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Solution to the Unsteady Seepage Model of Phreatic Water with Linear Variation in the Channel Water Level and Its Application

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Abstract: For a semi-infinite aquifer controlled by a river channel boundary, when the Laplace transform is used to solve a one-dimensional unsteady seepage model of phreatic water while considering the influence of the vertical water exchange intensity ε with the change in the river channel water level $f(t)$, a complicated and tedious integral transformation process is required. By replacing $f(t)$ with an operator, this study first derived the analytic formula of the ε term based on the properties of the Laplace transform without the direct participation of $f(t)$ in the transformation. By using $f(t)$ in the form of several types of linear functions, the Laplace transform and inverse transform laws were summarized. The analytical solution to the problem was easily obtained by applying the "integral property" of the transformation to the linear function term with time t. The relative error between the numerical solution and the analytical solution of the example was less than 0.2%, which verified the rationality of the model linearization method and the reliability of the analytical solution. For different boundary conditions, the process of establishing and applying the inflection point method and the curve-fitting method for calculating the model parameters by using dynamic monitoring data for phreatic water is presented with examples.

Keywords: river channel water level; unsteady seepage of phreatic water; Laplace transform; linear function; inflection point method; curve-fitting method

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1. Introduction

The exchange of water between surface water and groundwater in rivers and canals is a fundamental problem in groundwater dynamics [1] and hydrology [2]. Its solution is not only an essential and fundamental step in the evaluation of water resources [1–3], but is also an important part of the study of ecological problems in rivers [4–6]. To determine the water exchange relationship and processes between surface water and groundwater in rivers and canals, many methods have been developed, such as the temperature tracing method [7] and the isotope and hydro-chemical tracing methods [8–10]. However, methods that use groundwater dynamics for the study of such problems are still in the theoretical stage [11–18].

Regarding methods that use groundwater dynamics to study the water exchange relationship between surface water and groundwater in rivers and canals, the unsteady seepage model of phreatic water in a semi-infinite aquifer under the control of a river channel boundary is one of the most classical problems [12]. In the classical model, the river channel water level $f(t)$, which is a boundary function, is assumed to be a known constant ΔH (that is, the initial groundwater level changes instantaneously ΔH and remains constant). Under this assumption, the Laplace and Fourier transforms can be directly used to obtain the analytical solution of the model [1,2,19].

In practical problems, the boundary function $f(t)$ associated with the river and canal water levels is complex and variable, and the solution of the model involves a complicated and tedious integral transformation operation process with a change in the form of the $f(t)$ function [20–26]. In order to avoid the complex and tedious transformation process, in the literature [20], for the solution of a model of the unsteady flow of phreatic water near the boundary of a river and canal, the shortcut solution method, in which $f(t)$ is not directly involved in the transformation process, was proposed by using the differential properties of the Fourier transform.

However, there are still cases in which the model is difficult to directly solve when common function types are used as boundary functions. For instance, when $f(t)$ is an exponentially decaying function $\Delta T_0\, e^{-\lambda t}$, after the Laplace transform is used, the inverse problem of the combined product of the model's general solution and the image function $f(t)$ becomes difficult to directly solve, thus posing challenges in obtaining a direct solution. Taking advantage of $f(t)$ not being directly involved in the transformation process, previous studies [25–27] investigated the application of the shortcut solution method for the one-dimensional heat conduction model, in which $f(t)$ was set as an exponential decay function by using the Fourier and Laplace transforms.

In the shortcut solution method with the Laplace and Fourier transforms, $f(t)$ was replaced by operators in the transformation process, and a general theoretical solution was established on the basis of the differential properties of the transform, the convolution theorem, etc. The general theoretical solution for such models consists of the product of erfc(t) and $f(0)$ and the convolution of erfc(t) and the derivative of $f(t)$, and it is universally applicable. In practical problems, the actual solution of the problem is given by substituting $f(t)$ into the general theoretical solution under the condition that $f(t)$ is determined [20]. Compared with the traditional solution method, the shortcut method is simple and convenient and does not require a complicated and tedious integration and transformation process.

The models studied in the literature [20,25–27] did not consider the problem of the vertical exchange term, whereas the influence of vertical water exchange is often non-negligible in the study of phreatic seepage problems. Considering the problem of phreatic unsteady flow near rivers and canals under the influence of vertical water exchange, this study introduces a shortcut solution method involving the Laplace transform under the boundary condition of a linear function and presents analytical solutions for several common function types by using general theoretical solutions. Compared with the current method, this method can avoid the tedious process of finding the image function of the Laplace transform. The method was applied to an example case based on the solution of the inflection point method for calculating the model parameters by using dynamic monitoring data on the phreatic water level.

2. Basic Model and Its Linearization

An ideal straight river channel is shown in Figure 1, and the hydrogeological conditions can be summarized as follows:

(1) A homogeneous isotropic submerged aquifer with a horizontal lower confining bed and infinite spatial extension;
(2) A channel with a completely cut aquifer whose water level rapidly rises to a certain height and then remains constant for a long time with a water level rise of ΔH;
(3) An initial water level of the phreatic water $h(x,0)$ is horizontal;
(4) Phreatic flow can be regarded as a one-dimensional flow;
(5) The intensity of vertical water exchange ε is equally distributed throughout the region.

The above conceptual hydrogeological model is based on the classical Ferris model with the addition of condition (5) [1,2]. The problem can be formulated as a mathematical model (I) as follows:

$$\mu \frac{\partial h}{\partial t} = K \frac{\partial}{\partial x}\left[h \frac{\partial h}{\partial x}\right] + \varepsilon \quad (0 < x < +\infty, t > 0), \tag{1}$$

$$h(x,t)|_{t=0} = h(x,0) \quad (x > 0), \tag{2}$$

$$h(x,t)|_{x=0} = h(0,0) + f(t) \quad (t \geq 0), \tag{3}$$

$$h(x,t)|_{x\to\infty} = h(x,0) + \frac{\varepsilon}{\mu}t \quad (t \geq 0), \tag{4}$$

where μ is the specific yield of the aquifer; h is the phreatic water level (m); K is the infiltration coefficient (m/d); ε is the vertical water exchange intensity (m/d), in which the groundwater is recharged through positive processes such as precipitation infiltration, irrigation infiltration, etc., and it is discharged through negative processes such as phreatic water evaporation; $f(t)$ is a function of the change in the river and canal water level over time.

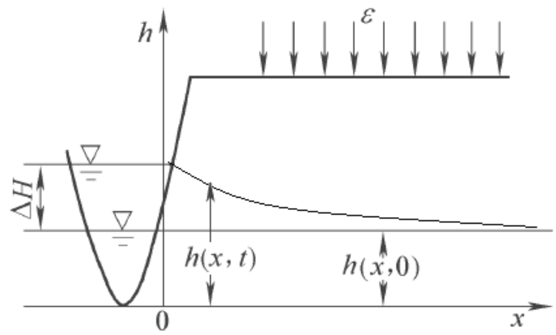

Figure 1. Seepage field near a river channel.

The generalized equation of model (I), Equation (1), is a nonlinear equation and needs to be linearized. When $h(x,t) - h(x,0) \leq 0.1\ h_m$. h_m is the average thickness of the phreatic flow during the calculation period, and most phreatic water problems can meet this condition in practice), this condition can be used for the first linearization of the Boussinesq Equation [1,2]:

$$\mu\frac{\partial h}{\partial t} = Kh_m\frac{\partial^2 h}{\partial x^2} + \varepsilon \tag{5}$$

For Equations (2) to (5), let $a = Kh_\mathrm{m}/\mu$ and let a be the pressure conductivity of the phreatic water aquifer (m^2/d); meanwhile, let $u(x,t) = h(x,t) - h(x,0) - \varepsilon t/\mu$, which leads to model (II):

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \quad (0 < x < +\infty, t > 0), \tag{6}$$

$$u(x,t)|_{t=0} = 0 \quad (x > 0), \tag{7}$$

$$u(x,t)|_{x=0} = f(t) - \frac{\varepsilon}{\mu}t \quad (t \geq 0), \tag{8}$$

$$u(x,t)|_{x\to\infty} = 0 \quad (t \geq 0), \tag{9}$$

3. Theoretical General Solution

Model (III) can be obtained by finding the Laplace transform with respect to model (II).

In model (III), U is the image function of the Laplace transform of u, s is the Laplace operator, L is the Laplace transform operator, and L^{-1} is the inverse transform operator.

In the above process, during the transformation of boundary condition (8) into boundary condition (11), $f(t)$ is not directly involved in the transformation process. In the transformation operation, the image function does not need to be calculated, and the Laplace transform of $f(t)$ is only symbolically denoted.

The general solution of Equation (7) in model (III) is

$$\frac{d^2\overline{u}}{dx^2} - \frac{s}{a}\overline{u} = 0, \tag{10}$$

$$\overline{u}|_{t=0} = L[f(t)] - \frac{\varepsilon}{\mu}\frac{1}{s^2}, \tag{11}$$

$$\overline{u}|_{t=0} = 0 \tag{12}$$

$$\overline{u}(x,s) = c_1 \exp\left(\sqrt{\frac{s}{a}}x\right) + c_2 \exp\left(-\sqrt{\frac{s}{a}}x\right), \tag{13}$$

where c_1 and c_1 are undetermined constants. According to the definite solution conditions in Equations (10) and (11), the specific solution of model (III) is

$$\overline{u}(x,s) = \left[L[f(t)] - \frac{\varepsilon}{\mu}\frac{1}{s^2}\right]\exp\left(-\sqrt{\frac{s}{a}}x\right), \tag{14}$$

Under the condition that $L[f(t)]$ is not directly inversed, $L[f(t)]$ is used as an operator in the inverse transformation process. We have

$$\begin{aligned} u(x,t) &= L^{-1}[\overline{u}(x,s)] \\ &= L^{-1}\left[L(f(t))\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right] - L^{-1}\left[\tfrac{\varepsilon}{\mu}\tfrac{1}{s^2}\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right], \end{aligned} \tag{15}$$

The Laplace inversion function from the "complementary error function erfc(u)" is

$$L^{-1}\left[\frac{1}{s}\exp\left(-\sqrt{\frac{s}{a}}x\right)\right] = \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right), \tag{16}$$

In Equation (15), when the second term on the right side is used to calculate the inverse Laplace transform, attention should be paid to the "integral property" of the Laplace transform for the $1/s^2$ term and to the application of Equation (16). Furthermore,

$$\begin{aligned} L^{-1}\left\{\tfrac{\varepsilon}{\mu}\tfrac{1}{s}\left[\tfrac{1}{s}\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right]\right\} &= \tfrac{\varepsilon}{\mu}\int_0^t L^{-1}\left[\tfrac{1}{s}\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right]dt \\ &= \tfrac{\varepsilon}{\mu}\int_0^t \operatorname{erfc}\left(\tfrac{1}{2\sqrt{at}}\right)dt \end{aligned} \tag{17}$$

Substituting Equation (17) into Equation (15), and by using $u(x,t) = h(x,t) - h(x,0) - \varepsilon t/\mu$, we have

$$h(x,t) = h(x,0) + L^{-1}\left\{L[f(t)]\exp\left(-\sqrt{\frac{s}{a}}x\right)\right\} - \frac{\varepsilon}{\mu}\int_0^t \operatorname{erfc}\left(\frac{1}{2\sqrt{at}}\right)dt + \frac{\varepsilon}{\mu}t. \tag{18}$$

For terms 3 and 4 on the right side of Equation (18),

$$\begin{aligned} -\tfrac{\varepsilon}{\mu}\int_0^t \operatorname{erfc}\left(\tfrac{x}{2\sqrt{at}}\right)dt + \tfrac{\varepsilon}{\mu}t &= \tfrac{\varepsilon}{\mu}\int_0^t\left[1 - \operatorname{erfc}\left(\tfrac{x}{2\sqrt{at}}\right)\right]dt \\ &= \tfrac{\varepsilon}{\mu}\int_0^t \operatorname{erf}\left(\tfrac{x}{2\sqrt{at}}\right)dt \end{aligned} \tag{19}$$

By substituting Equation (19) into Equation (18), we have

$$h(x,t) = h(x,0) + L^{-1}\left[L(f(t))\exp\left(-\sqrt{\frac{s}{a}}x\right)\right] + \frac{\varepsilon}{\mu}\int_0^t \operatorname{erf}\left(\frac{1}{2\sqrt{at}}\right)dt. \tag{20}$$

For the second term on the right side of Equation (20), Wei et al. [21] attempted to find the solution of a one-dimensional heat conduction model based on the "differential property" of the Laplace transform and the convolution theorem, and they gave the following expression:

$$L^{-1}\left[L[f(t)]\exp\left(-\sqrt{\frac{s}{a}}x\right)\right] = f(t)|_{t=0}\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \int_0^t \frac{d[f(t)]}{dt}\mathrm{erfc}\left(\frac{x}{2\sqrt{a(t-\tau)}}\right)d\tau. \tag{21}$$

By substituting Equation (21) into Equation (20), a generalized theoretical solution to the one-dimensional unsteady flow model for phreatic water near a river channel under the influence of vertical water exchange is obtained as follows:

$$h(x,t) = h(x,0) + f(t)|_{t=0}\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \int_0^t \frac{d[f(t)]}{dt}\mathrm{erfc}\left(\frac{x}{2\sqrt{a(t-\tau)}}\right)d\tau + \frac{\varepsilon}{\mu}\int_0^t \mathrm{erf}\left(\frac{1}{2\sqrt{at}}\right)dt. \tag{22}$$

when $\varepsilon = 0$, Equation (22) is transformed into the general theoretical solution of the one-dimensional unsteady flow model for phreatic water near rivers and canals, as described in [20].

Under the condition that the product combination of the image function of $f(t)$ and the general solution is not directly inversed, $L[f(t)]$ is used as an operator in the inverse transformation process by using the "differential property" and "convolution theorem" of the inverse transformation, and the "general theoretical solution" for this type of problem is obtained. Furthermore, the solution to the problem can be obtained by substituting $f(t)$ into the general theoretical solution. This solution method is fast, simple, and widely applicable, and it is also applicable to complex function types. If $f(t)$ is the exponential decay function $\Delta T_0\, e^{-\lambda}$ and $\lambda > 0$, the image function of $f(t)$ is $\Delta T_0/(s + \lambda)$. Furthermore, the item on the right side of the above formula is $\Delta T_0\exp(-\sqrt{s/a}\cdot x)/(s + \lambda)$. Consequently, the inverse of the product combination of this image function and the general solution is difficult to determine, thus further hindering the acquisition of the solution through the direct application of the Laplace function. Under these circumstances, the shortcut solution method can be more convenient for determining the solution to the problem [6–8].

However, the third term on the right side of the general theoretical solution, Equation (22), belongs to the delayed integral, and the direct calculation of the delayed integral generally depends on numerical algorithms for approximation [28]. Therefore, the solution based on the general theoretical solution often requires further expansion. When the $f(t)$ function's type is relatively simple, such as a linear function, the solution of the problem can also be easily obtained by directly transforming the second term on the right side of Equation (19), which is more direct and convenient than the application of the general theoretical solution (22).

4. Solutions for Several Types of Linear Functions

4.1. Constant Value Function

In a constant value function, $f(t)$ is a constant: $f(t) = \Delta H$. The physical meaning of this condition is that it remains unchanged after $t \to 0^+$, which is the instantaneous amplitude of the water level in the boundary river channel ΔH. This is also the boundary condition of the classical Ferris model.

$$\begin{aligned} L^{-1}\left\{L[f(t)]\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right\} &= L^{-1}\left[L(\Delta H)\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right] \\ &= L^{-1}\left[\tfrac{\Delta H}{s}\exp\left(-\sqrt{\tfrac{s}{a}}x\right)\right] \\ &= \Delta H\mathrm{erfc}\left(\tfrac{x}{2\sqrt{at}}\right), \end{aligned} \tag{23}$$

Substituting Equation (23) into Equation (20), when $f(t) = \Delta H$, the solution of the model is

$$h(x,t) = \Delta H\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \frac{\varepsilon}{\mu}\int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{a\tau}}\right)d\tau. \tag{24}$$

Equation (23) is also the solution of the Ferris model.

4.2. Step Function

In the field of engineering technology, although the target variable changes continuously with time, the actual observation process is discrete. In other words, a certain time interval occurs between the measurement of the value of the river and channel water level data at each time and the measurement of the value at the previous time. Even when self-recording test data, excerpts are often necessary. Therefore, according to discrete measurement data, the change in a variable over time can be expressed by using a piecewise function [28]. Based on discrete measurement data, the process of a variable's change over time can be expressed by using a segmentation function [29].

Segmentation functions, which mainly include step functions and linear interpolation functions, can express the process of a variable's change over time.

For the river and channel water levels f_i and f_{i+1} at the beginning and end of $t_i - t_{i+1}$($i \geq 2$), the average river and channel water level in the period $(f_i + f_{i+1})/2$ or the increase in $f_{i+1} - f_i$ in each period after t_1 is used (Figure 2). The step function of $f(t)$ can be written as

$$f(t) = \Delta H + \sum_{i=2}^{n} (f_i - f_{i-1})\delta(t - t_{i-1}) \qquad (t > t_{i-1}, i \in N^*), \tag{25}$$

where $\delta(t - t_{i-1})$ is the Heaviside function [28], where $t < t_{i-1}$, $\delta(t - t_{i-1}) = 0$, $t \geq t_{i-1}$, and $\delta(t - t_{i-1}) = 1$.

Note: ΔH represents the beginning of $t \to 0^+$ and remains unchanged for a certain length of time, which is $t_1 - t_0$. Therefore, the superposition function in Equation (25) is $i = 2–n$.

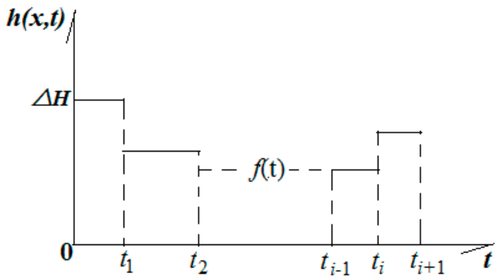

Figure 2. Discretization of the boundary function $f(t)$.

$$L\left[\Delta H + \sum_{i=2}^{n} (f_i - f_{i-1})\delta(t - t_{i-1})\right] = \frac{1}{s}\left[\Delta H + \sum_{i=2}^{n} (f_i - f_{i-1})\delta(t - t_{i-1})\right] \tag{26}$$

The right side of Equation (26) contains only the $1/s$ term. Referring to Equation (20), substituting Equation (23) into Equation (19), and considering the functional value of $\delta(t - t_{i-1})$ and the transformation characteristics of the function term series [29], we obtain

$$h(x,t) = \Delta H \mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \sum_{i=2}^{n} (f_i - f_{i-1}) \mathrm{erfc}\left(\frac{x}{2\sqrt{a(t - t_{i-1})}}\right) + \frac{\varepsilon}{\mu}\int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) dt. \tag{27}$$

4.3. Lagrange Linear Interpolation Function

The Lagrange linear interpolation equation is used as an example to discretize $f(t)$ [20]:

$$f(t) = \Delta H + \sum_{i=2}^{n} \frac{f_i - f_{i-1}}{t_i - t_{i-1}} (t - t_{i-1})\delta(t - t_{i-1}). \tag{28}$$

$$L\left[\Delta H + \sum_{i=2}^{n} \frac{f_i - f_{i-1}}{t_i - t_{i-1}} (t - t_{i-1})\delta(t - t_{i-1})\right] = \frac{\Delta H}{s} + \frac{1}{s^2}\sum_{i=2}^{n} \frac{f_i - f_{i-1}}{t_i - t_{i-1}} \delta(t - t_{i-1}), \tag{29}$$

To find L^{-1} for Equation (28), by applying the "integral nature" of the transformation to the $1/s^2$ term and paying attention to the nature of the $\delta(t-t_{i-1})$ function, we obtain

$$h(x,t) = \Delta H \mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) + \sum_{i=2}^{n} \frac{f_i - f_{i-1}}{t_i - t_{i-1}} \int_{t_{i-1}}^{t} \mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) dt + \frac{\varepsilon}{\mu} \int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) dt. \quad (30)$$

From the above, when the boundary function $f(t)$ is a simple function, such as a linear function, it is more direct and convenient to solve it by using Equation (19) than by using the general theoretical solution. In the process of solving this type of problem, the image function after the positive transformation of the constant is $1/s$, and the solution after the inverse transformation is the product of the constant term and the error function, including the linear function term: the time t. The image function after the transformation is the $1/s^2$ term. Regarding the "integral property" of the transformation, when solving the superposition function, attention should be paid to the properties of the Heaviside function.

5. Verification and Application of the Solution

5.1. Numerical Verification of Analytical Solutions

In this study, the boundary function $f(t)$ was set as a linear function, and the Laplace transform method was used to solve the one-dimensional unsteady seepage problem of phreatic water near river channels. The generalized solution of the mathematical model was based on the linearization of the Boussinesq equation. The solution of the model is based on the first linearization of the Boussinesq equation. Although the first linearization method is most widely used in the study of phreatic water problems, its rationality also needs to be verified. Literature [30–32] and others used MacCormack's explicit difference solution of the Boussinesq equation, and the solution results were compared with the analytical solution results of the equation linearization method to verify the rationality of linearization.

In this study, the solution contains the vertical water exchange term ε. In order to verify the validity of the linearization method and the reliability of the analytical solution, the Boussinesq Equation in the model was numerically solved by using a finite difference algorithm in the MacCormack scheme, and the analytical solution was compared with the numerical solution for a mutual comparison. Assuming that the water level $h_{k,n}$ at each node at time n is known, the water level $h_{k,n+1}$ at each node at time n + 1 is solved by using the two-step method of prediction correction in the MacCormack scheme [33–36] for time advancement.

The relatively simple expression in (23) is taken as an example. In the first step, the spatial derivatives are differenced forward in the prediction to obtain a prediction scheme for h, which is obtained by replacing the spatiotemporal derivatives with the forward differences [30–32] as follows:

$$h^{*}_{k,n+1} = h_{k,n} + \frac{\varepsilon}{\mu}\Delta t + \frac{k}{\mu}\frac{\Delta t}{\Delta x^2} \times \left[h_{k+1,n}(h_{k+1,n} - h_{k,n}) - h_{k,n}(h_{k,n} - h_{k-1,n})\right] \quad (31)$$

In the second step, the partial derivatives of x are replaced by the backward differential approximation, while the partial derivatives of t are replaced by the forward differential approximation, and Equation (1) can be written as follows [27,30–32]:

$$h^{**}_{k,n+1} = h_{k,n} + \frac{\varepsilon}{\mu}\Delta t + \frac{k}{\mu}\frac{\Delta t}{\Delta x^2} \times \left[h^{*}_{k,n+1}\left(h^{*}_{k+1,n+1} - h^{*}_{k,n+1}\right) - h^{*}_{k-1,n+1}\left(h^{*}_{k,n+1} - h^{*}_{k-1,n+1}\right)\right] \quad (32)$$

Finally, the correction value for $h_{\mathrm{k,n+1}}$ is the arithmetic mean of $h^{*}_{k,n+1}$ and $h^{**}_{k,n+1}$ [27,30–32]:

$$h_{k,n+1} = \frac{1}{2}\left\{h_{k,n} + h^{*}_{k,n+1} + \frac{\varepsilon}{\mu}\Delta t + \frac{k}{\mu}\frac{\Delta t}{\Delta x^2} \times \left[h^{*}_{k,n+1}\left(h^{*}_{k+1,n+1} - h^{*}_{k,n+1}\right) - h^{*}_{k-1,n+1}\left(h^{*}_{k,n+1} - h^{*}_{k-1,n+1}\right)\right]\right\} \quad (33)$$

Equations (31)–(33) constitute the finite difference scheme of the MacCormack scheme for nonlinear Equation (1). The stability criterion is met by adjusting the subdivision mesh [27,30–32]:

$$\frac{h_m \Delta t}{\frac{\mu}{k}\Delta x^2} = \frac{a\Delta t}{\Delta x^2} \le 0.5 \tag{34}$$

According to Equations (31)–(34), referring to the hydrogeological characteristics of the Huaibei plain, three aquifer media of coarse sand, medium sand, and fine sand, with specific yields of 0.30, 0.22, and 0.17 [27] and permeability coefficients of 30 m/d, 20 m/d, and 7 m/d [27], respectively, were selected. The initial phreatic water level was 27.0 m. During the calculation period, the average thickness of phreatic flow was taken as 4.0 m. The calculation period was 5 days, assuming that ε = 12 mm/d within the calculation period. By using the above parameters, the calculation results for the relative error of phreatic water level at 24 h and 48 h were obtained and are shown in Figures 3a and 3b, respectively.

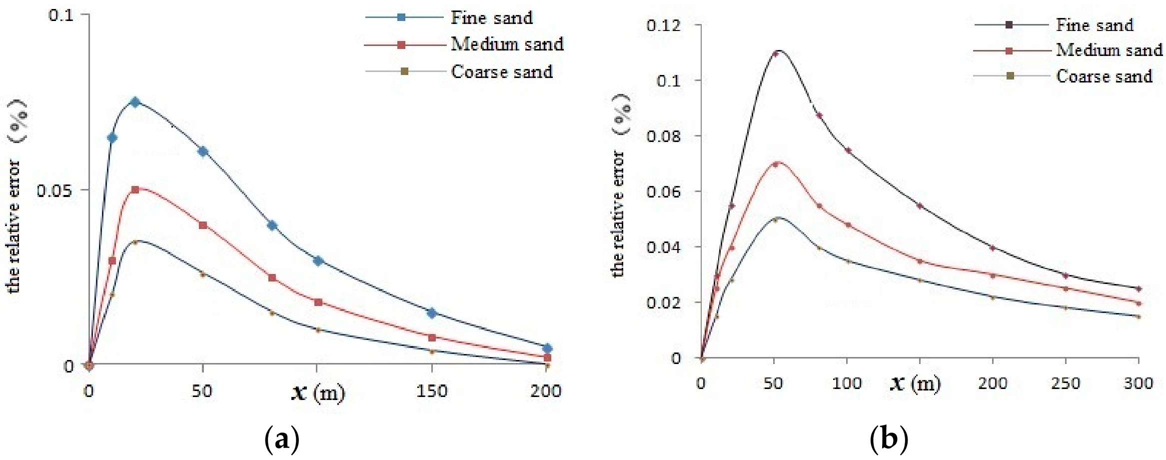

Figure 3. Relative error between the analytical and numerical solutions of the phreatic water level. (**a**) t = 24 h. (**b**) t = 48 h.

As shown in Figure 3, the agreement between the analytical and numerical solutions with different hydrogeological parameters was relatively good, and the relative error did not exceed 0.2%, which indicated that the linearization method of the Boussinesq equation in the model's solution process was feasible, and this verified the reliability of the analytical solution of the model.

5.2. Calculation Method for the Model Parameters

Another important purpose of studying such problems is to apply dynamic monitoring data on the phreatic water level for the calculation of model parameters. Existing studies [1,2,20–27,37,38] showed that, according to the derivative of the model's solution—the process of variation in the phreatic water level $h(x,t)$ over time—it is convenient to establish the method of inversion of the model parameters by using dynamic monitoring data on the phreatic water level.

By using Equation (23) with a relatively simple expression as an example, the establishment and application steps of the method for calculating the model parameter a were explored by using the process of variation in the phreatic water level $h(x, t)$ over time $h(x, t) - t$ and the process of variation in the speed of phreatic level change $\partial h(x,t)/\partial t$ over time $\partial h(x,t)/\partial t - t$.

5.2.1. Curve-Fitting Method of $\varphi(x,t) - t$ with $\varepsilon \neq 0$ and $\Delta H = 0$

By using Equation (24), the derivative with respect to t and the speed of change in the phreatic water level $\varphi(x,t) = \partial h(x,t)/\partial t$ at the distance from the boundary x was found. Furthermore,

$$\varphi(x,t) = \frac{\partial h}{\partial t} = \Delta H \frac{x t^{-3/2}}{2\sqrt{\pi a}} \exp\left(-\frac{x^2}{4at}\right) + \frac{\varepsilon}{\mu} \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right), \tag{35}$$

when $\varepsilon \neq 0$ and $\Delta H = 0$, Formula (31) transforms into

$$\varphi(x,t) = \frac{\varepsilon}{\mu} \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right), \tag{36}$$

According to the observation well with a distance boundary x (x is the known value), and according to Equation (32), $\varepsilon/\mu[\mathrm{erf}(x,t)]$ can be obtained, from which the corresponding theoretical curve family of $\varepsilon/\mu[\mathrm{erf}(x,t)] - t$ is also obtained. From the measured speed of water level change at the measuring point v, the measurement curve $\varphi(x,t)$ can be drawn.

When the a-value of an actual aquifer is equal to that of a curve in the family of the theoretical curves of $\varepsilon/\mu[\mathrm{erf}(x,t)] - t$, the a-value of the aquifer can be determined from the difference between the measurement curve of $\varphi(x,t) - t$ and the theoretical curve of $\varepsilon/\mu[\mathrm{erf}(x,t)] - t$ with the same a-value, which should be identical in form and coincide exactly. According to this principle, the a-value of the aquifer can be determined using the above-mentioned suitable line between the measured curve and the family of theoretical curves (Figure 4).

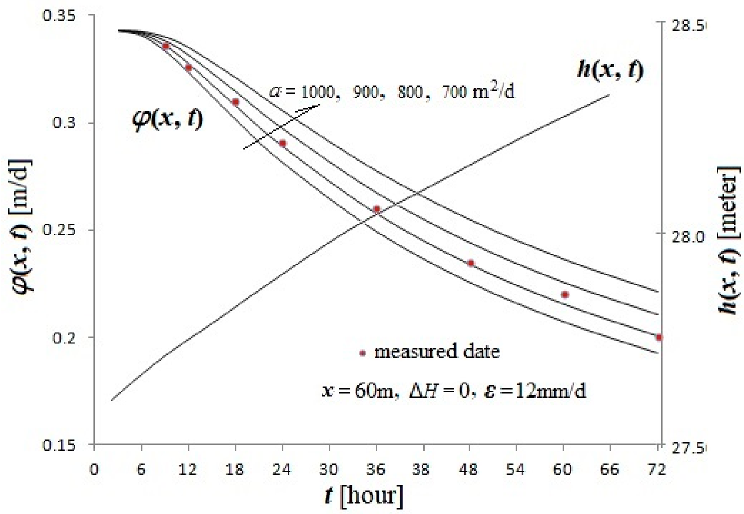

Figure 4. Real curve fitting method of $\varphi(x, t) - t$ for resolving a.

5.2.2. Curve-Fitting Method for $h(x,t) - t$ with $\varepsilon = 0$ and $\Delta H \neq 0$

When $\varepsilon = 0$ and $\Delta H \neq 0$, Formula (32) transforms into

$$h(x,t) = \Delta H \mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right). \tag{37}$$

According to the principle of the curve-fitting method, the a-value of the aquifer can be determined by fitting the measured $\Delta H[\mathrm{erf}(x,t)] - t$ with the theoretical curve of the $\Delta H[\mathrm{erf}(x,t)] - t$ family with different a values.

5.2.3. Inflection Point Method of $\varphi(x,t) - t$ with $\varepsilon = 0$ and $\Delta H \neq 0$

When $\varepsilon = 0$ and $\Delta H \neq 0$, Equation (30) transforms into

$$\frac{\partial[\varphi(x,t)]}{\partial t} = \frac{\Delta H x t^{-5/2}}{2\sqrt{\pi a}} \exp\left(-\frac{x^2}{4at}\right)\left(-\frac{3}{2} + \frac{x^2}{4at}\right). \tag{38}$$

According to Equation (38), there is an inflection point on the $\varphi(x,t) - t$ curve (Figure 5). Let the time at the inflection point be t_g, and let $\partial\varphi(x,t)/\partial t = 0$ at the inflection point; then

$$a = x^2/6t_g \tag{39}$$

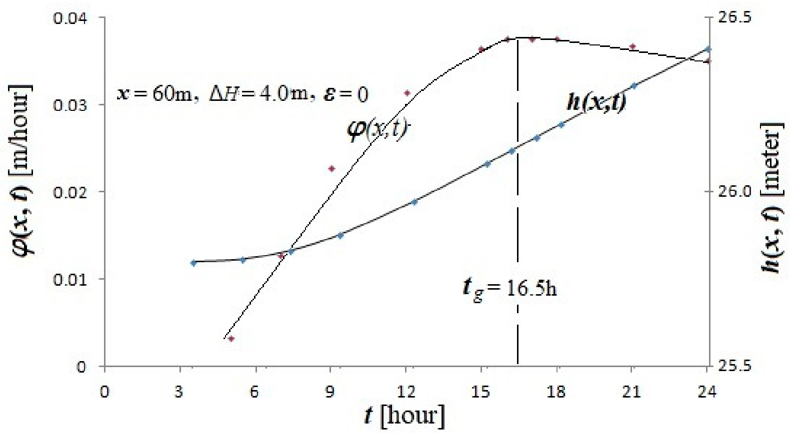

Figure 5. Real curve of $\varphi(x,t) - t$ in the inflection point method for resolving a.

Equation (39) is the Ferris model's equation for finding a by using the time of inflection [1,2].

5.2.4. Ratio Method of $\varphi(x,t)$ with $\varepsilon \neq 0$ and $\Delta H \neq 0$

Assuming that there are two observation wells, the distances from the boundary are x_1 and x_2, respectively, and under the condition $x_1/\sqrt{t_1} = x_2/\sqrt{t_2}$, the water-level variation rates of the two holes, $\varphi_1(x_1,t_1)$ and $\varphi_2(x_2,t_2)$, are taken; these can be obtained from Equation (32):

$$\mathrm{erf}\left(\frac{x_1}{2\sqrt{at_1}}\right) = \mu\frac{\varphi_1 t_1 - \varphi_2 t_2}{\varepsilon(t_1 - t_2)} \tag{40}$$

when ε and μ are known, erf[z] and z can be determined from Equation (33); x_1 and t_1 are definite values, and then a can be determined.

5.3. Case Study

In the central part of the Huaibei Plain in Anhui Province, a submerged aquifer, which is mainly powdered sand, has been widely developed; it has a thickness of approximately 8.0 m, and the bottom generally constitutes an incomplete continuous clay layer, which can be regarded as the lower confining bed of the submerged aquifer. In the study area, the depth of submergence was shallow: 2.5–3.0 m. In this study, the specific yield μ (0.035) was used for the field test data.

The area had a relatively complete agricultural irrigation canal system, and the depth of the trunk canal essentially reached the lower confining bed. The canal spacing of the trunk canal was about 2.0 km, and the head of the trunk canal was mostly controlled by a control gate. A national self-registering observation well for monitoring the groundwater level was located at a straight-line distance of 60 m from the trunk canal, and the ground elevation near the observation well was 30.72 m.

On 22 August 2022, the phreatic water level was basically the same as the channel level, which was 27.56 m. From 22 August to 24 August, the infiltration intensity associated with diversion irrigation was approximately 12.0 mm/d. The groundwater level monitoring data from the observation well (Table 1) are plotted in Figure 2.

Table 1. Dynamics of the phreatic water level from 22 August to 24 August.

t/h	3	6	9	12	18	24	36	48	60	72
$h(x,t)$ (m)	27.61	27.65	27.69	27.73	27.80	27.87	28.01	28.12	28.23	28.33
$\varphi(x,t)$ (m/h)	0.360	0.360	0.336	0.304	0.280	0.291	0.275	0.220	0.220	0.200

Note: x = 60 m, ground elevation is 30.72 m.

In the example with $\Delta H = 0$ and $\varepsilon = 12$ mm/d, based on the principle of the $\varphi(x,t) - t$ matching method, as shown in Figure 4, the point of $\varphi(x,t)$ that was calculated by using the actual data was basically located near the line of $a = 900$ m^2/d. Accordingly, the a-value obtained by using the curve-fitting method is 900 m^2/d.

On 6 October 2022, the phreatic water level was basically the same as the channel level (25.80 m). The water level in the river and channel rose by about 4.0 m within a relatively short period of time when the gates were closed and the water was diverted on October 6. The groundwater level monitoring data from the observation well (Table 2) are plotted in Figure 5.

Table 2. Dynamics of the phreatic water level on October 6.

t/h	3	5	7	9	12	15	16	17	18	21	24
$h(x,t)$ (m)	25.80	25.81	25.83	25.88	25.97	26.08	26.12	26.16	26.20	26.31	26.41
$\varphi(x,t)$ (mm/h)		0.34	1.27	2.29	3.15	3.64	3.76	3.77	3.76	3.68	3.52

Note: x = 60 m, ground elevation is 30.72 m.

As the depth of the groundwater level was approximately 4.5–5.0 m, phreatic water evaporation could be neglected ($\varepsilon = 0$). In the example, $x = 60$ m, $\mu = 0.035$, $\Delta H = 4.0$ m, and $\varepsilon = 0$. As shown in Figure 5, at the inflection point, $t_g = 16.5$ h = 0.69 d. Based on Equation (35), the value of a that was calculated with the inflection point method was 870 m^2/d.

From the above, the calculation results of the inflection point method and the curve-fitting method were basically consistent with those in the literature [27] (860 m^2/d).

6. Conclusions

This study investigated the application of the Laplace transform to solving a one-dimensional unsteady flow model of stream/phreatic water under the influence of vertical water exchange intensity, with the water level of the river channel exhibiting linear variations. The following conclusions were obtained:

(1) For the unsteady flow model of phreatic water near rivers and canals under the influence of vertical water exchange, the analytic equations of the boundary water level $f(t)$ and the vertical water exchange intensity ε were separately solved based on the properties of the Laplace transform, and the two analytic equations were then superimposed to obtain the solution.

(2) When the boundary function $f(t)$ is a simple function, such as a constant value function or a linear function, the image function after the positive transformation of the constant is $1/s$, and the solution after the inverse transformation is the product of the constant term and the error function. For the linear function term with time t, the transformed image function is the $1/s^2$ term. Regarding the "integral property" [39] of the transformation, when solving the superposition function, attention should be paid to the application of the Heaviside function.

(3) By using the process of variation in the phreatic water level $h(x,t)$ over time and the process of variation in the speed of change in the phreatic water level $\partial h(x,t)/\partial t$ over

time, the inflection point method and the curve-fitting method can be used to calculate the model parameter *a* under different conditions.

It is noteworthy that the principle of the method in this study is also applicable to the solution of the Fourier transform. Due to constraints on the length and focus of this study, the applicability of the method to other transforms will be discussed in another study.

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