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Some Globally Stable Fixed Points in *b*-Metric Spaces

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Received: 8 October 2018; Accepted: 24 October 2018; Published: 30 October 2018

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Abstract: In this paper, the existence and uniqueness of globally stable fixed points of asymptotically contractive mappings in complete *b*-metric spaces were studied. Also, we investigated the existence of fixed points under the setting of a continuous mapping. Furthermore, we introduce a contraction mapping that generalizes that of Banach, Kanan, and Chatterjea. Using our new introduced contraction mapping, we establish some results on the existence and uniqueness of fixed points. In obtaining some of our results, we assume that the space is associated with a partial order, and the *b*-metric function has the regularity property. Our results improve, and generalize some current results in the literature.

Keywords: fixed point; globally stable fixed point; asymptotically contractive mapping; regularity condition; order preserving mapping; altering distance function

1. Introduction

The research area of fixed point theory is playing an important role in finding solutions for some nonlinear equations (differential equations). The stability of a solution(fixed point) determines the long term effectiveness of the solution when subjected to a perturbation(usually small).

The early fixed point theorems were published between 1910–1945 [1]. The early fixed points theorems were established by Brouwer (1912) [2], Banach (1922) [3], Schauder (1930) [4], and Kakutani (1941) [5], see also [1]. Later in 1955, Tarski (Knaster-Tarski) fixed point theorem emerged with an inclusion of order relation [6]. The advent of Tarski fixed point theorem brought an alternative to the usage of a continuous or contractive mappings to establish the existence of a fixed point. Since then, many researchers establish results that combine the usage of an order and weaker contractive conditions on the mappings, see [7–9].

In the area of fixed point theory, the importance of famous Banach contraction mapping theorem [3] can never be over emphasized. Banach fixed point theorem/principle centered around the contraction of the mapping in discussion. Another importance of the Banach contraction principle is that, it allows the sequence of the successive approximation (picard iterations) to converge to a solution of the problem in discussion [1]. The successive approximations developed by Picard in 1980 can solve both linear and nonlinear problems [10]. Many authors established an analogue, generalization, and improvement of Banach fixed point theorem, both from the perspective of the spaces and the mapping in consideration, see [7–9,11–14]. In establishing the existence and uniqueness of a fixed point, the mapping in discussion is very important.

In the same direction, Kannan in 1969 [15] brought to light a fixed point theorem with a different contraction mapping compare to that of Banach [3]; i.e., he proves the existence of a fixed point in a complete metric space (X, d) with a mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq \lambda(d(x, Tx) + d(y, Ty)), \ \ \forall\, x, y \in X \text{ and } \lambda \in [0, \frac{1}{2}).$$

Furthermore, Chatterjea in 1972 [16] introduce another fixed point theorem with a different contraction mapping, if compare with both that of Banach [3] and Kannan [15]; i.e., he proves the existence of a fixed point in a complete metric space (X, d) with a mapping $T : X \to X$ that satisfy

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx)), \ \ \forall\, x, y \in X \text{ and } \lambda \in [0, \frac{1}{2}).$$

Very recently, in 2018, Zhou et al. [17] extend the result of Chatterjea [16] to a complete b-partial metric space.

On the other hand, it is from the work of Bourbaki [18], and Bakhtin [12] that, the idea/concept of a b-metric was initiated. Later in 1993, Czerwik [19] provide an axiom that is weaker than the triangular inequality, and formally defined a b-metric space with a sole motive of generalizing the Banach contraction mapping theorem [3]. Subsequently, the concept was improved by many authors [20], others generalized the concept [21,22] and established some fixed point existence results in b-metric spaces.

In 2013, Kamihigashi and Stachurski proved some existence and uniqueness theorems of a fixed point in a complete metric space [8]. In 2017, Rezai and Dinarvand [23] established the existence of a fixed point using a setting that generalizes the Chatterjea contraction mapping [16]. Recently in 2018, Yusuf and Kumam [9] extend the work of Kamihigashi and Stachurski to a partial metric space. On the other hand, in 2018, Du et al. [24] establish the existence results of a fixed point that generalizes results of Banach [3], Kannan [15] and Chatterjea [16]. In this paper, motivated by Kamihigashi et al. [8], Du et al. [24], Zhou et al. [17], and Yusuf et al. [9], we establish the existence of fixed points in a complete b-metric space associated with a partial order. We also investigated the global stability of the fixed points of an asymptotically contractive mapping.

2. Preliminaries

Let X be a non empty set, $\mathbb{R}_+$ be the set of non negative real numbers and $\mathbb{R}$ be the set of real numbers. The following definitions can be found in [8] unless otherwise stated.

Definition 1. *Let $\preceq$ be a binary relation on the set X then, the relation $\preceq$ is*

1. ***Reflexive** if $x \preceq x, \ \forall x \in X$.*
2. ***Antisymmetric** if $x \preceq y$ and $y \preceq x \Longrightarrow x = y, \ \ \forall x, y \in X$.*
3. ***Transitive** if $x \preceq y$ and $y \preceq z \Longrightarrow x \preceq z, \ \ \forall x, y, z \in X$.*

The binary relation $\preceq$ is called a **partial order** if it satisfies all of the above conditions (1–3), we call the pair (X,$\preceq$) a partial ordered set.

Definition 2. *In view of Kamihigashi et al. [8], a function* $\Psi : X \times X \to \mathbb{R}_+$ *is **Regular** if whenever* $x \preceq y \preceq z$*, then* $\max\{\Psi(x,y), \Psi(y,z)\} \leq \Psi(x,z)$*,* $\forall x,y,z \in X$*, where* $(X, \preceq)$ *is an ordered space,* max *function is from* $\mathbb{R}_+ \times \mathbb{R}_+$ *to* $\mathbb{R}_+$*.*

Definition 3. *Let* $(X, \preceq)$ *be an ordered space. Two elements* $x, y \in X$ *are said to be comparable if* $x \preceq y$ *or* $y \preceq x$*. A mapping* $T : X \to X$ *is order preserving if* $x \preceq y \Longrightarrow Tx \preceq Ty$ *for all* $x, y \in X$*. We say that a sequence* $\{x_i\}_{\in \mathbb{N}} \subseteq X$ *is increasing if* $x_i \preceq x_{i+1}, \forall i \in \mathbb{N}$*.*

Definition 4. *A **metric** on X is a function* $d : X \times X \to \mathbb{R}_+$ *such that,*
(D1) $\forall x,y \in X,\ d(x,y) = 0 \iff x = y$.
(D2) $\forall x,y \in X,\ d(x,y) = d(y,x)$.
(D3) $\forall x,y,z \in X,\ d(x,y) \leq d(x,z) + d(z,y)$.

Definition 5. *A **b-metric** on X is a function* $d_b : X \times X \to \mathbb{R}_+$ *such that,*
(D_b1) $\forall x,y \in X,\ d_b(x,y) = 0 \iff x = y$.
(D_b2) $\forall x,y \in X,\ d_b(x,y) = d_b(y,x)$.
(D_b3) *There exist a real number* $s \geq 1$*, for which* $d_b(x,y) \leq s\,[d_b(x,z) + d_b(z,y)]\,,\ \forall x,y,z \in X$.
It is clear to see that, every metric is a b-metric with $s = 1$*, see [12].*

Example 1. *Consider the* $l_p(0 < p < 1)$ *space* $l_p(0 < p < 1)$*,*

$$l_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with the function

$$d : l_p \times l_p \to \mathbb{R},\ d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^p,$$

where $x = (x_n)$*,* $y = (y_n) \in l_p$*. Then* l_p *is a b-metric space, and* $d(x,z\ ) \leq 2^{\frac{1}{p}}[d(x,y\ ) + d(y,z)]$*. Thus,* $s = 2^{p-1}$ *[12,25].*

Lemma 1. *Let* $a, b \in \mathbb{R}$*,* $n \in 2\mathbb{N}$*. Then,* $(a+b)^n \leq 2^{n-1}(a^n + b^n)$*.*

The proof follows from the well known inequality

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2}$$

which follows from the Jensen's inequality [26] since the function $g(x) = x^n$ is a convex on $\mathbb{R}$.

Example 2. *Let* $X = \mathbb{R}$*,* $n \in 2\mathbb{N}$*. Define* $d_b : X \times X \to \mathbb{R}_+$ *by* $d_b = (x-y)^n$*,* $\forall x,y \in X$*. Then,* d_b *is a b-metric with* $s = 2^{n-1}$*, and* d_b *is not a metric.*

Proof. The conditions D_b1 and D_b2 are trivial for all $x, y \in X$, and $n \in 2\mathbb{N}$. Condition D_b3 can be seen as follows. Let $x, y, z \in \mathbb{R}$. Then,

$$\begin{aligned} d_b(x,y) &= (x - z + z - y)^n && (1) \\ &= ((x-z) + (z-y))^n\,. && (2) \end{aligned}$$

Let $a = (x - z)$, and $b = (z - y)$. Without lost of generality, we assume $a \leq b$, from Lemma 1, (1) and (2), we have

$$\begin{aligned} d_b(x,y) &= ((x-z)+(z-y))^n \\ &= (a+b)^n \\ &\leq 2^{n-1}(a^n+b^n) \\ &= 2^{n-1}\left(d_b(x,z)+d_b(z,y)\right). \end{aligned}$$

Thus, D_b3 is satisfied.

Furthermore, for all $n \in 2\mathbb{N}$, $2^{n-1} \neq 1$. Hence, d_b is not a metric. □

Definition 6. *In view of Kamihigashi et al. [8], a mapping $T : X \to X$ is* ***asymptotically contractive in a b-metric space*** *(X, d_b) if*

$$d_b(T^n x, T^n y) \to 0, \ \forall x, y \in X. \tag{3}$$

Definition 7. *A fixed point $\hat{x} \in X$ of an asymptotically contractive mapping T in a b-metric space (X, d_b) is globally stable if*

$$d_b(\hat{x}, T^n y) \to 0. \tag{4}$$

Example 3. *Let $x, y \in X = [0, 1)$. Define a mapping $T : X \to X$ by $T(x) = \dfrac{x}{x+1}$, and $d_b : X \times X \to \mathbb{R}_+$ by $d_b(x, y) = (x-y)^{10}$.*

Clearly, d_b is a b-metric, T is an asymptotically contractive mapping. Also, $0 \in X$ is the fixed point of T. Furthermore, 0 is a globally stable fixed point of T.

Definition 8. *[27] An altering distance function $\psi : [0, \infty) \to [0, \infty)$ is the function satisfying the following properties:*

1. *ψ is continuous and nondecreasing.*
2. *$\psi(t) = 0$ iff $t = 0$.*

Definition 9. *Suppose ψ is an altering distance function, and $\phi : [0, +\infty) \times [0, \infty) \to [0, \infty)$ satisfies both (5) and (6),*

$$\phi(x, y) = 0 \Longleftrightarrow x = y, \ \text{and} \tag{5}$$

$$\phi(\lim_{n\to+\infty} \inf a_n, \lim_{n\to+\infty} \inf b_n) \leq \lim_{n\to+\infty} \inf \phi(a_n, b_n). \tag{6}$$

Then, a mapping $f : X \to X$ is a $(\psi, \phi)_s-$ weakly C-contractive if

$$\psi(sd(fx, fy)) \leq \psi\left(\frac{d(x, fy) + d(y, fx)}{s+1}\right) - \phi(d(x, fy), d(y, fx)). \tag{7}$$

3. Main Results

In this section, the bellow assumptions were considered.

Assumption 1. *Let d_b be regular, and $\preceq$ is a reflexive order defined on X.*

Assumption 2. *For any increasing sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ converging to $x \in X$, we have $x_i \preceq x \ \ \forall i \in \mathbb{N}$, and if there exists $y \in X$ such that, $x_i \preceq y \ \ \forall i \in \mathbb{N}$, then $x \preceq y$.*

Theorem 1. *Suppose (X, d_b) is a complete b-metric space, and for any $x, y \in X$, we have*

$$x \preceq y \Longrightarrow d_b(T^i x, T^i y) \to 0. \tag{8}$$

Suppose also there exist $u, v \in X$ with T order preserving such that,

$$\begin{aligned} u &\preceq Tu, & (9)\\ T^i u &\preceq v, \ \forall i \in \mathbb{N}. & (10) \end{aligned}$$

Then T has a fixed point.

Proof of Theorem 1. Now, let $x_i = T^i u$, $\forall i \in \mathbb{N}$. It follows from (9) and order preserving condition on T that, $\{x_i\}_{i \in \mathbb{N}}$ is increasing. Next we show $\{x_i\}$ is Cauchy using (8)–(10), and regularity of d_b. Let $\epsilon > 0$, from (8)–(10) there exists $m \in \mathbb{N}$ such that $d_b(T^m u, T^m v) < \epsilon$. Let $j, k \in \mathbb{N}$ such that, $k > j > m$ and $N = k - m$. Using $x_m \preceq x_j \preceq x_k$, we have

$$\begin{aligned} d_b(x_j, x_k) &\leq d_b(x_m, x_k) \\ &= d_b(T^m u, T^k u) \\ &= d_b(T^m u, T^m T^N u) \\ &\leq d_b(T^m u, T^m v) \\ &< \epsilon. \end{aligned}$$

Hence, $\lim_{j,k \to \infty} d_b(x_j, x_k) = 0$, which implies that $\{x_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence. By completeness of (X, d_b), there exists $\hat{x} \in X$ such that $x_i \to \hat{x}$, i.e., $\lim_{n \to \infty} d_b(x_i, \hat{x}) = 0$.

Now, using Assumption 2, (9), and the order preserving condition on T, we have

$$u \preceq T^i u \preceq \hat{x}, \ \ \forall i \in \mathbb{N}, \tag{11}$$

by applying the order preserving property of T in (11), we have

$$T^i u \preceq T^{i+1} \preceq T\hat{x}, \ \ \forall i \in \mathbb{N}. \tag{12}$$

Using the regularity property of d_b, (11) and (12), we proceed as

$$\begin{aligned} d_b(T\hat{x}, \hat{x}) &\leq \lim_{i \to \infty} s\left(d_b(T\hat{x}, x_i) + d_b(x_i, \hat{x})\right) \\ &= s \lim_{i \to \infty} d_b(T\hat{x}, x_i) \\ &\leq s \lim_{i \to \infty} d_b(T^i \hat{x}, T^i u) \\ &= 0. \end{aligned}$$

The above relation permit us to conclude that, $\hat{x}$ is a fixed point of the mapping T. □

Theorem 2. *Suppose the mapping $T : X \to X$ is asymptotically contractive, $\hat{x} \in X$ is a fixed point of T, and $x_n = T^n u$ for some $u \in X$ and $n \in \mathbb{N}$. Then, we have*

$$d_b(x_n, \hat{x}) \to 0 \Longleftrightarrow d_b(x_n, Tx_n) \to 0.$$

Proof of Theorem 2. Let $\hat{x} \in X$ be a fixed point of T and $s \geq 1$.

The forward case: Let $d_b(x_n, \hat{x}) \to 0$, we have

$$\begin{aligned} d_b(x_n, Tx_n) &\leq s(d_b(x_n, \hat{x}) + d_b(\hat{x}, Tx_n)) \\ &= s(d_b(x_n, \hat{x}) + d_b(\hat{x}, x_{n+1})) \\ &\longrightarrow 0. \end{aligned}$$

The backward case: Let $d_b(x_n, Tx_n) \to 0$, we have

$$\begin{aligned} d_b(x_n, \hat{x}) &\leq s(d_b(x_n, Tx_n) + d_b(Tx_n, \hat{x})) \\ &= s(d_b(x_n, Tx_n) + d_b(Tx_n, T^n\hat{x})) \\ &= s(d_b(x_n, Tx_n) + d_b(T^n(Tu), T^n\hat{x})) \\ &\longrightarrow 0. \end{aligned}$$

□

Theorem 3. *Suppose T is asymptotically contractive self mapping in a b-metric space (X, d_b), and $z \in X$ is a fixed point of T. Then, z is unique and globally stable.*

Proof of Theorem 3. Let $z, y \in X$ be any two fixed points of T. For T asymptotically contractive mapping we have

$$d_b(z, y) = d_b(T^i z, T^i y) \to 0, \tag{13}$$

hence, the fixed point is unique.

Also, let $z \in X$ be a fixed point of T and $y \in X$ be any point. For T asymptotically contractive mapping we have

$$d_b(z, T^i y) = d_b(T^i z, T^i y) \to 0, \tag{14}$$

hence, z is a globally stable fixed point of T. □

Corollary 1. *[8] Suppose (X, d) is a complete metric space, and for any $x, y \in X$ we have*

$$x \preceq y \Longrightarrow d(T^i x, T^i y) \to 0. \tag{15}$$

Suppose also there exist $u, v \in X$ with T order preserving such that (9) and (10) are satisfied. Then, T has a fixed point.

Corollary 2. *[8] Let $(X, \preceq)$ be a partially ordered set, (X,d) be a complete metric space, and $\Psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{i \to \infty} \Psi^i(t) = 0$ for each $t > 0$. Suppose that, for any comparable $x, y \in X$ we have*

$$x \preceq y \Longrightarrow d(T^i x, T^i y) \leq \Psi^i(d(x, y)). \tag{16}$$

Then, T has a fixed point.

By dropping Assumption 2, the below existence theorem follows.

Theorem 4. *Suppose (X, d_b) is a complete b-metric space, and for any comparable $x, y \in X$ we have*

$$x \preceq y \Longrightarrow d_b(T^i x, T^i y) \to 0. \tag{17}$$

Suppose also there exist $u, v \in X$ with T continuous and order preserving such that, (9) and (10) are satisfied. Then, T has a fixed point.

Proof of Theorem 4. For showing the sequence $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy, we use similar arguments as those given in the proof of Theorem 1. The limit of the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ can easily be seen as the fixed point of T using the continuity of T. □

Furthermore, the uniqueness and global stability of the fixed point can be established with T continuous and asymptotically contractive without Assumption 2.

Theorem 5. *Let (X,d_b) be a complete b-metric space with $s \geq 1$, and associated with a partial order $\preceq$. Suppose for all comparable elements $x, y \in X$, the mapping $T : X \to X$ is order preserving and satisfies the below condition*

$$\begin{aligned}\psi(d_b(Tx,Ty)) \quad &\leq \quad \psi\left(\alpha\left(\frac{d_b(x,y)+d_b(x,Tx)+d_b(x,Ty)+d_b(y,Ty)+d_b(y,Tx)}{s+4}\right)\right)\\ &\quad - \min\{\phi(\{d_b(x,Tx),d_b(y,Ty)),\phi(\{d_b(x,Ty),d_b(y,Tx))\}\end{aligned} \tag{18}$$

for some $\alpha \in [0,\gamma)$, where $\gamma = \min\{\frac{1}{s^2}, \frac{s+4}{5s}\}$, ϕ satisfy (5), and ψ a distance altering function. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Proof. Let us start by showing the uniqueness of the fixed point of T. For the sake of contradiction, we assume that, $x_1, x_2 \in X$ are two distinct fixed points of T. Then,

$$\begin{aligned}\psi(d_b(x_1,x_2)) &= \psi(d_b(Tx_1,Tx_2))\\ &\leq \psi\left(\alpha\left(\frac{d_b(x_1,x_2)+d_b(x_1,Tx_1)+d_b(x_1,Tx_2)+d_b(x_2,Tx_2)+d_b(x_2,Tx_1)}{s+4}\right)\right)\\ &\quad - \min\{\phi(d_b(x_1,Tx_1),d_b(x_2,Tx_2)),\phi(d_b(x_1,Tx_2),d_b(x_2,Tx_1))\}\\ &\leq \psi\left(\alpha\left(\frac{d_b(x_1,x_2)+d_b(x_1,Tx_1)+d_b(x_1,Tx_2)+d_b(x_2,Tx_2)+d_b(x_2,Tx_1)}{s+4}\right)\right)\\ &= \psi\left(\alpha\left(\frac{d_b(x_1,x_2)+d_b(x_1,x_2)+d_b(x_2,x_1)}{s+4}\right)\right)\\ &= \psi\left(\frac{3\alpha}{s+4}d_b(x_1,x_2)\right)\\ &< \psi\left(\frac{3}{s+4}d_b(x_1,x_2)\right)\\ &\leq \psi\left(\frac{3}{5}d_b(x_1,x_2)\right).\end{aligned} \tag{19}$$

Thus, from the property of ψ, inequality (19) implies $d_b(x_1,x_2) < \frac{3}{5}d_b(x_1,x_2)$. Hence, a contradiction. Therefore, if a fixed point of T exist, then it is unque.

Next, we show the existence of the fixed point. Let $x_0 \in X$ be such that, $x_0 \preceq Tx_0$. If $x_0 = Tx_0$ then x_0 is the fixed point. Suppose that $x_0 \neq Tx_0$. Then, define a sequence $x_n \subseteq X$ by $x_n = Tx_{n-1}$, $\forall\, n \in \mathbb{N}$. For T being order preserving and $x_0 \preceq Tx_0$, we have

$$x_0 \preceq Tx_0 = x_1,\ \ x_1 \preceq Tx_1 = x_2,\ \ x_2 \preceq Tx_2 = x_3,\ \ \cdots, x_n \preceq Tx_n = x_{n+1}.$$

By transitivity of $\preceq$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

If $x_n = Tx_n$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Suppose $x_n \neq Tx_n$ for all $n \in \mathbb{N}$.

Now, let $g_n = d_b(x_n, x_{n+1})$, we show that, g_n is a non-increasing sequence and

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0.$$

So, we proceed as follows,

$$\begin{aligned}
\psi(d_b(x_n, x_{n+1})) &= \psi(d_b(Tx_{n-1}, Tx_n)) \\
&\leq \psi\left(\alpha\left(\frac{d_b(x_{n-1}, x_n) + d_b(x_{n-1}, Tx_{n-1}) + d_b(x_{n-1}, Tx_n) + d_b(x_n, Tx_n) + d_b(x_n, Tx_{n-1})}{s+4}\right)\right) \\
&\quad - \min\{\phi(d_b(x_{n-1}, Tx_{n-1}), d_b(x_n, Tx_n)), \phi(d_b(x_{n-1}, Tx_n), d_b(x_n, Tx_{n-1}))\} \\
&= \psi\left(\alpha\left(\frac{d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_{n+1}) + d_b(x_n, x_{n+1}) + 0}{s+4}\right)\right) \\
&\quad - \min\{\phi(d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1})), \phi(d_b(x_{n-1}, x_{n+1}), 0)\} \\
&\leq \psi\left(\alpha\left(\frac{2d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_{n+1}) + d_b(x_n, x_{n+1})}{s+4}\right)\right) \\
&\leq \psi\left(s\alpha\left(\frac{2d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1}) + d_b(x_n, x_{n+1})}{s+4}\right)\right) \\
&= \psi\left(s\alpha\left(\frac{3d_b(x_{n-1}, x_n) + 2d_b(x_n, x_{n+1})}{s+4}\right)\right).
\end{aligned}$$

For ψ nondecreasing coupled with the immediate above inequality, we have

$$d_b(x_n, x_{n+1}) \leq s\alpha\left(\frac{3d_b(x_{n-1}, x_n) + 2d_b(x_n, x_{n+1})}{s+4}\right). \tag{20}$$

From the above inequality (20), we have

$$d_b(x_n, x_{n+1}) \leq \frac{3s\alpha}{s+4-2s\alpha} d_b(x_{n-1}, x_n). \tag{21}$$

From inequality (21), and for $\alpha \in [0, \gamma)$, we conclude that $\{g_n\}_{n=1}^{\infty}$ is a nonincreasing sequence in X which is bounded below by 0. Thus, $\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0$ [28].

Next we show that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

$$\begin{aligned}
\psi(d_b(x_n, x_m)) &= \psi(d_b(Tx_{n-1}, Tx_{m-1})) \\
&\leq \psi\left(\alpha\left(\frac{d_b(x_{n-1}, x_{m-1}) + d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_m) + d_b(x_{m-1}, x_m) + d_b(x_{m-1}, x_n)}{s+4}\right)\right) \\
&\quad - \min\{\phi(d_b(x_{n-1}, x_n), d_b(x_{m-1}, x_m)), \phi(d_b(x_{n-1}, x_m), d_b(x_{m-1}, x_n))\} \\
&\leq \psi\left(s\alpha\left(\frac{d_b(x_{n-1}, x_n) + d_b(x_n, x_{m-1}) + d_b(x_{n-1}, x_n) + d_b(x_{n-1}, x_n) + d_b(x_n, x_m)}{s+4}\right)\right) \\
&\quad + \psi\left(s\alpha\left(\frac{d_b(x_{m-1}, x_m) + d_b(x_{m-1}, x_n)}{s+4}\right)\right) \\
&= \psi\left(s\alpha\left(\frac{3d_b(x_{n-1}, x_n) + 2d_b(x_n, x_{m-1}) + d_b(x_n, x_m) + d_b(x_{m-1}, x_m)}{s+4}\right)\right) \\
&\leq \psi\left(s^2\alpha\left(\frac{3d_b(x_{n-1}, x_n) + 2d_b(x_n, x_m) + 2d_b(x_m, x_{m-1}) + d_b(x_n, x_m) + d_b(x_{m-1}, x_m)}{s+4}\right)\right) \\
&= \psi\left(s^2\alpha\left(\frac{3d_b(x_{n-1}, x_n) + 3d_b(x_n, x_m) + 3d_b(x_m, x_{m-1})}{s+4}\right)\right).
\end{aligned} \tag{22}$$

Now, from the property of ψ and inequality (22), we have

$$d_b(x_n, x_m) \leq \alpha s^2\left(\frac{3d_b(x_{n-1}, x_n) + 3d_b(x_n, x_m) + 3d_b(x_m, x_{m-1})}{s+4}\right), \tag{23}$$

inequality (23) implies

$$d_b(x_n, x_m) \leq 3\alpha s^2 \left(\frac{d_b(x_{n-1}, x_n) + d_b(x_m, x_{m-1})}{s + 4 - 3\alpha s^2} \right). \tag{24}$$

From the fact that, $\alpha \in [0, \gamma)$, we have $s + 4 - 3\alpha s^2 > 0$. Taking the limits of both sides of (24), we have

$$\lim_{n,m\to\infty} d_b(x_n, x_m) = 0. \tag{25}$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. For (X, d_b) being complete, there exist $\hat{x} \in X$ such that $\lim_{n\to\infty} x_n = \hat{x}$. Next, we show that, $T\hat{x} = \hat{x}$. We proceed as follows,

$$\begin{aligned} \psi(d_b(Tx_n, T\hat{x})) &\leq \psi\left(\alpha\left(\frac{d_b(x_n,\hat{x}) + d_b(x_n, Tx_n) + d_b(x_n, T\hat{x}) + d_b(\hat{x}, T\hat{x}) + d_b(\hat{x}, Tx_n)}{s+4}\right)\right) \\ &\quad - \min\{\phi(d_b(x_n, Tx_n), d_b(\hat{x}, T\hat{x})), \phi(d_b(x_n, T\hat{x}), d_b(\hat{x}, Tx_n))\} \\ &\leq \psi(K_1), \end{aligned} \tag{26}$$

where $K_1 = \alpha\left(\frac{d_b(x_n,\hat{x}) + d_b(x_n, x_{n+1}) + d_b(x_n, T\hat{x}) + d_b(\hat{x}, T\hat{x}) + d_b(\hat{x}, x_{n+1})}{s+4}\right)$. So, from the property of ψ and (26), we have

$$\begin{aligned} d_b(Tx_n, T\hat{x}) &\leq \alpha\left(\frac{d_b(x_n,\hat{x}) + d_b(x_n, x_{n+1}) + d_b(x_n, T\hat{x}) + d_b(\hat{x}, T\hat{x}) + d_b(\hat{x}, x_{n+1})}{s+4}\right) \\ &\leq \alpha\left(\frac{(s+1)d_b(x_n,\hat{x}) + d_b(x_n, x_{n+1}) + (s+1)d_b(\hat{x}, T\hat{x}) + d_b(\hat{x}, x_{n+1})}{s+4}\right) \end{aligned}$$

Therefore, from the above inequality we have

$$\begin{aligned} d_b(\hat{x}, T\hat{x}) &\leq s(d_b(\hat{x}, x_{n+1}) + d_b(x_{n+1}, T\hat{x})) \\ &= s(d_b(\hat{x}, x_{n+1}) + d_b(Tx_n, T\hat{x})) \\ &\leq s\left(d_b(\hat{x}, x_{n+1}) + \alpha(K_2)\right), \end{aligned}$$

where $K_2 = \left(\frac{d_b(x_n,\hat{x}) + d_b(x_n, x_{n+1}) + s(d_b(x_n,\hat{x}) + d_b(\hat{x}, T\hat{x})) + d_b(\hat{x}, T\hat{x}) + d_b(\hat{x}, x_{n+1})}{s+4}\right)$. By further simplification, we have

$$d_b(\hat{x}, T\hat{x}) \leq \frac{s^2 + 4s}{s + 4 - \alpha s - \alpha s^2}\left(\frac{(s+4+\alpha)d_b(\hat{x}, x_{n+1}) + \alpha(1+s)d_b(x_n,\hat{x}) + \alpha d_b(x_n, x_{n+1})}{s+4}\right) \tag{27}$$

It is clear that, for $\alpha \in [0, \gamma)$ $s + 4 - \alpha s - \alpha s^2 > 0$. So, by taking limit of both sides in (27), we have $d_b(\hat{x}, T\hat{x}) = 0$. Thus, $\hat{x}$ is a fixed point of T. $\square$

Corollary 3. *([15], Theorem 2) Let T be a continuous mapping of a compact metric space (X, d) into itself, and T satisfies*

$$d(Tx, Ty) \leq \frac{1}{2}\left(d(x, Tx) + d(y, Ty)\right),$$

for all $x, y \in X$. Also, if $a \in X$ then the sequence of iterates of a by T will be written as $\{T^n a\}$. Then, T has a unique fixed point in (X, d).

Corollary 4. *[16] Let (X, d) be a complete metric space. Suppose that $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx)),$$

for all $x, y \in X$, and $\lambda \in [0, \frac{1}{2})$. Then, T has a unique fixed point.

Corollary 5. *Let (X,d) be a complete metric space associated with a partial order $\preceq$. Suppose for all comparable elements $x, y \in X$, the mapping $T : X \to X$ is order preserving and satisfies the below condition*

$$\begin{aligned}\psi(d(Tx, Ty)) \quad \leq \quad & \psi\left(\alpha\left(\frac{d(x,y)+d(x,Tx)+d(x,Ty)+d(y,Ty)+d(y,Tx)}{s+4}\right)\right) \\ & -\min\{\phi(\{d(x,Tx), d(y,Ty)), \phi(\{d(x,Ty), d(y,Tx))\}\end{aligned}$$

for some $\alpha \in [0, \gamma)$, where $\gamma = \min\{\frac{1}{s^2}, \frac{s+4}{5s}\}$, ϕ a function satisfying (5), and ψ a distance altering function. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Corollary 6. *([29], Theorem 2.1) Let $(X, \preceq, d)$ be an ordered complete metric space. Let $f : X \to X$ be a continuous nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have*

$$\psi(d(fx, fy)) \leq \psi\left(\frac{d(x,fy)+d(y,fx)}{2}\right) - \phi(d(x,fy), d(y,fx)),$$

where

1. *$\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function.*
2. *$\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function with $\phi(x, y) = 0$ if and only if $x = y = 0$.*

If there exists $x_0 \in X$ such that, $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 7. *([23], Theorem 3) Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$. Let $f : X \to X$ be a continuous, and nondecreasing mapping with respect to $\preceq$. Suppose that, f is a $(\psi, \phi)_s$-weakly C-contractive mapping. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

4. Conclusions

In the first theorem of our main results, the existence of a fixed point $\hat{x} \in X$ of the mapping $T : X \to X$ in a complete b-metric space is guided upon the existence of some important two elements $u, v \in X$, satisfying the conditions provided in Theorem 1. The uniqueness and global stability of the fixed point $\hat{x} \in X$ of T can be obtained if the mapping T is asymptotically contractive. Furthermore, our result in Theorem 5 generalizes the result of Rezai and Dinarvand ([23], Theorem 3), and extends both the result of Du et al. ([24], Theorem 8) and results of Shatanawi ([29], Theorem 2.1).

Author Contributions: The authors contributed equally in writing this article. All authors read and approved the final manuscript.

Funding: Petchra Pra Jom Klao Doctoral Scholarship for Ph.D. program of King Mongkut's University of Technology Thonburi (KMUTT). Theoretical and Computational Science (TaCS) Center. King Mongkut's University of Technology North Bangkok, Contract no. KMUTNB-61-GOV-B-18.

Acknowledgments: This project was supported by Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation research Cluster (CLASSIC), Faculty of Science, KMUTT. The first author thanks for the support of the Petchra Pra Jom Klao Doctoral Scholarship Academic for Ph.D. Program at King Mongkut's University of Technology Thonburi (KMUTT). This research also was funded by King Mongkut's University of Technology North Bangkok, Contract no. KMUTNB-61-GOV-B-18. Moreover, this research work was financially supported by King Mongkut's University of Technology Thonburi through the KMUTT 55th Anniversary Commemorative Fund.

Conflicts of Interest: The authors declare no conflict of interest.

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