1}^{2}}\sum\limits_{1\le i_1<\cdots<i_2\le 4}\left(1-\sum\limits_{j=1}^{2}\omega_{i_j}\right)\dfrac{1}{\sum\limits_{j=1}^{2}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{2}}\right)^{\frac{1}{2}}}, \\ \dfrac{1}{1+\left(\dfrac{2}{C_{4-1}^{2}}\sum\limits_{1\le i_1<\cdots<i_2\le 4}\left(1-\sum\limits_{j=1}^{2}\omega_{i_j}\right)\dfrac{1}{\sum\limits_{j=1}^{2}\left(\frac{v_{i_j}}{1-v_{i_j}}\right)^{2}}\right)^{\frac{1}{2}}} \end{array}\right)$$

$$= \left(\begin{array}{c} 1-\dfrac{1}{1+\left(\frac{2}{3}\left(\begin{array}{c} \frac{1-0.2-0.3}{\left(\frac{1-0.8}{0.8}\right)^2+\left(\frac{1-0.6}{0.6}\right)^2}+\frac{1-0.2-0.4}{\left(\frac{1-0.8}{0.8}\right)^2+\left(\frac{1-0.7}{0.7}\right)^2} \\ +\frac{1-0.2-0.1}{\left(\frac{1-0.8}{0.8}\right)^2+\left(\frac{1-0.4}{0.4}\right)^2}+\frac{1-0.3-0.4}{\left(\frac{1-0.6}{0.6}\right)^2+\left(\frac{1-0.7}{0.7}\right)^2} \\ +\frac{1-0.3-0.1}{\left(\frac{1-0.6}{0.6}\right)^2+\left(\frac{1-0.4}{0.4}\right)^2}+\frac{1-0.4-0.1}{\left(\frac{1-0.7}{0.7}\right)^2+\left(\frac{1-0.4}{0.4}\right)^2} \end{array}\right)\right)^{\frac{1}{2}}}, \\ 1-\dfrac{1}{1+\left(\frac{2}{3}\left(\begin{array}{c} \frac{1-0.2-0.3}{\left(\frac{0.2}{1-0.2}\right)^2+\left(\frac{0.1}{1-0.1}\right)^2}+\frac{1-0.2-0.4}{\left(\frac{0.2}{1-0.2}\right)^2+\left(\frac{0.3}{1-0.3}\right)^2} \\ +\frac{1-0.2-0.1}{\left(\frac{0.2}{1-0.2}\right)^2+\left(\frac{0.2}{1-0.2}\right)^2}+\frac{1-0.3-0.4}{\left(\frac{0.1}{1-0.1}\right)^2+\left(\frac{0.3}{1-0.3}\right)^2} \\ +\frac{1-0.3-0.1}{\left(\frac{0.1}{1-0.1}\right)^2+\left(\frac{0.2}{1-0.2}\right)^2}+\frac{1-0.4-0.1}{\left(\frac{0.3}{1-0.3}\right)^2+\left(\frac{0.2}{1-0.2}\right)^2} \end{array}\right)\right)^{\frac{1}{2}}} \end{array}\right)$$

$$= (0.5302, 0.1952)$$

At last, we get $\mathrm{IFWDHM}_{\omega}^{(2)}(k_1, k_2, k_3, k_4) = (0.5302, 0.1952)$.

3.3. The IFDDHM Operator

Wu et al. [59] proposed the dual Hamy mean (DHM) operator.

Definition 10. *The DHM operator is defined as follows [59]:*

$$\mathrm{DHM}^{(x)}(k_1,k_2,\cdots,k_n)=\left(\prod_{1\le i_1<\ldots<i_x\le n}\left(\frac{\sum_{j=1}^{x}k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \tag{40}$$

where x is a parameter and $x=1,2,\ldots,n$, $i_1,i_2,\ldots,i_n$ *are x integer values taken from the set* $\{1,2,\ldots,n\}$ *of n integer values,* C_n^x *denotes the binomial coefficient and* $C_k^x=\frac{k!}{x!(k-x)!}$.

In this section, we will propose the intuitionistic fuzzy Dombi dual Hamy mean DHM (IFDDHM) operator.

Definition 11. *Let* $k_i=(\mu_{i_j},\nu_{i_j})(i=1,2,\cdots,n)$ *be a collection of IFNs, then we can define IFDDHM operator as follows:*

$$\mathrm{IFDDHM}^{(x)}(k_1,k_2,\cdots,k_n)=\left(\underset{1\le i_1<\cdots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \tag{41}$$

where x is a parameter and $x=1,2,\ldots,n$, $i_1,i_2,\ldots,i_n$ *are x integer values taken from the set* $\{1,2,\ldots,n\}$ *of n integer values,* C_n^x *denotes the binomial coefficient and* $C_k^x=\frac{k!}{x!(k-x)!}$.

Theorem 3. *Let* $k_i=(\mu_i,\nu_i)(i=1,2,\cdots,n)$ *be a collection of the IFNs, then the aggregate result of Definition 10 is still an IFNs, and have*

$$\begin{aligned}&\mathrm{IFDDHM}^{(x)}(k_1,k_2,\cdots,k_n)=\left(\underset{1\le i_1<\cdots<i_x\le n}{\otimes}\left(\frac{\overset{x}{\underset{j=1}{\oplus}}k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}\\&=\left(\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)\end{aligned} \tag{42}$$

Proof.

(1) First of all, we prove (42) is kept. According to the operational laws of IFNs, we get

$$\overset{x}{\underset{j=1}{\oplus}}k_{i_j}=\left(1-\frac{1}{1+\left(\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}},\frac{1}{1+\left(\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{43}$$

$$\frac{\underset{j=1}{\overset{x}{\oplus}} k_{i_j}}{x} = \left(1 - \frac{1}{1 + \left(\frac{1}{x}\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{1}{x}\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{44}$$

Moreover,

$$\begin{aligned} &\underset{1\le i_1<\cdots<i_x\le n}{\otimes}\left(\frac{\underset{j=1}{\overset{x}{\oplus}} k_{i_j}}{x}\right) \\ &= \left(\frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right) \end{aligned} \tag{45}$$

Furthermore,

$$\begin{aligned} &\mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \left(\underset{1\le i_1<\cdots<i_x\le n}{\otimes}\left(\frac{\underset{j=1}{\overset{x}{\oplus}} k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} \\ &= \left(\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right) \end{aligned} \tag{46}$$

(2) Next, we prove (42) is an IFN.
Let

$$a = \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, b = 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Then we need to prove that the following two conditions which are satisfied.
(i) $0 \le a \le 1, 0 \le b \le 1$;
(ii) $0 \le a + b \le 1$.
(i) Since $\mu_{i_j} \in [0,1]$, we can get

$$1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}} \ge 1 \Rightarrow \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \in [0,1]$$

Therefore, $0 \le a \le 1$. Similarly, we can get $0 \le b \le 1$.
(ii) Obviously, $0 \le a+b \le 1$, then

$$\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$
$$\le\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}+1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=1$$

We get $0 \le a+b \le 1$. so the aggregated result of Definition 10 is still an IFN. Next we will discuss some properties of IFDDHM operator. □

Property 9 (Idempotency). *If $k_i(1,2\cdots,n)$ and k are IFNs, and $k_i=k=(\mu_i,\nu_i)$ for all $i=1,2\cdots,n$, then we get*

$$\mathrm{IFDDHM}^{(x)}(k_1,k_2,\cdots k_n)=k \tag{47}$$

Proof. Since $k=(\mu,\nu)$, based on Theorem 3, we have

$$\mathrm{IFDDHM}^{(x)}(k_1,k_2,\cdots,k_n)=\left(\bigotimes_{1\le i_1<\cdots<i_x\le n}\left(\frac{\bigoplus_{j=1}^{x}k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$
$$=\left(\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$
$$=\left(\frac{1}{1+\left(\frac{1}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\frac{1}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$
$$=\left(\frac{1}{1+\left(\frac{1}{\left(\frac{\mu_i}{1-\mu_i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\frac{1}{\left(\frac{1-\nu_i}{\nu_i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)=\left(\frac{1}{1+\frac{1}{\frac{\mu_i}{1-\mu_i}}},1-\frac{1}{1+\frac{1}{\frac{1-\nu_i}{\nu_i}}}\right)$$
$$=(\mu_i,\nu_i)=(\mu,\nu)=k$$

□

Property 10 (Monotonicity). *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ *be two sets of IFNs. If* $\mu_{i_j} \geq \mu_{\theta_j}, \nu_{i_j} \leq \nu_{\theta_j}$*, for all j, then*

$$\mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) \geq \mathrm{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{48}$$

Proof. Since $x \geq 1, \mu_{i_j} \geq \mu_{\theta_j} \geq 0, \nu_{\theta_j} \geq \nu_{i_j} \geq 0$, then

$$\begin{aligned}
&\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \leq \sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \geq 1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda} \\
&\Rightarrow \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \geq \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda} \\
&\Rightarrow 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \geq \sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda} \\
&\Rightarrow \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq \frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda} \\
&\Rightarrow 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq 1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \geq \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \\
&\Rightarrow 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \leq 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n} 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}
\end{aligned}$$

Let $k = \mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n)$, $\pi = \mathrm{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$ and $S(k), S(\pi)$ be the score values of k and π respectively. Based on the score value of IFN in (2) and the above inequality, we can imply that $S(k) \geq S(\pi)$, and then we discuss the following cases:

(1) If $S(k) > S(\pi)$, then we get $\mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) > \mathrm{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$.

(2) If $S(k) = S(\pi)$, then

$$\begin{aligned}
&\frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \\
&= \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}
\end{aligned}$$

Since $\mu_{i_j} \geq \mu_{\theta_j} \geq 0, \nu_{\theta_j} \geq \nu_{i_j} \geq 0$, we can deduce that

$$\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} = \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

and

$$1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} = 1-\frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Therefore, it follows that

$$\begin{aligned} H(k) &= \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \\ &= \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} = H(\pi) \end{aligned}$$

Then $\mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$. □

Property 11 (Boundedness). *Let $k_i = (\mu_{i_j}, \nu_{i_j}), k^+ = (\mu_{\max\ i_j}, \nu_{\max\ i_j})(i = 1, 2, \cdots, k)$ be a set of IFNs, and $k^- = (\mu_{\min\ i_j}, \nu_{\min\ i_j})$ then*

$$k^- < \mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) < k^+ \tag{49}$$

Proof. Based on Properties 9 and 10, we have

$$\begin{aligned} \mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_k) &\geq \mathrm{IFDDHM}^{(x)}(k^-, k^-, \cdots, k^-) = k^-, \\ \mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_k) &\leq \mathrm{IFDDHM}^{(x)}(k^+, k^+, \cdots, k^+) = k^+. \end{aligned}$$

Then we have $k^- < \mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) < k^+$. □

Property 12 (Commutativity). *Let $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ be two sets of IFNs. Suppose $(\pi_1, \pi_2, \cdots, \pi_n)$ is any permutation of $(k_1, k_2, \cdots, k_n)$, then*

$$\mathrm{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{50}$$

Proof. Because $(\pi_1, \pi_2, \cdots, \pi_n)$ is any permutation of $(k_1, k_2, \cdots, k_n)$, then

$$\left(\bigotimes_{1\leq i_1<\cdots<i_x\leq n}\left(\frac{\overset{x}{\underset{j=1}{\oplus}} k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}} = \left(\bigotimes_{1\leq i_1<\cdots<i_x\leq n}\left(\frac{\overset{x}{\underset{j=1}{\oplus}} \pi_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$

thus

$$\text{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \text{IFDDHM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$$

□

Example 7. *Let* $k_1 = (0.7, 0.3), k_2 = (0.4, 0.1), k_3 = (0.5, 0.2), k_4 = (0.6, 0.2)$ *be four IFNs. Then we use the proposed IFDDHM operator to aggregate four IFNs (suppose* $x = 2, \lambda = 2$*).*

Let

$$\text{IFDDHM}^{(x)}(k_1, k_2, \cdots, k_n) = \left(\bigotimes_{1 \le i_1 < \cdots < i_x \le n} \left(\frac{\overset{x}{\underset{j=1}{\oplus}} k_{i_j}}{x} \right) \right)^{\frac{1}{C_n^x}}$$

$$= \left(\frac{1}{1+\left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{x}{C_n^x} \sum_{1 \le i_1 < \cdots < i_x \le n} \frac{1}{\sum_{j=1}^{x} \left(\frac{1-v_{i_j}}{v_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \right)$$

$$= \left(\frac{1}{1+\left(\frac{2}{C_4^2} \sum_{1 \le i_1 < \cdots < i_2 \le 4} \frac{1}{\sum_{j=1}^{4} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{2}} \right)^{\frac{1}{2}}}, 1 - \frac{1}{1+\left(\frac{2}{C_4^2} \sum_{1 \le i_1 < \cdots < i_4 \le 4} \frac{1}{\sum_{j=1}^{4} \left(\frac{1-v_{i_j}}{v_{i_j}} \right)^{4}} \right)^{\frac{1}{2}}} \right)$$

$$= \left(\begin{array}{c} \dfrac{1}{1+\left(\frac{2}{6} \times \left(\begin{array}{c} \frac{1}{\left(\frac{0.7}{1-0.7}\right)^2+\left(\frac{0.4}{1-0.4}\right)^2} + \frac{1}{\left(\frac{0.7}{1-0.7}\right)^2+\left(\frac{0.5}{1-0.5}\right)^2} + \\ \frac{1}{\left(\frac{0.7}{1-0.7}\right)^2+\left(\frac{0.6}{1-0.6}\right)^2} + \frac{1}{\left(\frac{0.4}{1-0.4}\right)^2+\left(\frac{0.5}{1-0.5}\right)^2} + \\ \frac{1}{\left(\frac{0.4}{1-0.4}\right)^2+\left(\frac{0.6}{1-0.6}\right)^2} + \frac{1}{\left(\frac{0.5}{1-0.5}\right)^2+\left(\frac{0.6}{1-0.6}\right)^2} \end{array} \right) \right)^{\frac{1}{2}}}, \\ 1 - \dfrac{1}{1+\left(\frac{2}{6} \times \left(\begin{array}{c} \frac{1}{\left(\frac{1-0.3}{0.3}\right)^2+\left(\frac{1-0.1}{0.1}\right)^2} + \frac{1}{\left(\frac{1-0.3}{0.3}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2} + \\ \frac{1}{\left(\frac{1-0.3}{0.3}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2} + \frac{1}{\left(\frac{1-0.1}{0.1}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2} + \\ \frac{1}{\left(\frac{1-0.1}{0.1}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2} + \frac{1}{\left(\frac{1-0.2}{0.2}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2} \end{array} \right) \right)^{\frac{1}{2}}} \end{array} \right)$$

$$= (0.5617, 0.1596)$$

At last, we get $\text{IFDDHM}^{(4)}(k_1, k_2, k_3, k_4) = (0.5617, 0.1596)$.

3.4. The IFWDDHM Operator

The weights of attributes play an important role in practical decision making, and they can influence the decision result. Therefore, it is necessary to consider attribute weights in aggregating information. It is obvious that the IFWDDHM operator fails to consider the problem of attribute weights. In order to overcome this defect, we propose the IFWDDHM operator.

Definition 12. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right)(i = 1, 2, \cdots, n)$ *be a group of IFNs,* $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$ *be the weight vector for* $(i = 1, 2, \cdots, n)$*, which satisfies* $\omega_i \in [0.1]$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$*, then we can define IFWDDHM operator as follows:*

$$\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \begin{cases} \dfrac{\underset{1 \le i_1 < \cdots < i_x \le n}{\otimes} \left(1 - \sum\limits_{i=1}^{x} \omega_i\right) \left(\overset{x}{\underset{j=1}{\oplus}} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}} & (1 \le x < n) \\ \overset{x}{\underset{i=1}{\oplus}} k_i^{\frac{1-\omega_i}{n-1}} & (x = n) \end{cases} \tag{51}$$

Theorem 4. *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right)(i = 1, 2, \cdots, n)$ *be a group of IFNs, and their weight vector meet* $\omega_i \in [0, 1]$*, then the result from Definition 12 is still an IFN, and has*

$$\begin{aligned} &\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \frac{\underset{1 \le i_1 < \cdots < i_x \le n}{\otimes} \left(1 - \sum\limits_{i=1}^{x} \omega_i\right) \left(\overset{x}{\underset{j=1}{\oplus}} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}} \\ &= \left(\begin{array}{c} \dfrac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \end{array} \right) (1 \le x < n) \end{aligned} \tag{52}$$

or

$$\begin{aligned} &\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \overset{x}{\underset{i=1}{\oplus}} k_i^{\frac{1-\omega_i}{n-1}} \\ &= \left(1 - \frac{1}{1 + \left(\sum\limits_{i=1}^{x} \left(\frac{1-\omega_i}{n-1}\right) \left(\frac{\mu_i}{1-\mu_i}\right)^{\lambda} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\sum\limits_{i=1}^{x} \left(\frac{1-\omega_i}{n-1}\right) \left(\frac{1-\nu_i}{\nu_i}\right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) (x = n) \end{aligned} \tag{53}$$

Proof.

$$\overset{x}{\underset{j=1}{\oplus}} k_{i_j} = \left(1 - \frac{1}{1 + \left(\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) \tag{54}$$

$$\left(\overset{x}{\underset{j=1}{\oplus}} k_{i_j}\right)^{\frac{1}{x}} = \left(1 - \frac{1}{1 + \left(\frac{1}{x} \sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{1}{x} \sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}} \right)^{\lambda} \right)^{\frac{1}{\lambda}}} \right) \tag{55}$$

Thereafter,

$$\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\left(\bigoplus_{j=1}^{x}k_{i_j}\right)^{\frac{1}{x}}=\left(\begin{array}{c}\dfrac{1}{1+\left(\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{\nu_{i_j}}{1-\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},\\ 1-\dfrac{1}{1+\left(\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\mu_{i_j}}{\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\end{array}\right) \tag{56}$$

Moreover,

$$\bigotimes_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)\left(\bigoplus_{j=1}^{x}k_{i_j}\right)^{\frac{1}{x}}=\left(\begin{array}{c}\dfrac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},\\ 1-\dfrac{1}{1+\left(\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\end{array}\right) \tag{57}$$

Therefore,

$$\begin{aligned}&\frac{1}{C_{n-1}^{x}}\bigotimes_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{i=1}^{x}\omega_i\right)\left(\bigoplus_{j=1}^{x}k_{i_j}\right)^{\frac{1}{x}}\\&=\left(\begin{array}{c}\dfrac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}},\\ 1-\dfrac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{i=1}^{x}\omega_i\right)\dfrac{x}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\end{array}\right)\end{aligned} \tag{58}$$

For the second case, when $(x=n)$, we get

$$k_i^{\frac{1-\omega_i}{n-1}}=\left(\frac{1}{1+\left(\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\mu_i}{\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}},1-\frac{1}{1+\left(\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\nu_i}{1-\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{59}$$

Then,

$$\bigoplus_{i=1}^{x}k_i^{\frac{1-\omega_i}{n-1}}=\left(1-\frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\mu_i}{1-\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}},\frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\nu_i}{\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}\right) \tag{60}$$

Next, we prove the (52) and (53) are IFNs. For the first case, when $1\le x<n$,

Let

$$a = \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$

$$b = 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}}$$

Then we need prove the following two conditions.
(I) $0 \le a \le 1, 0 \le b \le 1$;
(II) $0 \le a + b \le 1$.
(I) Since $a \in [0,1]$, we can get

$$\begin{aligned} & 1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}} > 1 \\ \Rightarrow & \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{x}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \in [0,1] \end{aligned}$$

Therefore, $0 \le a \le 1$. Similarly, we can get

$$1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \in [0,1]$$

Therefore, $0 \le b \le 1$.
(II) Since $0 \le a + b \le 1$, we can get the following inequality.

$$\begin{aligned} & \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{1-\nu_{i_j}}{\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \\ \le + & \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\mu_{i_j}}{1-\mu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} + 1 - \frac{1}{1 + \left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \le i_1 < \cdots < i_x \le n} \left(1 - \sum\limits_{i=1}^{x} \omega_i \right) \frac{1}{\sum\limits_{j=1}^{x} \left(\frac{\nu_{i_j}}{1-\nu_{i_j}} \right)^{\lambda}} \right)^{\frac{1}{\lambda}}} \\ = & 1 \end{aligned}$$

For the second case, when $x = n$, we can easily prove that it is kept. So the aggregation result produced by Definition 9 is still an IFN. Next, we shall deduce some desirable properties of IFWDDHM operator. $\square$

Property 13 (Idempotency). *If $k_i (i = 1, 2, \cdots, n)$ are equal, i.e., $k_i = k = (\mu, \nu)$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{k} \omega_i = 1$ then*

$$\text{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = k \tag{61}$$

Proof. Since $k_i = k = (\mu, \nu)$, based on Theorem 4, we get

(1) For the first case, when $1 \leq x < n$.

$$\begin{aligned}
&\text{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n)\\
&= \left(\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \leq i_1 < \cdots < i_x \leq n} \left(1-\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}} \sum\limits_{1 \leq i_1 < \cdots < i_x \leq n} \left(1-\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)\\
&= \left(\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \sum\limits_{1 \leq i_1 < \cdots < i_x \leq n} \left(1-\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\left(\frac{\mu}{1-\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \sum\limits_{1 \leq i_1 < \cdots < i_x \leq n} \left(1-\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\left(\frac{1-\nu}{\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)\\
&= \left(\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_{n}^{x}-C_{n-1}^{x-1}\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\left(\frac{\mu}{1-\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_{n}^{x}-C_{n-1}^{x-1}\sum\limits_{i=1}^{x} \omega_i\right) \frac{1}{\left(\frac{1-\nu}{\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)
\end{aligned}$$

Since $\sum\limits_{i=1}^{k} \omega_i = 1$, we can get

$$\begin{aligned}
&\text{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n)\\
&= \left(\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_{n}^{x}-C_{n-1}^{x-1}\right) \frac{1}{\left(\frac{\nu}{1-\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{C_{n-1}^{x}} \left(C_{n}^{x}-C_{n-1}^{x-1}\right) \frac{1}{\left(\frac{1-\mu}{\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)\\
&= \left(\frac{1}{1+\left(\frac{1}{\frac{(n-1)!}{x!(k-1-x)!}} \frac{(n-1)!}{x!(k-1-x)!} \frac{1}{\left(\frac{\mu}{1-\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{\frac{(n-1)!}{x!(k-1-x)!}} \frac{(n-1)!}{x!(k-1-x)!} \frac{1}{\left(\frac{1-\nu}{\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)\\
&= \left(\frac{1}{1+\left(\frac{1}{\left(\frac{\mu}{1-\mu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{\left(\frac{1-\nu}{\nu}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \right)\\
&= (\mu, \nu)\\
&= k
\end{aligned}$$

(2) For the second case, when $x = n$,

$$\begin{aligned}&\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n)\\&= \left(\begin{array}{c} 1 - \frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{\mu_i}{1-\mu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\sum\limits_{i=1}^{x}\left(\frac{1-\omega_i}{n-1}\right)\left(\frac{1-\nu_i}{\nu_i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right) = \left(\begin{array}{c} 1 - \frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{\mu}{1-\mu}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{1-\nu}{\nu}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right) = (\mu, \nu) = k\end{aligned}$$

which proves the idempotency property of the IFWDDHM operator. □

Property 14 (Monotonicity). *Let $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right)(i = 1, 2, \cdots, n)$ be two sets of IFNs. If $\mu_{i_j} \geq \mu_{\theta_j}, \nu_{i_j} \leq \nu_{\theta_j}$ for all j, and weight vector meets $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{k} \omega_i = 1$, the k and π are equal, then we have*

$$\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \geq \mathrm{IFWDDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{62}$$

Proof. Since $x \geq 1, \mu_{i_j} \geq \mu_{\theta_j} \geq 0, \nu_{\theta_j} \geq \nu_{i_j} \geq 0$, then

$$\begin{aligned}
&\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \geq \sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \leq 1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \leq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda} \leq \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\\
&\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)1/\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \geq \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)1/\sum\limits_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \geq \sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda} \Rightarrow 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq 1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq \left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda} \leq \frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\\
&\Rightarrow 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq 1+\left(\frac{x}{C_{n-1}^{x}}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x}\omega_{i_j}\right)1/\sum_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Let $a = \mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n)$, $\pi = \mathrm{IFWDDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$ and $S(k)$, $S(\pi)$ be the score values of a and π respectively. Based on the score value of IFN in (2) and the above inequality, we can imply that $S(k) \geq S(\pi)$, and then we discuss the following cases:

(1) If $S(k) > S(\pi)$, then we can get

$$\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) > \mathrm{IFWDDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{63}$$

(2) If $S(k) = S(\pi)$, then

$$\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}-\left(1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$

$$=\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}-\left(1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$

Since $\mu_{i_j} \ge \mu_{\theta_j} \ge 0, \nu_{\theta_j} \ge \nu_{i_j} \ge 0$, and based on the Equations (2) and (3), we can deduce

$$\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{\theta_j}}{1-\mu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

and

$$1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}=1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{\theta_j}}{\nu_{\theta_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}$$

Therefore, it follows that $\mathrm{H}(k) = \mathrm{H}(\pi)$, $\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDDHM}_{\omega}^{(x)}$ $(\pi_1, \pi_2, \cdots, \pi_n)$, When $x = n$, we can prove it in a similar way. □

Property 15 (Boundedness). *Let $k_i = (\mu_{i_j}, \nu_{i_j}), k^+ = (\mu_{\max i_j}, \nu_{\max i_j})(i = 1, 2, \cdots, n)$ be a set of IFNs, and $k^- = (\mu_{\min i_j}, \nu_{\min i_j})$, and weight vector meets $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$*

$$k^- \le \mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le k^+ \tag{64}$$

Proof. Based on Properties 13 and 14, we have

$$\begin{aligned}\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \ge \mathrm{IFWDDHM}_{\omega}^{(x)}(k^-, k^-, \cdots, k^-) = k^-\\ \mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le \mathrm{IFWDDHM}_{\omega}^{(x)}(k^+, k^+, \cdots, k^+) = k^+\end{aligned}$$

Then we have $k^- \le \mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) \le k^+$. □

Property 16 (Commutativity). *Let* $k_i = \left(\mu_{i_j}, \nu_{i_j}\right), \pi_i = \left(\mu_{\theta_j}, \nu_{\theta_j}\right) (i = 1, 2, \cdots, n)$ *be two sets of IFNs. Suppose* $(\pi_1, \pi_2, \cdots, \pi_n)$ *is any permutation of* $(k_1, k_2, \cdots, k_n)$*, and weight vector meets* $\omega_i \in [0, 1]$ *and* $\sum\limits_{i=1}^{k} \omega_i = 1$*, the k and* π *are equal, then we have*

$$\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n) \tag{65}$$

Proof. Because $(\pi_1, \pi_2, \cdots, \pi_n)$ is any permutation of $(k_1, k_2, \cdots, k_n)$, then

$$\begin{aligned} &\frac{\underset{1\le i_1<\cdots<i_x\le k}{\otimes}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\oplus}} k_{i_j}\right)^{\frac{1}{x}}}{C_{k-1}^{x}} = \frac{\underset{1\le i_1<\cdots<i_x\le k}{\otimes}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\oplus}} \pi_{i_j}\right)^{\frac{1}{x}}}{C_{k-1}^{x}} && (1 \le x < k) \\ &\overset{x}{\underset{i=1}{\oplus}} k_i^{\frac{1-\omega_i}{k-1}} = \overset{x}{\underset{i=1}{\oplus}} \pi_i^{\frac{1-\omega_i}{k-1}} && (x = k) \end{aligned}$$

Thus, $\mathrm{IFWDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \mathrm{IFWDHM}_{\omega}^{(x)}(\pi_1, \pi_2, \cdots, \pi_n)$. □

Example 8. *Let* $k_1 = (0.6, 0.1), k_2 = (0.5, 0.4), k_3 = (0.8, 0.3), k_4 = (0.7, 0.2)$ *be four IFNs. Then we use the IFWDDHM operator to fuse four IFNs, the weighting vector of attributes be* $\omega = \{0.2, 0.3, 0.4, 0.1\}$ *(suppose* $x = 2, \lambda = 2$*).*

Let

$$\begin{aligned} &\mathrm{IFWDDHM}_{\omega}^{(x)}(k_1, k_2, \cdots, k_n) = \frac{\underset{1\le i_1<\cdots<i_x\le n}{\otimes}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\left(\overset{x}{\underset{j=1}{\oplus}} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}} \\ &= \left(\begin{array}{c} \frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(\frac{x}{C_{n-1}^{x}}\sum\limits_{1\le i_1<\cdots<i_x\le n}\left(1-\sum\limits_{j=1}^{x}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{x}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{array}\right) \\ &= \left(\begin{array}{c} \frac{1}{1+\left(\frac{2}{C_3^2}\sum\limits_{1\le i_1<\cdots<i_2\le 4}\left(1-\sum\limits_{j=1}^{2}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{2}\left(\frac{\mu_{i_j}}{1-\mu_{i_j}}\right)^{2}}\right)^{\frac{1}{2}}}, \\ 1-\frac{1}{1+\left(\frac{2}{C_3^2}\sum\limits_{1\le i_1<\cdots<i_2\le 4}\left(1-\sum\limits_{j=1}^{2}\omega_{i_j}\right)\frac{1}{\sum\limits_{j=1}^{2}\left(\frac{1-\nu_{i_j}}{\nu_{i_j}}\right)^{2}}\right)^{\frac{1}{2}}} \end{array}\right) \\ &= \left(\begin{array}{c} \frac{1}{1+\left(\frac{2}{3}\times+\left(\begin{array}{c}\frac{1-0.2-0.3}{\left(\frac{0.6}{1-0.6}\right)^2+\left(\frac{0.5}{1-0.5}\right)^2}+\frac{1-0.2-0.4}{\left(\frac{0.6}{1-0.6}\right)^2+\left(\frac{0.8}{1-0.8}\right)^2}+\\ \frac{1-0.2-0.1}{\left(\frac{0.6}{1-0.6}\right)^2+\left(\frac{0.7}{1-0.7}\right)^2}+\frac{1-0.3-0.4}{\left(\frac{0.5}{1-0.5}\right)^2+\left(\frac{0.8}{1-0.8}\right)^2}+\\ \frac{1-0.3-0.1}{\left(\frac{0.5}{1-0.5}\right)^2+\left(\frac{0.7}{1-0.7}\right)^2}+\frac{1-0.4-0.1}{\left(\frac{0.8}{1-0.8}\right)^2+\left(\frac{0.7}{1-0.7}\right)^2}\end{array}\right)\right)}, \\ 1-\frac{1}{1+\left(\frac{2}{3}\times\left(\begin{array}{c}\frac{1-0.2-0.3}{\left(\frac{1-0.1}{0.1}\right)^2+\left(\frac{1-0.4}{0.4}\right)^2}+\frac{1-0.2-0.4}{\left(\frac{1-0.1}{0.1}\right)^2+\left(\frac{1-0.3}{0.3}\right)^2}+\\ \frac{1-0.2-0.1}{\left(\frac{1-0.1}{0.1}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2}+\frac{1-0.3-0.4}{\left(\frac{1-0.4}{0.4}\right)^2+\left(\frac{1-0.3}{0.3}\right)^2}+\\ \frac{1-0.3-0.1}{\left(\frac{1-0.4}{0.4}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2}+\frac{1-0.4-0.1}{\left(\frac{1-0.3}{0.3}\right)^2+\left(\frac{1-0.2}{0.2}\right)^2}\end{array}\right)\right)} \end{array}\right) \\ &= (0.6592, 0.2153) \end{aligned}$$

At last, we get $\mathrm{IFWDDHM}_{\omega}^{(2)}(k_1, k_2, k_3, k_4) = (0.6592, 0.2153)$.

4. A MAGDM Approach Based on the Proposed Operators

In this section, we will apply the proposed IFWDHM (IFWDDHM) operator to cope with the MAGDM problem with IFNs. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a set of alternatives, and $C = \{c_1, c_2, \cdots, c_n\}$ be a set of attributes, the weighting vector of attributes be $\omega = \{\omega_1, \omega_2 \cdots, \omega_n\}$, meet $\omega_j \in [0,1], j = 1,2,\cdots,n, \sum_{j=1}^{n} \omega_j = 1$. There are experts $Y = \{y_1, y_2, \cdots y_z\}$ who are invited to give the evaluation information, and their weighting vector is $w = \{w_1, w_2, \cdots w_z\}^T$ with $w_t \in [0,1], (t = 1,2,\cdots,z), \sum_{t=1}^{z} w_t = 1$. The expert y_t evaluates each attribute c_j of each alternative x_i by the form of IFN $a_{ij}^t = \left(\mu_{ij}^t, v_{ij}^t\right)(i = 1,2,\cdots,m, j = 1,2,\cdots,n)$, and then the decision matrix $A_t = \left(\widetilde{a}_{ij}^t\right)_{m\times n} = \left(\left(\mu_{ij}^t, v_{ij}^t\right)\right)_{m\times n}(t = 1,2,\cdots,z)$ is constructed. The ultimate goal is to give a ranking of all alternatives.

Then, we will give the steps for solving this problem.

Step1: Calculate the collective evaluation value of each attribute for each alternative by

$$\widetilde{a}_{ij}^t = \text{IFWDHM}_{\omega}^{(x)}\left(\widetilde{a}_{ij}^1, \widetilde{a}_{ij}^2, \cdots \widetilde{a}_{ij}^z\right) \text{ and } \widetilde{a}_{ij}^t = \text{IFWDDHM}_{\omega}^{(x)}\left(\widetilde{a}_{ij}^1, \widetilde{a}_{ij}^2, \cdots \widetilde{a}_{ij}^z\right)$$

Step2: Calculate the overall value of each alternative with the IFWDHM (IFWDDHM) operator

$$\widetilde{a}_i = \text{IFWDHM}_{\omega}^{(x)}(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \cdots \widetilde{a}_{in}),\ \widetilde{a}_i = \text{IFWDDHM}_{\omega}^{(x)}(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \cdots \widetilde{a}_{in})$$

Step3: Calculate the $S(\widetilde{a})$ and $H(\widetilde{a})$.
Step4: Sort all alternatives $\{x_1, x_2, \cdots, x_m\}$ and choose the best one.

5. An Illustrate Example

In this section, we give an example to explain the proposed method. A transportation company wants to pick a car and there are four cars as candidates $M_i = (M_1, M_2, M_3, M_4)$. We evaluate each supplier from four aspects $E_i = (E_1, E_2, E_3, E_4)$, which are "production price", "production quality", "production's service performance", and "risk factor". The weight vector of attributes is $\omega = (0.1, 0.4, 0.3, 0.2)^T$. There are four experts, and the weight vector of the experts is $(0.3, 0.4, 0.2, 0.1)^T$. Then the decision matrix $R_t = \left(a_{ij}^t\right)_{4\times 4}(t = 1,2,3,4)$ are shown in Tables 1–4, and our goal is to rank four cars and select the best one.

Table 1. Decision matrix R_1.

	E_1	E_2	E_3	E_4
M_1	(0.5 0.3)	(0.6 0.3)	(0.5 0.2)	(0.6 0.4)
M_2	(0.7 0.3)	(0.9 0.1)	(0.8 0.1)	(0.7 0.2)
M_3	(0.7 0.2)	(0.5 0.4)	(0.6 0.1)	(0.4 0.2)
M_4	(0.5 0.3)	(0.3 0.4)	(0.5 0.4)	(0.4 0.5)

Table 2. Decision matrix R_2.

	E_1	E_2	E_3	E_4
M_1	(0.6 0.2)	(0.7 0.1)	(0.6 0.2)	(0.6 0.3)
M_2	(0.9 0.1)	(0.8 0.2)	(0.7 0.1)	(0.6 0.4)
M_3	(0.5 0.2)	(0.6 0.3)	(0.7 0.2)	(0.8 0.1)
M_4	(0.7 0.2)	(0.4 0.3)	(0.5 0.5)	(0.6 0.3)

Table 3. Decision matrix R_3.

	E_1	E_2	E_3	E_4
M_1	(0.6 0.4)	(0.7 0.2)	(0.6 0.3)	(0.5 0.4)
M_2	(0.8 0.6)	(0.7 0.1)	(0.6 0.4)	(0.9 0.1)
M_3	(0.5 0.2)	(0.4 0.5)	(0.4 0.3)	(0.5 0.4)
M_4	(0.2 0.5)	(0.5 0.4)	(0.7 0.2)	(0.5 0.4)

Table 4. Decision matrix R_4.

	E_1	E_2	E_3	E_4
M_1	(0.6 0.2)	(0.5 0.4)	(0.6 0.4)	(0.4 0.5)
M_2	(0.7 0.3)	(0.8 0.1)	(0.6 0.2)	(0.9 0.1)
M_3	(0.6 0.4)	(0.3 0.6)	(0.2 0.6)	(0.5 0.3)
M_4	(0.3 0.5)	(0.2 0.7)	(0.5 0.4)	(0.3 0.6)

5.1. Decision-Making Processes

Step 1: Since the four attributes are of the same type, thus, we don't need to normalize the matrix $R_1 \sim R_4$.
Step 2: Use IFWDHM operator to fuse four decision matrix $R_t = \left(a_{ij}^t\right)_{m\times n}$ into a collective matrix $R = \left(a_{ij}^t\right)_{m\times n}$ which is shown in Table 5 (suppose $x = 2$, $\lambda = 2$).

Table 5. The collective decision matrix R.

	G_1	G_2	G_3	G_4
A_1	(0.2976 0.3875)	(0.3504 0.2771)	(0.2156 0.2689)	(0.3996 0.3304)
A_2	(0.5818 0.2103)	(0.5000 0.1638)	(0.4172 0.1781)	(0.4554 0.2073)
A_3	(0.3282 0.2872)	(0.4554 0.3079)	(0.3095 0.2316)	(0.2857 0.3472)
A_4	(0.2411 0.5299)	(0.3671 0.2684)	(0.1813 0.2504)	(0.0371 0.7363)

Use IFWDDHM operator to aggregate four decision matrixes $R_t = \left(a_{ij}^t\right)_{m\times n}$ into a collective matrix $R = \left(a_{ij}^t\right)_{m\times n}$, which is shown in Table 6 (suppose $x = 2$, $\lambda = 2$).

Table 6. The collective decision matrix R.

	G_1	G_2	G_3	G_4
A_1	(0.5372 0.0288)	(0.6100 0.0063)	(0.5618 0.0043)	(0.6535 0.0116)
A_2	(0.7000 0.0021)	(0.6667 0.0006)	(0.6422 0.0009)	(0.6646 0.0012)
A_3	(0.6300 0.0066)	(0.6344 0.0105)	(0.6066 0.0025)	(0.5969 0.0196)
A_4	(0.5827 0.1360)	(0.6729 0.0052)	(0.5818 0.0035)	(0.4172 0.6485)

Step 3: Use the IFWDHM (IFWDDHM) operator to aggregate all the attribute values $a_{ij}, a'_{ij} (j = 1, 2, 3, 4)$ and get the comprehensive evaluation value (suppose $x = 2$, $\lambda = 2$).

$$a_1 = (0.0694, 0.4051), a_2 = (0.5357, 0.2264), a_3 = (0.1464, 0.3736), a_4 = (0.0330, 0.6366).$$
$$a'_1 = (0.8010, 0.0103), a'_2 = (0.9380, 0.0011), a'_3 = (0.8584, 0.0087), a'_4 = (0.6690, 0.0290).$$

Step 4: Obtain the score values.

$$S(a_1) = -0.3357, S(a_2) = 0.3093, S(a_3) = -0.2272, S(a_4) = -0.6036.$$
$$S\left(a'_1\right) = 0.7907, S\left(a'_2\right) = 0.9369, S\left(a'_3\right) = 0.8497, S\left(a'_4\right) = 0.6399.$$

Step 5: Rank all alternatives. $a_2 \succ a_3 \succ a_1 \succ a_4$, then the best choice is a_2.

Considering the different parameter values of an IFWDHM operator that may have an impact on the ranking results, we calculated the scores produced from the different x and the results are listed in Table 7.

Table 7. Score and ranking of the alternatives with different parameter values x.

x	Score of $S(\tilde{a}_i)$	Ranking
$x = 1$	$S(a_1) = -0.0073, S(a_2) = 0.0352,$ $S(a_3) = -0.0002, S(a_4) = -0.0383.$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 2$	$S(a_1) = -0.3357, S(a_2) = 0.3093,$ $S(a_3) = -0.2272, S(a_4) = -0.6036$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 3$	$S(a_1) = -0.2988, S(a_2) = -0.1265,$ $S(a_3) = -0.2595, S(a_4) = -0.4853.$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 4$	$S(a_1) = 0.1391, S(a_2) = 0.3007,$ $S(a_3) = 0.1860, S(\tilde{a}_4) = -0.1440.$	$a_2 \succ a_3 \succ a_1 \succ a_4$

Considering the different parameter values of an IFWDDHM operator that may have an impact on the ordering results, we calculated the scores with different x and the results are listed in Table 8.

Table 8. Score and order of the alternatives with different parameter values x.

x	Score of $S(a_i)$	Ranking
$x = 1$	$S(a_1) = 0.0437, S(a_2) = 0.0915,$ $S(a_3) = 0.0548, S(a_4) = 0.0110.$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 2$	$S(a_1) = 0.7907, S(a_2) = 0.9369,$ $S(a_3) = 0.8497, S(a_4) = 0.6399.$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 3$	$S(a_1) = 0.2059, S(a_2) = 0.3597,$ $S(a_3) = 0.2397, S(a_4) = 0.0320.$	$a_2 \succ a_3 \succ a_1 \succ a_4$
$x = 4$	$S(a_1) = 0.2398, S(a_2) = 0.3695,$ $S(a_3) = 0.2477, S(a_4) = 0.0424.$	$a_2 \succ a_3 \succ a_1 \succ a_4$

From Tables 7 and 8, we get following conclusions.

When $x = 1$, the sorting of alternatives is $a_2 \succ a_3 \succ a_1 \succ a_4$, and the best choice is a_2.

When $x = 2, 3, 4$, the sorting of alternatives is $a_2 \succ a_3 \succ a_1 \succ a_4$, and the best choice is a_2.

Although there is the same best selection, the ranking is different. When $x = 1$, the interrelationship between the attributes is not considered, and when $x = 2, 3, 4$, we can consider the interrelationship for different number of attributes. So these results are reasonable for these two conditions.

5.2. Comparative Analysis

Following this, we compare the proposed method with IFWA operator [4], IFWG operator [5], IFWMM operator [62], and IFDWMM operator [62] and the comparative results are depicted in Table 9.

Table 9. Ordering of the green suppliers.

	Ordering
IFWA	$a_2 \succ a_3 \succ a_1 \succ a_4$
IFWG	$a_2 \succ a_3 \succ a_1 \succ a_4$
IFWMM	$a_2 \succ a_3 \succ a_1 \succ a_4$
IFDWMM	$a_2 \succ a_3 \succ a_1 \succ a_4$

From above analysis, we arrived at the same results. However, the existing operators, such as IFWA operator and IFWG operator do not consider the relationship between arguments, and thus cannot eliminate the corresponding influence of unfair arguments on decision result. The IFWMM operator, IFDWMM operator, IFWDHM and IFWDDHM operators consider the relationship among the arguments.

6. Conclusions

In this paper, we investigated the MADM problems with IFNs. Following this, we utilized the HM operator, DHM operator, DDHM operator, WDHM operator, and WDDHM operator to develop some novel operators with IFNs: Intuitionistic fuzzy DHM (IFDHM) operator, intuitionistic fuzzy WDHM operator, intuitionistic fuzzy DDHM (IFDDHM) operator, and intuitionistic fuzzy WDDHM (IFWDDHM) operator. The prominent characteristic of these proposed operators were studied. Moreover, we have utilized these operators to develop some models to solve the MAGDM problems with IFNs. Finally, a practical example for the selection of a car car supplier was given. In the future, the application of the IFNs needs to be explored in decision-making processes [63–72], risk analysis [73,74], and other fuzzy environments [75–80].

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References

1. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
2. Atanassov, K. More on intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *33*, 37–46. [CrossRef]
3. Burillo, P.; Bustince, H. Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. *Fuzzy Sets Syst.* **2001**, *118*, 305–316. [CrossRef]
4. Xu, Z.S. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
5. Xu, Z.S.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gener. Syst.* **2006**, *35*, 417–433. [CrossRef]
6. Xu, Z.S. Intuitionistic preference relations and their application in group decision making. *Inform. Sci.* **2007**, *177*, 2363–2379. [CrossRef]
7. Li, D.F. Extension of the LINMAP for multiattribute decision making under Atanassov's intuitionistic fuzzy environment. *Fuzzy Optim. Decis. Mak.* **2008**, *7*, 17–34. [CrossRef]
8. Xu, Z.S. Choquet integrals of weighted intuitionistic fuzzy information. *Inform. Sci.* **2010**, *180*, 726–736. [CrossRef]
9. Ye, J. Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Math. Comput. Model.* **2011**, *53*, 91–97. [CrossRef]
10. Li, D.-F.; Ren, H.-P. Multi-attribute decision making method considering the amount and reliability of intuitionistic fuzzy information. *J. Intell. Fuzzy Syst.* **2015**, *28*, 1877–1883.
11. Wei, G.W. Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Appl. Soft Comput.* **2010**, *10*, 423–431. [CrossRef]
12. Wei, G.W.; Zhao, X.F. Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making. *Expert Syst. Appl.* **2012**, *39*, 2026–2034. [CrossRef]
13. Wei, G. Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making. *Expert Syst. Appl.* **2011**, *38*, 11671–11677. [CrossRef]
14. Wei, G.W. GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting. *Knowl.-Based Syst.* **2010**, *23*, 243–247. [CrossRef]

15. Zhao, X.; Wei, G. Some Intuitionistic Fuzzy Einstein Hybrid Aggregation Operators and Their Application to Multiple Attribute Decision Making. *Knowl.-Based Syst.* **2013**, *37*, 472–479. [CrossRef]
16. Garg, H. Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. *Comput. Ind. Eng.* **2016**, *101*, 53–69. [CrossRef]
17. Chu, J.; Liu, X.; Wang, Y.-M.; Chin, K.-S. A group decision making model considering both the additive consistency and group consensus of intuitionistic fuzzy preference relations. *Comput. Ind. Eng.* **2016**, *101*, 227–242. [CrossRef]
18. Wan, S.-P.; Wang, F.; Dong, J.-Y. A novel risk attitudinal ranking method for intuitionistic fuzzy values and application to MADM. *Appl. Soft Comput.* **2016**, *40*, 98–112. [CrossRef]
19. Zhao, J.; You, X.-Y.; Liu, H.-C.; Wu, S.-M. An Extended VIKOR Method Using Intuitionistic Fuzzy Sets and Combination Weights for Supplier Selection. *Symmetry* **2017**, *9*, 169. [CrossRef]
20. Liu, P. Multiple Attribute Decision-Making Methods Based on Normal Intuitionistic Fuzzy Interaction Aggregation Operators. *Symmetry* **2017**, *9*, 261. [CrossRef]
21. Shi, Y.; Yuan, X.; Zhang, Y. Constructive methods for intuitionistic fuzzy implication operators. *Soft Comput.* **2017**, *21*, 5245–5264. [CrossRef]
22. Otay, I.; Öztaysi, B.; Onar, S.Ç.; Kahraman, C. Multi-expert performance evaluation of healthcare institutions using an integrated intuitionistic fuzzy AHP&DEA methodology. *Knowl.-Based Syst.* **2017**, *133*, 90–106.
23. Ai, Z.; Xu, Z. Multiple Definite Integrals of Intuitionistic Fuzzy Calculus and Isomorphic Mappings. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 670–680. [CrossRef]
24. Montes, I.; Pal, N.R.; Montes, S. Entropy measures for Atanassov intuitionistic fuzzy sets based on divergence. *Soft Comput.* **2018**, *22*, 5051–5071. [CrossRef]
25. Liu, Q.; Wu, C.; Lou, L. Evaluation research on commercial bank counterparty credit risk management based on new intuitionistic fuzzy method. *Soft Comput.* **2018**, *22*, 5363–5375. [CrossRef]
26. Li, Y.H.; Olson, D.L.; Zheng, Q. Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis. *Pattern Recognit. Lett.* **2007**, *28*, 278–285. [CrossRef]
27. Szmidt, E.; Kacprzyk, J. Distances between intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **2000**, *114*, 505–518. [CrossRef]
28. Szmidt, E.; Kacprzyk, J. A new concept of a similarity measure for intuitionistic fuzzy sets and its use in group decision making. In *Lecture Notes in Computer Science, (Subseries LNAI)*; Springer: Cham, Switzerland, 2005; Volume 3558, pp. 272–282.
29. Hung, W.L.; Yang, M.S. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognit. Lett.* **2004**, *25*, 1603–1611. [CrossRef]
30. Ziemba, P. NEAT F-PROMETHEE—A new fuzzy multiple criteria decision making method based on the adjustment of mapping trapezoidal fuzzy numbers. *Expert Syst. Appl.* **2018**, *110*, 363–380. [CrossRef]
31. Hung, W.L.; Yang, M.S. Similarity measures of intuitionistic fuzzy sets based on Lp metric. *Int. J. Approx. Reason.* **2007**, *46*, 120–136. [CrossRef]
32. Xu, Z.S.; Xia, M.M. Some new similarity measures for intuitionistic fuzzy values and their application in group decision making. *J. Syst. Sci. Eng.* **2010**, *19*, 430–452.
33. Li, Z.; Wei, G.; Gao, H. Methods for Multiple Attribute Decision Making with Interval-Valued Pythagorean Fuzzy Information. *Mathematics* **2018**, *6*, 228. [CrossRef]
34. Rajarajeswari, P.; Uma, N. Intuitionistic fuzzy multi similarity measure based on cotangent function. *Int. J. Eng. Res. Technol.* **2013**, *2*, 1323–1329.
35. Hwang, C.-M.; Yang, M.-S.; Hung, W.-L. New similarity measures of intuitionistic fuzzy sets based on the Jaccard index with its application to clustering. *Int. J. Intell. Syst.* **2018**, *33*, 1672–1688. [CrossRef]
36. Ye, J. Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes. *J. Intell. Fuzzy Syst.* **2016**, *30*, 151–158. [CrossRef]
37. Ye, J. Multicriteria group decision-making method using vector similarity measures for trapezoidal intuitionistic fuzzy numbers. *Group Decis. Negotiat.* **2012**, *21*, 519–530. [CrossRef]
38. Wei, G.W. Some similarity measures for picture fuzzy sets and their applications. *Iran. J. Fuzzy Syst.* **2018**, *15*, 77–89.
39. Wei, G.W.; Gao, H. The generalized Dice similarity measures for picture fuzzy sets and their applications. *Informatica* **2018**, *29*, 1–18. [CrossRef]

40. Wei, G.W.; Wei, Y. Similarity measures of Pythagorean fuzzy sets based on cosine function and their applications. *Int. J. Intell. Fuzzy Syst.* **2018**, *33*, 634–652. [CrossRef]
41. Wei, G.W. Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making. *Informatica* **2017**, *28*, 547–564. [CrossRef]
42. Wang, C.-Y.; Chen, S.-M. A new multiple attribute decision making method based on linear programming methodology and novel score function and novel accuracy function of interval-valued intuitionistic fuzzy values. *Inf. Sci.* **2018**, *438*, 145–155. [CrossRef]
43. Merigo, J.M.; Gil-Lafuente, A.M. Fuzzy induced generalized aggregation operators and its application in multi-person decision making. *Expert Syst. Appl.* **2011**, *38*, 9761–9772. [CrossRef]
44. Wei, G.W. Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Fundam. Inform.* **2018**, *157*, 271–320. [CrossRef]
45. Wei, G.W.; Alsaadi, F.E.; Hayat, T.; Alsaedi, A. Picture 2-tuple linguistic aggregation operators in multiple attribute decision making. *Soft Comput.* **2018**, *22*, 989–1002. [CrossRef]
46. Gao, H.; Lu, M.; Wei, G.W.; Wei, Y. Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making. *Fundam. Inform.* **2018**, *159*, 385–428. [CrossRef]
47. Wei, G.W.; Gao, H.; Wei, Y. Some q-Rung Orthopair Fuzzy Heronian Mean Operators in Multiple Attribute Decision Making. *Int. J. Intell. Syst.* **2018**, *33*, 1426–1458. [CrossRef]
48. Mohagheghi, V.; Mousavi, S.M.; Vahdani, B. Enhancing decision-making flexibility by introducing a new last aggregation evaluating approach based on multi-criteria group decision making and Pythagorean fuzzy sets. *Appl. Soft Comput.* **2017**, *61*, 527–535. [CrossRef]
49. Wei, G.W. Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making. *Kybernetes* **2017**, *46*, 1777–1800. [CrossRef]
50. Zhang, L.; Zhan, J.; Alcantud, J.C.R. Novel classes of fuzzy soft β-coverings-based fuzzy rough sets with applications to multi-criteria fuzzy group decision making. *Soft Comput.* **2018**. [CrossRef]
51. Wei, G.W.; Gao, H.; Wang, J.; Huang, Y.H. Research on Risk Evaluation of Enterprise Human Capital Investment with Interval-valued bipolar 2-tuple linguistic Information. *IEEE Access* **2018**, *6*, 35697–35712. [CrossRef]
52. Ziemba, P.; Jankowski, J.; Watróbski, J. Online Comparison System with Certain and Uncertain Criteria Based on Multi-criteria Decision Analysis Method. In Proceedings of the Computational Collective Intelligence ICCCI 2017, Nicosia, Cyprus, 27 September 2017; pp. 579–589.
53. Wang, J.; Wei, G.; Lu, M. TODIM Method for Multiple Attribute Group Decision Making under 2-Tuple Linguistic Neutrosophic Environment. *Symmetry* **2018**, *10*, 486. [CrossRef]
54. Dombi, J. A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets Syst.* **1982**, *8*, 149–163. [CrossRef]
55. Liu, P.D.; Liu, J.L.; Chen, S.M. Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making. *J. Oper. Res. Soc.* **2018**, *69*. [CrossRef]
56. Chen, J.Q.; Ye, J. Some Single-Valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple Attribute Decision-Making. *Symmetry* **2017**, *9*, 82. [CrossRef]
57. Wei, G.; Wei, Y. Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2018**, *35*, 2001–2013. [CrossRef]
58. Hara, T.; Uchiyama, M.; Takahasi, S.E. A refinement of various mean inequalities. *J. Inequal. Appl.* **1998**, *2*, 387–395. [CrossRef]
59. Wu, S.; Wang, J.; Wei, G.; Wei, Y. Research on Construction Engineering Project Risk Assessment with Some 2-Tuple Linguistic Neutrosophic Hamy Mean Operators. *Sustainability* **2018**, *10*, 1536. [CrossRef]
60. Chen, S.M.; Tan, J.M. Handling multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy Sets Syst.* **1994**, *67*, 163–172. [CrossRef]
61. Hong, D.H.; Choi, C.H. Multicriteria fuzzy problems based on vague set theory. *Fuzzy Sets Syst.* **2000**, *114*, 103–113. [CrossRef]
62. Liu, P.D.; Li, D.F. Some Muirhead Mean Operators for Intuitionistic Fuzzy Numbers and Their Applications to Group Decision Making. *PLoS ONE* **2017**, *12*, e0168767. [CrossRef] [PubMed]
63. Gao, H.; Wei, G.W.; Huang, Y.H. Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *IEEE Access* **2018**, *6*, 11508–11522. [CrossRef]

64. Merigó, J.M.; Gil-Lafuente, A.M. Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making. *Inf. Sci.* **2013**, *236*, 1–16. [CrossRef]
65. Wei, G.W.; Alsaadi, F.E.; Hayat, T.; Alsaedi, A. Projection models for multiple attribute decision making with picture fuzzy information. *Int. J. Mach. Learn. Cybern.* **2018**, *9*, 713–719. [CrossRef]
66. Chen, T.-Y. The inclusion-based TOPSIS method with interval-valued intuitionistic fuzzy sets for multiple criteria group decision making. *Appl. Soft Comput.* **2015**, *26*, 57–73. [CrossRef]
67. Li, Z.; Wei, G.; Lu, M. Pythagorean Fuzzy Hamy Mean Operators in Multiple Attribute Group Decision Making and Their Application to Supplier Selection. *Symmetry* **2018**, *10*, 505. [CrossRef]
68. Al-Quran, A.; Hassan, N. The Complex Neutrosophic Soft Expert Relation and Its Multiple Attribute Decision-Making Method. *Entropy* **2018**, *20*, 101. [CrossRef]
69. Wei, G.W.; Alsaadi, F.E.; Hayat, T.; Alsaedi, A. Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making. *Int. J. Fuzzy Syst.* **2018**, *20*, 1–12. [CrossRef]
70. Verma, R. Multiple attribute group decision making based on generalized trapezoid fuzzy linguistic prioritized weighted average operator. *Int. J. Mach. Learn. Cybern.* **2017**, *8*, 1993–2007. [CrossRef]
71. Wang, J.; Wei, G.; Lu, M. An Extended VIKOR Method for Multiple Criteria Group Decision Making with Triangular Fuzzy Neutrosophic Numbers. *Symmetry* **2018**, *10*, 497. [CrossRef]
72. Wang, J.; Wei, G.; Gao, H. Approaches to Multiple Attribute Decision Making with Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Information. *Mathematics* **2018**, *6*, 201. [CrossRef]
73. Wei, Y.; Liu, J.; Lai, X.; Hu, Y. Which determinant is the most informative in forecasting crude oil market volatility: Fundamental, speculation, or uncertainty? *Energy Econ.* **2017**, *68*, 141–150. [CrossRef]
74. Wei, Y.; Yu, Q.; Liu, J.; Cao, Y. Hot money and China's stock market volatility: Further evidence using the GARCH-MIDAS model. *Phys. A Stat. Mech. Appl.* **2018**, *492*, 923–930. [CrossRef]
75. Wei, G.W. TODIM method for picture fuzzy multiple attribute decision making. *Informatica* **2018**, *29*, 555–566.
76. Deng, X.M.; Wei, G.W.; Gao, H.; Wang, J. Models for safety assessment of construction project with some 2-tuple linguistic Pythagorean fuzzy Bonferroni mean operators. *IEEE Access* **2018**, *6*, 52105–52137. [CrossRef]
77. Akram, M.; Shahzadi, S. Novel intuitionistic fuzzy soft multiple-attribute decision-making methods. *Neural Comput. Appl.* **2018**, *29*, 435–447. [CrossRef]
78. Huang, Y.H.; Wei, G.W. TODIM Method for Pythagorean 2-tuple Linguistic Multiple Attribute Decision Making. *J. Intell. Fuzzy Syst.* **2018**, *35*, 901–915. [CrossRef]
79. Hao, Y.; Chen, X. Study on the ranking problems in multiple attribute decision making based on interval-valued intuitionistic fuzzy numbers. *Int. J. Intell. Syst.* **2018**, *33*, 560–572. [CrossRef]
80. José Carlos, R. Alcantud: Some formal relationships among soft sets, fuzzy sets, and their extensions. *Int. J. Approx. Reason.* **2016**, *68*, 45–53.