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Stochastic Modelling of Small-Scale Perturbation

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Abstract: In this paper we propose a stochastic model reduction procedure for deterministic equations from geophysical fluid dynamics. Once large-scale and small-scale components of the dynamics have been identified, our method consists in modelling stochastically the small scales and, as a result, we obtain that a transport-type Stratonovich noise is sufficient to model the influence of the small scale structures on the large scales ones. This work aims to contribute to motivate the use of stochastic models in fluid mechanics and identifies examples of noise of interest for the reduction of complexity of the interaction between scales. The ideas are presented in full generality and applied to specific examples in the last section.

Keywords: stochastic model reduction; Wong-Zakai principle; transport noise

1. Introduction

This work deals with stochastic models in fluid mechanics. The literature on the subject is very large, but it is mostly of theoretical nature. Having in mind potential applications, two main questions arise: (i) Why should we use stochastic models in fluid mechanics? (ii) Which noise is more interesting, the classical additive noise or other forms? Among various answers to these questions, one is based on stochastic model reduction, the topic discussed in this work. In a sentence, it claims that stochastic models may reduce the complexity of interaction between scales and the noise arising from such a reduction is not the classical additive noise added to the equations in most of the literature (which however is interesting for other reasons), but a multiplicative one of transport type, described here and in related works. We address an audience made up of both mathematicians and practitioners, and our hope is to contribute to the understanding of fluid mechanics PDEs with transport noise and how they are related to applications.

Going into details, in this paper we are interested in general models of geophysical fluid-dynamics with the following form,

$$\begin{cases} \partial_t u + J(u,\nabla u) = -\frac{1}{\rho}\nabla p + \mathcal{D}(u) + f, \\ \partial_t \rho + \nabla\cdot(u\rho) = 0. \end{cases} \tag{1}$$

The spatial domain on which the equations are studied is denoted by D and, depending on the particular problem under investigation, can be either two- or three-dimensional: $\dim D = n = 2$ or 3. The unknowns of the equations above are the velocity vector field $u : \mathbb{R}_+ \times D \to \mathbb{R}^n$ and the pressure scalar field $p : \mathbb{R}_+ \times D \to \mathbb{R}$.

The quantity $\rho : \mathbb{R}_+ \times D \to \mathbb{R}_+$ describes the density of the fluid and is deduced from u by conservation of mass, mathematically translated as the continuity equation $\partial_t\rho + \nabla\cdot(u\rho) = 0$.

The term $J(u, \nabla u)$ represents the inertial force per unit of mass acting on the fluid due to advection, for instance,

$$J(u, \nabla u) = (u \cdot \nabla)u$$

for Navier–Stokes equations, but it can assume different forms in certain regimes, like the small aspect ratio regime proper of Primitive Equations.

$\mathcal{D}(u)$ represents the force per unit of mass of any dissipation mechanism acting on the fluid. Usually dissipation occurs via viscosity

$$\mathcal{D}(u) = \nu \Delta u, \quad \nu \geq 0,$$

or friction

$$\mathcal{D}(u) = -\alpha u, \quad \alpha \geq 0,$$

or a combination of the two, depending on the model under consideration. Indeed, in general viscosity is due to the interaction of the fluid particles with themselves and is therefore an intrinsic property of the fluid: typical experimental values of ν at room temperature and pressure are $\nu_{air} = 1.5 \times 10^{-5}$ m^2/s for air and $\nu_{water} = 1.1 \times 10^{-6}$ m^2/s for water. On the other hand, friction describes well the interaction of a roughly two-dimensional fluid with a solid bottom (or top) layer: for instance, for a fluid of depth h the value of α is related to ν via the relation

$$\alpha \propto \frac{\nu}{h^2}.$$

In particular, friction forces dominate viscous forces (at low wavenumbers) for values of h which are small compared to the other typical lengths of the fluid, while viscous forces dominate friction forces for larger values of h. It is worth mentioning that in certain idealised models dissipation is neglected: $\alpha = \nu = 0$, see, for instance, Euler equations and related models.

The term f represents any other force per unit of mass acting on the fluid, either inertial (e.g., the Coriolis force) or not.

System (1) is usually accompanied with suitable boundary conditions on u, depending on the geometry of the domain and physical meaning.

Under the assumption of incompressibility $\rho = \rho_0$ is constant and (1) assumes the form

$$\begin{cases} \partial_t u + J(u, \nabla u) = -\frac{1}{\rho_0} \nabla p + \mathcal{D}(u) + f, \\ \nabla \cdot u = 0, \end{cases} \tag{2}$$

so that continuity equation simplifies into the condition that the velocity vector field u is solenoidal: $\nabla \cdot u = 0$. Mathematically speaking, we assume in the following that J is a bilinear operator and $\mathcal{D}$ is a linear operator, possibly unbounded.

It is clear to everybody that atmospheric and oceanic dynamics show a superposition of structures of different sizes, ranging from continental, with order of magnitude of 1000 km, to human scale structures of size 1 m. In this paper we propose a stochastic model reduction procedure for deterministic geophysical fluid dynamics models of the form (2), which, in our opinion, is able to isolate the evolution of large scale structures via a closed equation which is a stochastic modification of (2), where a transport-type Stratonovich noise is sufficient to model, with a certain degree of approximation, the influence of the small scale structures on the large scales ones.

The literature on the topic of either stochastic or deterministic model reduction is wide and the motivations beyond the interest in reduction procedures for geophysical fluid-dynamics models are several.

From the numerical point of view, especially when one is interested in the simulations of complex turbulent flows like weather forecast, one necessarily has to deal with the fact that limited computational power often implies an under-representation of the real physical processes with spatial or temporal scale smaller than a certain threshold, typically the length of the grid parametrisation and the time discretisation step. However, these small scale processes may have a non-trivial impact on the large scales ones, and thus it is important to take this impact into the account in order to obtain accurate description of the evolution of the simulated process, see in [1] and the references therein.

Another field of application is climate prediction [2–5]; indeed, the high complexity of real geophysical models allows an accurate forecast only for relatively short time intervals, that is, it is impossible to have good weather predictions over a time span greater than a few days. On the other hand, decreasing the complexity of a model allows for better error control in long-term simulations, thus opening the way to the study of climate tendency.

By the theoretical point of view, model reduction has always played a primary role in geophysics and, more generally, in fluid mechanics; here, model reduction is meant in the broad sense, as the operation of reducing the complexity of a model in order to conveniently describe certain phenomena. For example, if one is interested in the evolution of a certain geophysical flow on a relatively small portion of Earth's surface, then the spherical geometry of the problem is usually not so important and the use of spherical coordinates is an unnecessary complication: it is way more convenient to study the problem in Cartesian coordinates. The dynamical effects of Earth's rotation are therefore captured with the so-called f-plane approximation [6] (and more generally with the β-plane approximation), which constitutes a nice simplification of the problem yet capable of describing very interesting phenomena, like the motion of cyclonic flows at geostrophic balance and the Taylor–Proudman effect.

Our reduction procedure consists in splitting (2) into a system of two coupled equations, describing the evolution of the large scale component u_{L} and the small scale component u_{S} separately. As already explained, we are not interested in solving explicitly the equation for the small scale process u_{S}, which instead is modelled stochastically as described in Section 2. This operation can be performed whenever the structures produced in a geophysical system have a wide range of spatial scales, which corresponds to a wide range of temporal scales. For the sake of modelling, among the various temporal scales, we select three particular of them satisfying certain relations, see below for details. The stochastic modelling depends on a parameter ϵ describing the separation between these temporal scales, and our result, obtained by taking the limit of infinite separation of time scales, consists in the convergence of the large scale velocity u_{L} towards the solution of the stochastic equation

$$\begin{cases} \partial_t u_{\mathrm{L}} + J(u_{\mathrm{L}}, \nabla u_{\mathrm{L}}) = -\frac{\sigma}{\gamma} J(\circ \dot{W}, \nabla u_{\mathrm{L}}) - \frac{1}{\rho_0} \nabla p_{\mathrm{L}} + \mathcal{D}(u_{\mathrm{L}}) + f_{\mathrm{L}}, \\ \nabla \cdot u_{\mathrm{L}} = 0, \end{cases}$$

where W is a Brownian motion, σ and γ are suitable coefficients and f_{L} denotes the large scale forces acting on the fluid. Our results therefore add further motivation to the study of transport-type noise in equations from fluid-mechanics, which started with the works in [7–10] and has received a lot of attention in the last years, see in [11–15] and more recently in [16–18].

Our approach differs from the many already available in the literature for being purely infinite-dimensional. In fact, although finite-dimensional models are usually sufficient to provide good numerical simulations of the real geophysical processes, for the theoretical motivations explained above it is important to have reduction procedures that act directly on the infinite-dimensional model under investigation. In our particular case, the special form of the limiting equation (stochastic PDE with Stratonovich transport noise) gives access to a vast range of results and techniques from stochastic analysis to study some properties of a geophysical system like, for instance, the existence of invariant measures, ergodicity, Large Deviations estimates for small intensity of the noise, and others.

2. Main Results

First of all, we clarify from the beginning that the theory illustrated in this work applies to systems with a wide range of space-time scales, this sentence to be understood as explained below. Among this variety of scales, for the sake of modelling we identify three reference scales that constitutes the basis of our analysis.

Concerning the time scales, we need a small time scale $\mathcal{T}_{\mathrm{S}}$, which is the characteristic time of the small scale dynamics, we need an intermediate scale $\mathcal{T}_{\mathrm{M}}$, and then we need a third, large time scale $\mathcal{T}_{\mathrm{L}}$ typical of the large scale dynamics. The following relation will play a role,

$$\frac{\mathcal{T}_{\mathrm{S}}}{\mathcal{T}_{\mathrm{M}}} = \frac{\mathcal{T}_{\mathrm{M}}}{\mathcal{T}_{\mathrm{L}}} = \epsilon. \tag{3}$$

In terms of spatial scales, we take three reference scales: one small scale $\mathcal{X}_{\mathrm{S}}$, one intermediate scale $\mathcal{X}_{\mathrm{M}}$ and one large scale $\mathcal{X}_{\mathrm{L}}$. The scales $\mathcal{X}_{\mathrm{S}}$ and $\mathcal{X}_{\mathrm{L}}$ are understood, respectively, as the characteristic length of small-scale and large-scale dynamics.

The specific values of scales $\mathcal{T}_{\mathrm{S}}$, $\mathcal{T}_{\mathrm{M}}$ $\mathcal{T}_{\mathrm{L}}$, $\mathcal{X}_{\mathrm{S}}$, $\mathcal{X}_{\mathrm{M}}$ and $\mathcal{X}_{\mathrm{L}}$ are not fundamental in our analysis, and can be modified for other applications of our arguments. Relations between spatial and temporal scales are specified below.

An example, although ideal, may be the lower-layer atmospheric fluid over a large region, which interacts with the irregularities of the ground. This system can be described, with a certain degree of approximation, by means of the ideal model (2):

$$\begin{cases} \partial_t u + J(u, \nabla u) = -\frac{1}{\rho_0}\nabla p + \mathcal{D}(u) + f, \\ \nabla \cdot u = 0. \end{cases}$$

Suppose we are observing our system at a certain combination of space-time scales $\mathcal{X}$ and $\mathcal{T}$. Dimensional analysis of (2) above gives the following identity,

$$\frac{\mathcal{U}}{\mathcal{T}} + \frac{\mathcal{U}^2}{\mathcal{X}} \sim \mathcal{F},$$

where $\mathcal{U}$ is the reference order of magnitude of velocities and $\mathcal{F}$ is the reference order of magnitude of forces per unit of mass in the system (2). Hereafter, we adopt the natural choice

$$\mathcal{U} = \frac{\mathcal{X}}{\mathcal{T}}, \quad \mathcal{F} = \frac{\mathcal{X}}{\mathcal{T}^2}.$$

The last reference quantity we introduce is reference mass $\mathcal{M}$, which for convenience we take as

$$\mathcal{M} = \rho_0 \mathcal{X}^3. \tag{4}$$

Following [6], Equations (2) can be non-dimensionalised via the substitutions:

$$\mathbf{u} = \frac{u}{\mathcal{U}}, \quad \mathbf{x} = \frac{x}{\mathcal{X}}, \quad \mathbf{t} = \frac{t}{\mathcal{T}}, \quad \mathbf{f} = \frac{f}{\mathcal{F}}, \quad \mathbf{p} = \frac{p}{\mathcal{F}\mathcal{M}\mathcal{X}^{-2}}, \quad \boldsymbol{\rho}_0 = \frac{\rho_0}{\mathcal{M}\mathcal{X}^{-3}},$$

and take the form

$$\begin{cases} \partial_t \mathbf{u} + J(\mathbf{u}, \nabla \mathbf{u}) = -\nabla \mathbf{p} + \mathcal{D}(\mathbf{u}) + \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where ∂_t and ∇ are nondimensional derivatives with respect to variables $\mathbf{t}$ and $\mathbf{x}$, and the non-dimensionalised density $\boldsymbol{\rho}_0$ is unitary thanks to (4).

2.1. Small Scale

By small scale we mean the system observed by the point of view of an observer whose characteristic unit of measure are small, that is,

$$\mathcal{X} = \mathcal{X}_{\mathrm{S}}, \quad \mathcal{T} = \mathcal{T}_{\mathrm{S}}.$$

Assume we split the initial conditions according to some reasonable rule (geometric, spectral...), in large and small scales

$$u|_{t=0} = u_{\mathrm{L}}(0) + u_{\mathrm{S}}(0).$$

Small scales describe the fluid fluctuations at space distances of order $\mathcal{X}_{\mathrm{S}}$; large scales those which impact at the regional level (national, continental), namely, with structures with size of order $\mathcal{X}_{\mathrm{L}}$. We assume this separation of scales at time $t = 0$.

Given this separation of the initial datum, we split (2) into the following system of equations,

$$\begin{cases} \partial_t u_{\mathrm{L}} + J(u_{\mathrm{L}}, \nabla u_{\mathrm{L}}) = -J(u_{\mathrm{S}}, \nabla u_{\mathrm{L}}) - \frac{1}{\rho_0}\nabla p_{\mathrm{L}} + \mathcal{D}(u_{\mathrm{L}}) + f_{\mathrm{L}}, \\ \partial_t u_{\mathrm{S}} + J(u_{\mathrm{L}}, \nabla u_{\mathrm{S}}) = -J(u_{\mathrm{S}}, \nabla u_{\mathrm{S}}) - \frac{1}{\rho_0}\nabla p_{\mathrm{S}} + \mathcal{D}(u_{\mathrm{S}}) + f_{\mathrm{S}}, \\ \nabla \cdot u_{\mathrm{L}} = 0, \quad \nabla \cdot u_{\mathrm{S}} = 0, \end{cases} \tag{5}$$

where f_{L} corresponds to large scale external forces and f_{S} incorporates the small scale inputs due to ground irregularities. We assume that f_{S} acts on small scale, namely, it includes variations at distances of order $\mathcal{X}_{\mathrm{S}}$, with changes in time in a range of order of $\mathcal{T}_{\mathrm{S}}$. The property above can be reformulated in the following way; the non-dimensionalisation of f_{S} with reference magnitude given by $\mathcal{F}_{\mathrm{S}}$ is of order one

$$f_{\mathrm{S}} = \mathcal{F}_{\mathrm{S}}\,\mathbf{f}_{\mathrm{S}} = \frac{\mathcal{X}_{\mathrm{S}}}{\mathcal{T}_{\mathrm{S}}^2}\,\mathbf{f}_{\mathrm{S}}, \quad \text{with } \mathbf{f}_{\mathrm{S}} \text{ of order one,}$$

and $\mathbf{f}_{\mathrm{S}}$ has typical variations at distances and times of order one. We assume that similar properties hold for the small scale dissipation term $\mathcal{D}(u_{\mathrm{S}})$. In particular, under suitable assumptions on the initial condition $u_{\mathrm{S}}(0)$, the non-dimensionalised small scale velocity u_{S} with reference magnitude given by $\mathcal{U}_{\mathrm{S}}$ is of order one as well:

$$u_{\mathrm{S}} = \mathcal{U}_{\mathrm{S}}\,\mathbf{u}_{\mathrm{S}} = \frac{\mathcal{X}_{\mathrm{S}}}{\mathcal{T}_{\mathrm{S}}}\,\mathbf{u}_{\mathrm{S}}, \quad \text{with } \mathbf{u}_{\mathrm{S}} \text{ of order one.}$$

In addition, $\mathbf{u}_{\mathrm{S}}$ undergoes appreciable changes over time intervals and distances of order one, in formulae

$$\partial_t \mathbf{u}_{\mathrm{S}}, \nabla \mathbf{u}_{\mathrm{S}} \quad \text{of order one,}$$

where ∂_t and ∇ are nondimensional derivatives with respect to variables $\mathbf{t} = t/\mathcal{T}_{\mathrm{S}}$ and $\mathbf{x} = x/\mathcal{X}_{\mathrm{S}}$.

Remark 1. *It is easy to check that the splitting* (5) *is consistent with* (2)*, in the sense that if* (u_L, u_S) *is a solution of* (5)*, then* $u = u_L + u_S$ *is a solution of* (2)*. However, we point out that a priori one could have split the equation in a different way, for instance, exchanging the role of* $J(u_S, \nabla u_L)$ *and* $J(u_L, \nabla u_S)$*: both splittings would have been consistent with the initial equation.*

In other words, the physics only prescribes the evolution of $u_L + u_S$ *and not the evolution of* u_L *and* u_S *individually, and therefore the choice of a splitting for* (2) *corresponds de facto in the choice of a model for the evolution of* u_L *and* u_S *separately, and vice versa.*

The main issue here is that not every splitting is also consistent with the heuristic idea that the two components of the system model the dynamics of large and small structures separately. As far as this is concerned, the splitting (5) is part of the trend called location uncertainty [1], which prescribes the evolution of u_L in a way that is substantially equivalent to the splitting of (2) at the level of velocity.

Nevertheless, motivated by the works in [15,19–21], we also point out as a possible alternative approach the so-called stochastic advection by Lie transport. According to this scheme, the evolution of u_L is prescribed in a manner that is basically equivalent to the splitting of (2) at the level of vorticity, see, for instance, in [22] and Theorem 1 below.

For the reader's convenience, we rewrite system (5) in non-dimensionalised variables:

$$\begin{cases} \partial_t \mathbf{u}_L + J(\mathbf{u}_L, \nabla \mathbf{u}_L) = -J(\mathbf{u}_S, \nabla \mathbf{u}_L) - \nabla \mathbf{p}_L + \mathcal{D}(\mathbf{u}_L) + \mathbf{f}_L, \\ \partial_t \mathbf{u}_S + J(\mathbf{u}_L, \nabla \mathbf{u}_S) = -J(\mathbf{u}_S, \nabla \mathbf{u}_S) - \nabla \mathbf{p}_S + \mathcal{D}(\mathbf{u}_S) + \mathbf{f}_S, \\ \nabla \cdot \mathbf{u}_L = 0, \quad \nabla \cdot \mathbf{u}_S = 0. \end{cases} \tag{6}$$

2.2. Intermediate Scale

Let us observe the same system from the viewpoint of an observer whose reference unit of measure are

$$\mathcal{X} = \mathcal{X}_M, \quad \mathcal{T} = \mathcal{T}_M.$$

Assume that the order of magnitude of $\mathcal{U}_S$ and $\mathcal{U}_M$ are comparable:

$$\frac{\mathcal{X}_S}{\mathcal{T}_S} \sim \frac{\mathcal{X}_M}{\mathcal{T}_M}.$$

As a result, the non-dimensionalised velocity $\mathbf{u}_S$ has the same order of magnitude, independently of the choice of $\mathcal{U}_S$ or $\mathcal{U}_M$ as reference unit of measure. However, the typical time of the fluctuations of the small scale velocity u_S is $\mathcal{T}_S$: this implies that non-dimensionalising the velocity with respect to reference measure $\mathcal{U}_M$ gives a non-dimensionalised velocity process with fluctuations of typical period

$$\frac{\mathcal{T}_S}{\mathcal{T}_M} = \epsilon.$$

Similarly, as u_S changes in space over distances of order $\mathcal{X}_S$, the non-dimensionalised velocity process with respect to reference measure $\mathcal{U}_M$ changes in space over distances of order

$$\frac{\mathcal{X}_S}{\mathcal{X}_M} = \epsilon.$$

In formulae, the arguments above can be summarised as follows,

$$u_S = \mathcal{U}_M \, \mathbf{u}_S = \frac{\mathcal{X}_M}{\mathcal{T}_M} \mathbf{u}_S, \quad \text{with } \mathbf{u}_S \text{ of order one,}$$

and

$$\partial_t \mathbf{u}_S, \nabla \mathbf{u}_S \quad \text{of order } \epsilon^{-1},$$

where ∂_t and ∇ are nondimensional derivatives with respect to variables $\mathbf{t} = t/\mathcal{T}_M$ and $\mathbf{x} = x/\mathcal{X}_M$.

This motivates our main modelling assumption, see also in [23–25]. We replace the small scales by a stochastic equation, Gaussian conditionally to the large scales; that is, we replace the second equation in (6) by

$$\partial_t \mathbf{u}_S + J(\mathbf{u}_L, \nabla \mathbf{u}_S) = -\frac{\gamma}{\epsilon}\mathbf{u}_S - \nabla \mathbf{p}_S + \frac{\sigma}{\sqrt{\epsilon}}\dot{\mathbf{W}}_S, \tag{7}$$

where $\mathbf{W}_S$ is a Brownian motion on the velocity space $L^2(D, \mathbb{R}^n)$, with $\nabla \cdot \mathbf{W}_S = 0$ and possibly additional boundary conditions, and γ, σ are positive constants. The condition $\nabla \cdot \mathbf{W}_S = 0$ is not restrictive for our purpose, see Remark 4 below. For technical reasons we assume that the space covariance of $\mathbf{W}_S$ is sufficiently regular. For the sake of simplicity we take (cfr. also the discussion in [26])

$$\sigma \mathbf{W}_S(t, x) = \sum_{k\in\mathbb{N}} \sigma_k(x)\beta_t^k, \tag{8}$$

where $\{\beta^k\}_{k\in\mathbb{N}}$ is a family of independent standard Brownian motions on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\sigma_k \in C^2(D, \mathbb{R}^n)$ for every $k \in \mathbb{N}$ with

$$\sum_{k\in\mathbb{N}} \|\nabla\nabla\sigma_k\|_{L^\infty} < \infty. \tag{9}$$

We make this modelling choice for a number of reasons: first, we work under the implicit assumption that quickly varying fluctuations in the small scales dynamics are given by the combined effect of a large number of weakly coupled factors, so that Central Limit Theorem applies. Therefore, it is natural to model the self-interaction $-J(u_S, \nabla u_S)$ and the external forcing f_S with a Gaussian source of noise.

The presence of the damping term $-\frac{\gamma}{\epsilon}\mathbf{u}_S$ simulates dissipation, where γ is of order one. The coefficient ϵ^{-1} in front of the damping is motivated by the fact that, in the regime under investigation, the velocity $\mathbf{u}_S$ is of order one, while the dissipative forces acting on the fluid are of order

$$\frac{\mathcal{F}_S}{\mathcal{F}_M} = \epsilon^{-1},$$

and therefore a coefficient ϵ^{-1} is needed to make damping of the same order of magnitude as dissipation.

Finally, given the factor ϵ^{-1} in front of the damping, we observe that the coefficient in front of the random term $\frac{\sigma}{\sqrt{\epsilon}}\dot{\mathbf{W}}_S$, which models $-J(u_S, \nabla u_S)$ and f_S, is the only compatible with the fact that $\mathbf{u}_S$ is of order one, with typical period of fluctuation of order ϵ. Indeed, neglecting for simplicity the terms $J(\mathbf{u}_L, \nabla \mathbf{u}_S)$ and $\mathbf{p}_S$ in the equation for $\mathbf{u}_S$ and taking $\mathbf{u}_S(0) = 0$, one has

$$\mathbf{u}_S(t) = \int_0^t e^{-\gamma\epsilon^{-1}(t-r)}\frac{\sigma}{\sqrt{\epsilon}}\dot{\mathbf{W}}_S(r)dr$$

and the covariance between $\mathbf{u}_S(t)$ and $\mathbf{u}_S(s)$ is equal to

$$\mathrm{Cov}(\mathbf{u}_S(t), \mathbf{u}_S(s)) = \frac{\sigma^2}{2\gamma}\left(e^{-\gamma\epsilon^{-1}|t-s|} - e^{-\gamma\epsilon^{-1}(t+s)}\right),$$

in accordance with the fact that $\mathbf{u}_S$ is of order one, as its variance is approximately equal to $\frac{\sigma^2}{2\gamma}$, and has typical period of fluctuation of order ϵ, as the covariance between $\mathbf{u}_S(t)$ and $\mathbf{u}_S(s)$ decays approximately as $e^{-\gamma\epsilon^{-1}|t-s|}$. These two properties can not hold simultaneously with a random term of the form $\frac{\sigma}{\epsilon^\alpha}\dot{\mathbf{W}}_S$, $\alpha \neq 1/2$, thus motivating our choice.

Moreover, in [23] it is shown, under certain hypotheses on the spatial correlation of the noise, that a model similar to that considered here is capable of representing in silico the main statistical properties of two-dimensional turbulence: energy spectra, inverse energy cascade and direct enstrophy cascade. This fact adds further justification to our modelling choice.

We remark that, in addition to the physical motivation just discussed behind our modelisation, there is also a practical reason: indeed, the Ornstein–Uhlenbeck process $\mathbf{v}$ given by

$$\partial_t \mathbf{v} = -\frac{\gamma}{\epsilon}\mathbf{v} + \frac{\sigma}{\sqrt{\epsilon}}\dot{\mathbf{W}}_{\mathrm{S}}$$

is mathematically very treatable, thus making possible explicit computations for (7).

Remark 2. *A posteriori, we will see that the large scale non-dimensionalised large scale velocity process $\boldsymbol{u}_{\mathrm{L}}$ is of order ϵ when expressed with respect to reference measure $\mathcal{U}_{\mathrm{M}}$ (see subsection below). Using this, together with the fact that $\nabla \boldsymbol{u}_{\mathrm{S}}$ is of order ϵ^{-1} when expressed with respect to $\mathcal{U}_{\mathrm{M}}$, one has*

$$J(\boldsymbol{u}_{\mathrm{L}}, \nabla \boldsymbol{u}_{\mathrm{S}}) \quad \textit{of order one}$$

at intermediate scales. On the other hand, for the quadratic self-interaction and external forces we have

$$J(\boldsymbol{u}_{\mathrm{S}}, \nabla \boldsymbol{u}_{\mathrm{S}}), \boldsymbol{f}_{S} \quad \textit{of order } \epsilon^{-1}$$

at intermediate scales. This suggests that the scattering term $J(\boldsymbol{u}_{\mathrm{L}}, \nabla \boldsymbol{u}_{\mathrm{S}})$ plays only a minor role in the dynamics of $\boldsymbol{u}_{\mathrm{S}}$, which can be made rigorous in some particular case.

2.3. Large Scale

By this we mean the same system, lower atmospheric layer over a large region, observed by a satellite. The unit of measure are

$$\mathcal{X} = \mathcal{X}_{\mathrm{L}}, \quad \mathcal{T} = \mathcal{T}_{\mathrm{L}}.$$

We assume now the following relation,

$$\frac{\mathcal{U}_{\mathrm{M}}}{\mathcal{U}_{\mathrm{L}}} = \epsilon^{-1},$$

which corresponds to

$$\mathcal{X}_{\mathrm{M}} = \mathcal{X}_{\mathrm{L}}.$$

Equation (7) becomes

$$\partial_t \mathbf{u}_{\mathrm{S}} + J(\mathbf{u}_{\mathrm{L}}, \nabla \mathbf{u}_{\mathrm{S}}) = -\frac{\gamma}{\epsilon^2}\mathbf{u}_{\mathrm{S}} - \nabla \mathbf{p}_{\mathrm{S}} + \frac{\sigma}{\epsilon^2}\dot{\widetilde{\mathbf{W}}}_{\mathrm{S}}, \tag{10}$$

where $\widetilde{\mathbf{W}}_{\mathrm{S}}$ satisfies $\widetilde{\mathbf{W}}_{\mathrm{S}}(t,x) = \epsilon^{1/2}\mathbf{W}_{\mathrm{S}}(t/\epsilon, x)$, in particular $\widetilde{\mathbf{W}}_{\mathrm{S}}$ also is a Brownian motion.

Now go back to the equation for the large-scale velocity:

$$\partial_t u_{\mathrm{L}} + J(u_{\mathrm{L}}, \nabla u_{\mathrm{L}}) = -J(u_{\mathrm{S}}, \nabla u_{\mathrm{L}}) - \frac{1}{\rho_0}\nabla p_{\mathrm{L}} + \mathcal{D}(u_{\mathrm{L}}) + f_{\mathrm{L}}.$$

Assume that the typical order of magnitude of u_{L} is $\mathcal{U}_{\mathrm{L}}$, that is, the large-scale structures travel a distance of order $\mathcal{X}_{\mathrm{L}}$ in a time of order $\mathcal{T}_{\mathrm{L}}$. This means that the non-dimensionalisation of u_{L} with reference magnitude given by $\mathcal{U}_{\mathrm{L}}$ is of order one

$$u_{\mathrm{L}} = \mathcal{U}_{\mathrm{L}}\, \mathbf{u}_{\mathrm{L}} = \frac{\mathcal{X}_{\mathrm{L}}}{\mathcal{T}_{\mathrm{L}}}\, \mathbf{u}_{\mathrm{L}}, \quad \text{with } \mathbf{u}_{\mathrm{L}} \text{ of order one,}$$

and also

$$\partial_t \mathbf{u}_{\mathrm{L}}, \nabla \mathbf{u}_{\mathrm{L}} \quad \text{of order one,}$$

where ∂_t and ∇ are nondimensional derivatives with respect to variables $\mathbf{t} = t/\mathcal{T}_{\mathrm{L}}$ and $\mathbf{x} = x/\mathcal{X}_{\mathrm{L}}$.

Similarly, the forces acting on u_{L} due to pressure, dissipation and external sources are of magnitude $\mathcal{F}_{\mathrm{L}}$, so that their non-dimensionalisation with reference magnitude given by $\mathcal{F}_{\mathrm{L}}$ is of order one as well. Therefore, looking at the whole system in non-dimensionalised variables, with unit of measure $\mathcal{X}_{\mathrm{L}}, \mathcal{T}_{\mathrm{L}}$, it takes the following form (recall that by assumptions $\widetilde{\mathbf{W}}_{\mathrm{S}}$ is divergence-free),

$$\begin{cases} \partial_t \mathbf{u}_{\mathrm{L}} + J(\mathbf{u}_{\mathrm{L}}, \nabla \mathbf{u}_{\mathrm{L}}) = -J(\mathbf{u}_{\mathrm{S}}, \nabla \mathbf{u}_{\mathrm{L}}) - \nabla \mathbf{p}_{\mathrm{L}} + \mathcal{D}(\mathbf{u}_{\mathrm{L}}) + \mathbf{f}_{\mathrm{L}}, \\ \partial_t \mathbf{u}_{\mathrm{S}} + J(\mathbf{u}_{\mathrm{L}}, \nabla \mathbf{u}_{\mathrm{S}}) = -\frac{\gamma}{\epsilon^2} \mathbf{u}_{\mathrm{S}} - \nabla \mathbf{p}_{\mathrm{S}} + \frac{\sigma}{\epsilon^2} \dot{\widetilde{\mathbf{W}}}_{\mathrm{S}}, \\ \nabla \cdot \mathbf{u}_{\mathrm{L}} = 0, \quad \nabla \cdot \mathbf{u}_{\mathrm{S}} = 0. \end{cases} \tag{11}$$

To ease the notation we denote $\mathbf{W}_{\mathrm{S}} = \widetilde{\mathbf{W}}_{\mathrm{S}}$ in the following.

Remark 3. *Looking at* (11) *above at large scales, one immediately notice that all the terms in the equation for $\boldsymbol{u}_{\mathrm{L}}$ are of order one, except $J(\boldsymbol{u}_{\mathrm{S}}, \nabla \boldsymbol{u}_{\mathrm{L}})$. Indeed, as the non-dimensionalised small scale velocity $\boldsymbol{u}_{\mathrm{S}}$ is of order ϵ^{-1} when expressed with respect the reference velocity $\mathcal{U}_{\mathrm{L}}$, the term $J(\boldsymbol{u}_{\mathrm{S}}, \nabla \boldsymbol{u}_{\mathrm{L}})$ is of order ϵ^{-1} as well. However, for small ϵ, the quickly varying-in-time of $\boldsymbol{u}_{\mathrm{S}}$ has an averaging effect on the term $J(\boldsymbol{u}_{\mathrm{S}}, \nabla \boldsymbol{u}_{\mathrm{L}})$, which thus converges (in a suitable sense) to noise of transport type, see subsection below.*

2.4. Asymptotic Behaviour of Coupled System

As already said, we are interested in the large scale component $\mathbf{u}_{\mathrm{L}}$ of (11) above. In particular our goal is to find a new equation for $\mathbf{u}_{\mathrm{L}}$ which is closed in $\mathbf{u}_{\mathrm{L}}$, namely, we do not want to solve for $\mathbf{u}_{\mathrm{S}}$ in order to compute the coupling term $-J(\mathbf{u}_{\mathrm{S}}, \nabla \mathbf{u}_{\mathrm{L}})$. We notice that, in the limit as $\epsilon \to 0$, the small-scale velocity is well approximated by the stationary Ornstein–Uhlenbeck process $\mathbf{v}^\epsilon$ given by

$$\partial_t \mathbf{v}^\epsilon = -\frac{\gamma}{\epsilon^2} \mathbf{v}^\epsilon + \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{\mathrm{S}},$$

almost independently on the initial condition $\mathbf{u}_{\mathrm{S}}(0)$, as time correlation decays as $\exp(-\gamma\epsilon^{-2}t)$. The process $\mathbf{v}^\epsilon$ formally converges to a white-in-time noise, because of the following computation,

$$\begin{aligned} \int_0^t \mathbf{v}_s^\epsilon ds &= \int_0^t \mathbf{v}_0^\epsilon e^{-\gamma\epsilon^{-2}s} ds + \int_0^t \left(\int_0^s e^{-\gamma\epsilon^{-2}(s-r)} \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{\mathrm{S}}(r) dr \right) ds \\ &= \int_0^t \mathbf{v}_0^\epsilon e^{-\gamma\epsilon^{-2}s} ds + \int_0^t \left(\int_r^t e^{-\gamma\epsilon^{-2}(s-r)} \gamma\epsilon^{-2} ds \right) \frac{\sigma}{\gamma} \dot{\mathbf{W}}_{\mathrm{S}}(r) dr \\ &= \int_0^t \mathbf{v}_0^\epsilon e^{-\gamma\epsilon^{-2}s} ds + \int_0^t \left(1 - e^{-\gamma\epsilon^{-2}(t-r)} \right) \frac{\sigma}{\gamma} \dot{\mathbf{W}}_{\mathrm{S}}(r) dr \\ &= \frac{\sigma}{\gamma} \mathbf{W}_{\mathrm{S}}(t) + O(\epsilon). \end{aligned}$$

The asymptotic behaviour of (11) as $\epsilon \to 0$ can therefore be studied in a rigorous mathematical framework as an example of Wong–Zakai approximation principle for stochastic PDEs. Starting from the seminal work of Wong and Zakai [27], a number of results have been obtained in this direction:

we mention among others the works in [28–33] and, more recently, those in [34,35] based on rough path theory. The aforementioned results suggest, as a rule of thumb, to interpret the formal limit of $\mathbf{v}^\epsilon$ as a white-in-time noise in Stratonovich sense, that is, for every suitable process φ and some appropriate notion of convergence

$$\int_0^t \varphi_s \mathbf{v}_s^\epsilon ds \to \int_0^t \varphi_s \frac{\sigma}{\gamma} \circ \dot{\mathbf{W}}_{\mathrm{S}}(s) ds$$
$$= \lim_{\substack{\max|s_{i+1}-s_i|\to 0,\\ 0=s_1<\cdots<s_i<\cdots<s_N=t}} \sum_{i=0}^{N-1} \frac{\varphi_{s_{i+1}} + \varphi_{s_i}}{2} \frac{\sigma}{\gamma} \left(\mathbf{W}_{\mathrm{S}}(s_{i+1}) - \mathbf{W}_{\mathrm{S}}(s_i) \right),$$

where the latter is a limit in mean square.

Therefore, the candidate limit equation for the sole large scale velocity $\mathbf{u}_\mathrm{L}$ is the following,

$$\begin{cases} \partial_t \mathbf{u}_\mathrm{L} + J(\mathbf{u}_\mathrm{L}, \nabla \mathbf{u}_\mathrm{L}) = -\frac{\sigma}{\gamma} J(\circ \dot{\mathbf{W}}_\mathrm{S}, \nabla \mathbf{u}_\mathrm{L}) - \nabla \mathbf{p}_\mathrm{L} + \mathcal{D}(\mathbf{u}_\mathrm{L}) + \mathbf{f}_\mathrm{L}, \\ \nabla \cdot \mathbf{u}_\mathrm{L} = 0, \end{cases} \tag{12}$$

where $\circ\dot{\mathbf{W}}_\mathrm{S}$ stands for stochastic integration in the Stratonovich sense. In the particular case

$$\sigma \mathbf{W}_\mathrm{S}(t,x) = \sum_{k\in\mathbb{N}} \sigma_k(x) \beta_t^k,$$

by bilinearity of J Equation (12) above takes the more explicit form

$$\begin{cases} \partial_t \mathbf{u}_\mathrm{L} + J(\mathbf{u}_\mathrm{L}, \nabla \mathbf{u}_\mathrm{L}) = -\gamma^{-1} \sum_{k\in\mathbb{N}} J(\sigma_k, \nabla \mathbf{u}_\mathrm{L}) \circ \dot{\beta}_t^k - \nabla \mathbf{p}_\mathrm{L} + \mathcal{D}(\mathbf{u}_\mathrm{L}) + \mathbf{f}_\mathrm{L}, \\ \nabla \cdot \mathbf{u}_\mathrm{L} = 0. \end{cases}$$

Remark 4. *In the argument above we have used the approximation $\boldsymbol{u}_S \sim \boldsymbol{v}^\epsilon$, thus neglecting the terms $J(\boldsymbol{u}_L, \nabla \boldsymbol{u}_S)$ and $\nabla \boldsymbol{p}_S$, which is indeed the case if $\nabla \cdot \boldsymbol{W}_S = 0$. We point out that in the general case $\nabla \cdot \boldsymbol{W}_S \neq 0$ the process $\boldsymbol{u}_S$ does not converge to $\circ\dot{\boldsymbol{W}}_S$, but thank to the presence of the stochastic pressure term it converges to $\circ\dot{\boldsymbol{W}}_S^\sigma$, where $\boldsymbol{W}_S^\sigma$ is the solenoidal part in the Helmhotz decomposition of $\boldsymbol{W}_S$, satisfying*

$$\nabla \cdot \boldsymbol{W}_S^\sigma = 0, \quad \boldsymbol{W}_S - \boldsymbol{W}_S^\sigma \text{ is a gradient.}$$

In particular, the limit Equation (12) would have been the same, except for $\boldsymbol{W}_S^\sigma$ replacing $\boldsymbol{W}_S$, and therefore the assumption $\nabla \cdot \boldsymbol{W}_S = 0$ is not restrictive when investigating the limit behaviour of the large-scale velocity process.

Remark 5. *By a physical point of view, taking the limit $\epsilon \to 0$ in (11) corresponds to implicitly assume infinite separation of time scales. This hypothesis, although not matching strictly speaking with reality, constitutes a sufficiently good approximations and is a very practical working assumption. This also motivates the interest in the identification of the rate of convergence of Wong–Zakai approximations to their limits, see for instance [29,36]. However, we do not treat this problem here.*

We summarise the heuristic discussion above with the following.

Conjecture 1. *Fix $T > 0$. For every $\epsilon > 0$ denote $\boldsymbol{u}_L^\epsilon$ the solution of (11) on the time interval $[0,T]$ and let $\boldsymbol{u}_L$ be the solution of (12) on the time interval $[0,T]$. Then, the following convergence in probability holds,*

$$\boldsymbol{u}_L^\epsilon \to \boldsymbol{u}_L, \quad \text{as } \epsilon \to 0,$$

where $\boldsymbol{u}_L^\epsilon$ and $\boldsymbol{u}_L$ are intended as random variables in $C([0,T], L^1(D,\mathbb{R}^n))$.

A few remarks are in order.

First of all, we are tacitly assuming well-posedness of (11) and (12); otherwise, the result just conjectured may not have a precise meaning. A global well-posedness result in this abstract setting, however, is not available: the theory of incompressible equations from fluid-dynamics is very different depending on the dimension n, on the type of dissipation and on the boundary conditions. Therefore, it is impossible to unify everything in one single theorem, and each case must be treated separately.

At least two different situation are worth of special mention. The first is the case where well-posedness holds globally for (12), but only locally for (11), up to a (possibly random) time τ_ϵ converging to ∞ as $\epsilon \to 0$. In this case, the convergence of Conjecture 1 may still hold if we replace $\mathbf{u}_{\mathrm{L}}^{\epsilon}$ with its stopped version $\mathbf{u}_{\mathrm{L}}^{\epsilon}(\cdot \wedge \tau_\epsilon)$: we are in front of global well-posedness *in the limit*. The second scenario is when (12) is globally well-posed, but its deterministic counterpart (2) is not: in some sense, the presence of the noise regularises the equations. Regularisation by noise has been widely investigated (see in [37] and the references therein) and is still an active topic of research.

The second remark concerns the strategy of the proof of convergence $\mathbf{u}_{\mathrm{L}}^{\epsilon} \to \mathbf{u}_{\mathrm{L}}$. As already said, the validity of this result heavily depends on many factors, so we do not aim to give an universal approach to this problem, but rather some ideas. If dissipation in the large scale component of (11) is sufficiently strong, then the evolution semigroup $e^{\mathcal{D}t}$ is regularising and one can consider the mild formulation

$$\begin{aligned}\mathbf{u}_{\mathrm{L}}^{\epsilon}(t) = &-\int_0^t e^{\mathcal{D}(t-s)} J(\mathbf{u}_{\mathrm{L}}^{\epsilon}(s), \nabla \mathbf{u}_{\mathrm{L}}^{\epsilon}(s))ds - \int_0^t e^{\mathcal{D}(t-s)} J(\mathbf{u}_{\mathrm{S}}^{\epsilon}(s), \nabla \mathbf{u}_{\mathrm{L}}^{\epsilon}(s))ds \\ &- \int_0^t e^{\mathcal{D}(t-s)} \nabla \mathbf{p}_{\mathrm{L}}^{\epsilon}(s)ds + \int_0^t e^{\mathcal{D}(t-s)} \mathbf{f}_{\mathrm{L}}(s)ds + e^{\mathcal{D}t}\mathbf{u}_{\mathrm{L}}(0).\end{aligned}$$

If the quantities $e^{\mathcal{D}(t-s)} J(\mathbf{u}_{\mathrm{L}}^{\epsilon}(s), \nabla \mathbf{u}_{\mathrm{L}}^{\epsilon}(s))$, $e^{\mathcal{D}(t-s)} J(\mathbf{u}_{\mathrm{S}}^{\epsilon}(s), \nabla \mathbf{u}_{\mathrm{L}}^{\epsilon}(s))$, etc. are sufficiently well-behaved, then one can prove the desired convergence arguing as in some of the available works on Wong–Zakai principle we already mentioned.

Another strategy, that is specific for equations of transport type, may be the following. For simplicity, and having in mind Remark 1, we present the idea contained in [22] for 2D Euler equations on the two dimensional torus $D = \mathbb{T}^2$ in vorticity form

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = K * \omega, \end{cases}$$

where K is the Biot–Savart kernel on $\mathbb{T}^2$, so that $\nabla \cdot u = 0$ and $\nabla \times u = \omega$. Splitting the system above in large scale and small scale and non-dimensionalising we obtain

$$\begin{cases} \partial_t \boldsymbol{\omega}_{\mathrm{L}}^{\epsilon} + \mathbf{u}_{\mathrm{L}}^{\epsilon} \cdot \nabla \boldsymbol{\omega}_{\mathrm{L}}^{\epsilon} = -\mathbf{u}_{\mathrm{S}}^{\epsilon} \cdot \nabla \boldsymbol{\omega}_{\mathrm{L}}^{\epsilon}, \\ \partial_t \boldsymbol{\omega}_{\mathrm{S}} + \mathbf{u}_{\mathrm{L}}^{\epsilon} \cdot \nabla \boldsymbol{\omega}_{\mathrm{S}}^{\epsilon} = -\frac{\gamma}{\epsilon^2} \boldsymbol{\omega}_{\mathrm{S}}^{\epsilon} + \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{\mathrm{S}}, \\ \mathbf{u}_{\mathrm{L}}^{\epsilon} = K * \boldsymbol{\omega}_{\mathrm{L}}^{\epsilon}, \quad \mathbf{u}_{\mathrm{S}}^{\epsilon} = K * \boldsymbol{\omega}_{\mathrm{S}}^{\epsilon}. \end{cases} \tag{13}$$

The large-scale component of the system above has an explicit solution:

$$\boldsymbol{\omega}_{\mathrm{L}}^{\epsilon}(t,x) = \boldsymbol{\omega}_{\mathrm{L}}(0, (\Phi_t^{\epsilon})^{-1}(x)),$$

where $\boldsymbol{\omega}_{\mathrm{L}}(0)$ is the initial condition and $\Phi^{\epsilon}(x)$, $x \in \mathbb{T}^2$ are the characteristics, which are given by the solution of

$$\begin{cases} \partial_t \Phi_t^{\epsilon}(x) = \mathbf{u}_{\mathrm{L}}^{\epsilon}(t, \Phi_t^{\epsilon}(x)) + \mathbf{u}_{\mathrm{S}}^{\epsilon}(t, \Phi_t^{\epsilon}(x)), \\ \Phi_0^{\epsilon}(x) = x, \end{cases}$$

and a similar formula holds for the limit equation. Therefore, it is possible to deduce a convergence result for the vorticity $\boldsymbol{\omega}^{\epsilon}_{\mathrm{L}}$ (and as a consequence also for the velocity $\mathbf{u}^{\epsilon}_{\mathrm{L}}$) from a convergence result at the level of characteristics. To be precise, in [22] it is proved the following.

Theorem 1. *Let $T > 0$ and take $\boldsymbol{W}_S$ such that (8) and (9) hold. For a zero-mean initial vorticity $\boldsymbol{\omega}_L(0) \in L^{\infty}(\mathbb{T}^2)$, let $\boldsymbol{\omega}^{\epsilon}_L$ be the solution of (13) and let $\boldsymbol{\omega}_L$ be the unique solution of the stochastic equation*

$$\begin{cases} \partial_t \boldsymbol{\omega}_L + \boldsymbol{u}_L \cdot \nabla \boldsymbol{\omega}_L = \circ \boldsymbol{B} \cdot \nabla \boldsymbol{\omega}_L, \\ \boldsymbol{u}_L = K * \boldsymbol{\omega}_L, \\ \boldsymbol{\omega}_L|_{t=0} = \boldsymbol{\omega}_L(0), \end{cases}$$

where $\boldsymbol{B} = -K * \boldsymbol{W}_S$. Then, the process $\boldsymbol{\omega}^{\epsilon}_L$ converges as $\epsilon \to 0$ to $\boldsymbol{\omega}_L$ in the following sense; for every $f \in L^1(\mathbb{T}^2)$:

$$\mathbb{E}\left[\left|\int_{\mathbb{T}^2} \boldsymbol{\omega}^{\epsilon}_L(t,x) f(x)\, dx - \int_{\mathbb{T}^2} \boldsymbol{\omega}_L(t,x) f(x)\, dx\right|\right] \to 0$$

as $\epsilon \to 0$, for every fixed $t \in [0,T]$ and in $L^p([0,T])$ for every finite p. In addition, the velocity field $\boldsymbol{u}^{\epsilon}_L = K * \boldsymbol{\omega}^{\epsilon}_L$ converges as $\epsilon \to 0$, in mean value, to $\boldsymbol{u}_L = K * \boldsymbol{\omega}_L$, as random variables in $C([0,T], L^1(\mathbb{T}^2, \mathbb{R}^2))$.

3. Examples

In this section, we give some examples of the general scheme developed so far. We provide a stochastic model reduction for two systems: two-dimensional Navier–Stokes equations and three-dimensional Primitive equations. A mathematically rigorous derivation of the results stated in this section is omitted here; we will address this issue in future works.

3.1. 2D Navier–Stokes Equations

The first example we treat is the 2D Navier–Stokes equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nu \Delta u - \frac{1}{\rho_0} \nabla p + f, \\ \nabla \cdot u = 0, \end{cases} \tag{14}$$

The spatial domain is the two-dimensional sphere $D = R\,\mathbb{S}^2$, where R is the radius of the Earth. We aim to use this model to describe the evolution of the lower layer atmosphere. The interaction of the atmosphere with the surface of the Earth (either the ground or the ocean) affects the evolution of the former, introducing small scale perturbations of the flows via the external force f. We neglect all other external sources of force, that is a strong idealisation, but one can easily adapt the argument to the general case.

We assume, for this model, the following small scales.

$$\mathcal{T}_{\mathrm{S}} = 1\,\mathrm{s}, \quad \mathcal{X}_{\mathrm{S}} = 10\,\mathrm{m}.$$

To fit with our theory, the large-scale reference unit of measure must satisfy

$$\frac{\mathcal{T}_{\mathrm{L}}}{\mathcal{T}_{\mathrm{S}}} = \left(\frac{\mathcal{X}_{\mathrm{L}}}{\mathcal{X}_{\mathrm{S}}}\right)^2.$$

We therefore make the choice

$$\mathcal{T}_{\mathrm{L}} = 1\,\mathrm{d}, \quad \mathcal{X}_{\mathrm{L}} = 3\,\mathrm{km},$$

and as a consequence we have $\mathcal{T}_{\mathrm{M}} \sim 5\,\mathrm{min}$.

Splitting (14) into large scale and small scales, and assuming that f acts on small scales only, we obtain the following system.

$$\begin{cases} \partial_t u_{\mathrm{L}} + (u_{\mathrm{L}} \cdot \nabla) u_{\mathrm{L}} = -(u_{\mathrm{S}} \cdot \nabla) u_{\mathrm{L}} + \nu \Delta u_{\mathrm{L}} - \frac{1}{\rho_0} \nabla p_{\mathrm{L}}, \\ \partial_t u_{\mathrm{S}} + (u_{\mathrm{L}} \cdot \nabla) u_{\mathrm{S}} = -(u_{\mathrm{S}} \cdot \nabla) u_{\mathrm{S}} + \nu \Delta u_{\mathrm{S}} - \frac{1}{\rho_0} \nabla p_{\mathrm{S}} + f, \\ \nabla \cdot u_{\mathrm{L}} = 0, \quad \nabla \cdot u_{\mathrm{S}} = 0. \end{cases}$$

Arguing as in Section 2, looking at the previous system from the point of view

$$\mathcal{T} = \mathcal{T}_{\mathrm{L}}, \quad \mathcal{X} = \mathcal{X}_{\mathrm{L}},$$

we arrive to the following non-dimensionalised system.

$$\begin{cases} \partial_t \mathbf{u}_{\mathrm{L}} + (\mathbf{u}_{\mathrm{L}} \cdot \nabla) \mathbf{u}_{\mathrm{L}} = -(\mathbf{u}_{\mathrm{S}} \cdot \nabla) \mathbf{u}_{\mathrm{L}} + \nu_{\mathrm{L}} \Delta \mathbf{u}_{\mathrm{L}} - \nabla \mathbf{p}_{\mathrm{L}}, \\ \partial_t \mathbf{u}_{\mathrm{S}} + (\mathbf{u}_{\mathrm{L}} \cdot \nabla) \mathbf{u}_{\mathrm{S}} = -\frac{\gamma}{\epsilon^2} \mathbf{u}_{\mathrm{S}} - \nabla \mathbf{p}_{\mathrm{S}} + \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{\mathrm{S}}, \\ \nabla \cdot \mathbf{u}_{\mathrm{L}} = 0, \quad \nabla \cdot \mathbf{u}_{\mathrm{S}} = 0, \end{cases}$$

where $\epsilon^{-1} \sim 300$ and γ, σ, ν_{L} are non-dimensional coefficients. The spatial covariance of the noise $\mathbf{W}_{\mathrm{S}}$ takes account of the external sources of perturbation, like irregularities of the ground, so that the production of noise is more important near the locations of the sources.

According to our theory, the limit equation for the sole large scale velocity $\mathbf{u}_{\mathrm{L}}$ is given by

$$\begin{cases} \partial_t \mathbf{u}_{\mathrm{L}} + (\mathbf{u}_{\mathrm{L}} \cdot \nabla) \mathbf{u}_{\mathrm{L}} = -\frac{\sigma}{\gamma} (\circ \dot{\mathbf{W}}_{\mathrm{S}} \cdot \nabla) \mathbf{u}_{\mathrm{L}} + \nu_{\mathrm{L}} \Delta \mathbf{u}_{\mathrm{L}} - \nabla \mathbf{p}_{\mathrm{L}}, \\ \nabla \cdot \mathbf{u}_{\mathrm{L}} = 0, \end{cases}$$

which is a stochastic Navier–Stokes equation with Stratonovich transport noise. This equations have been studied on $D = \mathbb{R}^2$ in [10], and global well-posedness holds under relatively mild assumptions on the coefficients for every L^2 divergence-free initial datum $\mathbf{u}_{\mathrm{L}}(0)$, see also [8,38]. We remark that a Wong–Zakai result for 2D Navier–Stokes equations in a setting slightly different from ours has already been obtained, see for instance [34].

3.2. 3D Primitive Equations

The second example we present is the 3D Primitive Equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla_H) u + w \partial_z u = -\frac{1}{\rho_0} \nabla_H p + \nu \Delta u + f, \\ \partial_z p = -g \rho_0, \\ \nabla_H \cdot u + \partial_z w = 0. \end{cases} \tag{15}$$

The constant ν is the viscosity coefficient and g is the intensity of the gravitational acceleration on Earth, which is approximated to a constant: $g \sim 9.81 \ \mathrm{m/s^2}$.

These equations are used to describe the evolution of a three-dimensional fluid on domains whose ratio between the vertical and the horizontal typical length is small. A classical example is a portion of ocean with size of several hundreds of kilometres: neglecting the curvature of the Earth, the domain under investigation therefore has the form

$$D = \left\{ (\mathbf{x}, z) \in D' \times r\mathbb{R} : h(\mathbf{x}) \leq z \leq 0 \right\},$$

where $D' \subseteq r\mathbb{R}^2$ is a portion of Earth surface and $h : D' \to [-r, 0]$ is a function which describe the depth of the ocean. The quantity r is approximately 1 to 10 km and the typical diameter of D' is 100 to 1000 km.

In Equation (15), the symbol ∇_H stands for the horizontal gradient: $\nabla_H = (\partial_{\mathbf{x}_1}, \partial_{\mathbf{x}_2})$, while Δ denotes the full Laplacian: $\Delta = \partial^2_{\mathbf{x}_1} + \partial^2_{\mathbf{x}_2} + \partial^2_z$. The unknown $u : \mathbb{R}_+ \times D \to \mathbb{R}^2$ is the horizontal velocity, and the vertical velocity $w : \mathbb{R}_+ \times D \to \mathbb{R}$ is a diagnostic variable, i.e., it is obtained from u, continuity equation $\nabla_H \cdot u + \partial_z w = 0$ and boundary conditions:

$$w(t, \mathbf{x}, z) = w(t, \mathbf{x}, 0) + \int_z^0 \nabla_H \cdot u(t, \mathbf{x}, z')dz'.$$

A common choice of boundary conditions is the following,

$$\begin{cases} (u, w) \cdot \hat{n} = 0, \quad (\mathbf{x}, z) \in \partial D, \\ (u, w) = 0, \quad z = h(\mathbf{x}), \\ \partial_z u = 0, \quad z = 0. \end{cases} \tag{16}$$

The first line of (16) is the so-called impermeability condition, here $\hat{n}$ stands for the (outward) vector at point $(\mathbf{x}, z) \in \partial D$ normal to ∂D. The second line is a non-slip boundary condition at the ocean bottom and the last line of (16) is the free-layer boundary condition at the ocean surface: these last two conditions describe, respectively, the interaction of the ocean with the solid ground at the bottom and with the atmosphere on the top.

The forcing term f in (15) encodes external forces acting *horizontally* on the fluid, like, for instance, the Coriolis force:

$$f_{Cor} = -2\sin(\theta)\,\Omega_{Earth}\,\hat{z} \times u = -2\,\Omega \times u,$$

where θ is the latitude of the point $(\mathbf{x}, z)$, $\Omega_{Earth} \sim 7.27 \times 10^{-5} s^{-1}$ is the angular velocity of the Earth and $\Omega = \sin(\theta)\,\Omega_{Earth}\,\hat{z}$. As in the previous case denote f_S the inputs due to interaction of the fluid with coastal borders and ocean bottom. Other external sources can be taken into account but we decide to neglect them for the seek of simplicity, so that $f = f_{Cor} + f_S$.

We remark that the first equation in (15) is the momentum equation for the horizontal velocity u of the fluid only. The momentum equation for the vertical velocity w is replaced by the so-called hydrostatic balance condition $\partial_z p = -g\rho_0$. Hydrostasy can be derived by vertical momentum equation under the approximation of small vertical acceleration and advection compared to gravity, which is usually the case in the ocean.

We split (15) into large scale and small scale as usual: we assume that the small scale velocity field (u_S, w_S) satisfies boundary conditions (16) and the momentum equations:

$$\begin{cases} \partial_t u_S + (u_L \cdot \nabla_H)u_S + w_L \partial_z u_S = -(u_S \cdot \nabla_H)u_S - w_S \partial_z u_S \\ \qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{\rho_0}\nabla_H p_S + \nu\Delta u_S - 2\,\Omega \times u_S + f_S, \\ \partial_t w_S + (u_L \cdot \nabla_H)w_S + w_L \partial_z w_S = -(u_S \cdot \nabla_H)w_S - w_S \partial_z w_S \\ \qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{\rho_0}\partial_z p_S + \nu\Delta w_S - g_S, \\ \nabla_H \cdot u_S + \partial_z w_S = 0. \end{cases}$$

A little comment is in order. Indeed, the second equation of the system above is the vertical momentum equation for the small scale velocity field. We do not assume hydrostatic balance at this level, as here vertical acceleration and advection are not small compared to gravity, but rather the contrary is true. The term g_S stands for the vertical forces acting on the system at small scales and it is small. The full system has the following expression.

$$\begin{cases} \partial_t u_L + (u_L \cdot \nabla_H) u_L + w_L \partial_z u_L = -(u_S \cdot \nabla_H) u_L - w_S \partial_z u_L \\ \qquad\qquad - \frac{1}{\rho_0} \nabla_H p_L + \nu \Delta u_L - 2\,\Omega \times u_L, \\ \partial_t u_S + (u_L \cdot \nabla_H) u_S + w_L \partial_z u_S = -(u_S \cdot \nabla_H) u_S - w_S \partial_z u_S \\ \qquad\qquad - \frac{1}{\rho_0} \nabla_H p_S + \nu \Delta u_S - 2\,\Omega \times u_S + f_S, \\ \partial_t w_S + (u_L \cdot \nabla_H) w_S + w_L \partial_z w_S = -(u_S \cdot \nabla_H) w_S - w_S \partial_z w_S \\ \qquad\qquad - \frac{1}{\rho_0} \partial_z p_S + \nu \Delta w_S - g_S, \\ \partial_z p_L = -g\rho_0, \\ \nabla_H \cdot u_L + \partial_z w_L = 0, \quad \nabla_H \cdot u_S + \partial_z w_S = 0. \end{cases}$$

Notice that the equation $\partial_z p_L = -g\rho_0$ plays the role of vertical momentum equation for the large scale velocity. By imposing this condition, we have implicitly assumed $\partial_z p_S = 0$, that is indeed the case at points $(\mathbf{x}, z) \in D$ distant from the boundary ∂D (sources of randomness are localised at ∂D).

Applying our main modelling procedure to the small scale component of the system above and non-dimensionalizing with respect to large scale unit of measure we get

$$\begin{cases} \partial_t \mathbf{u}_L + (\mathbf{u}_L \cdot \nabla_H) \mathbf{u}_L + \mathbf{w}_L \partial_z \mathbf{u}_L = -(\mathbf{u}_S \cdot \nabla_H) \mathbf{u}_L - \mathbf{w}_S \partial_z \mathbf{u}_L \\ \qquad\qquad - \nabla_H \mathbf{p}_L + \nu_L \Delta \mathbf{u}_L - 2\,\mathbf{\Omega} \times \mathbf{u}_L, \\ \partial_t \mathbf{u}_S + (\mathbf{u}_L \cdot \nabla_H) \mathbf{u}_S + \mathbf{w}_L \partial_z \mathbf{u}_S = -\frac{\gamma}{\epsilon^2} \mathbf{u}_S - \nabla_H \mathbf{p}_S + \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{S,\mathbf{x}}, \\ \partial_t \mathbf{w}_S + (\mathbf{u}_L \cdot \nabla_H) \mathbf{w}_S + \mathbf{w}_L \partial_z \mathbf{w}_S = -\frac{\gamma}{\epsilon^2} \mathbf{w}_S - \partial_z \mathbf{p}_S + \frac{\sigma}{\epsilon^2} \dot{\mathbf{W}}_{S,z}, \\ \partial_z \mathbf{p}_L = -g_L, \\ \nabla_H \cdot \mathbf{u}_L + \partial_z \mathbf{w}_L = 0, \quad \nabla_H \cdot \mathbf{u}_S + \partial_z \mathbf{w}_S = 0. \end{cases}$$

The limit equation for the sole large scale velocity is

$$\begin{cases} \partial_t \mathbf{u}_L + (\mathbf{u}_L \cdot \nabla_H) \mathbf{u}_L + \mathbf{w}_L \partial_z \mathbf{u}_L = -\frac{\sigma}{\gamma} (\circ \dot{\mathbf{W}}_{S,\mathbf{x}} \cdot \nabla_H) \mathbf{u}_L - \frac{\sigma}{\gamma} \circ \dot{\mathbf{W}}_{S,z}\, \partial_z \mathbf{u}_L \\ \qquad\qquad - \nabla_H \mathbf{p}_L + \nu_L \Delta \mathbf{u}_L - 2\,\mathbf{\Omega} \times \mathbf{u}_L, \\ \partial_z \mathbf{p}_L = -g_L, \\ \nabla_H \cdot \mathbf{u}_L + \partial_z \mathbf{w}_L = 0, \end{cases}$$

which is a 3D stochastic Primitive equation with Stratonovich transport noise. This equations have been studied by [17], where well posedness in the case of flat topography is proved under suitable assumptions on the noise and the initial condition.

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