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Review

Beyond Scale-by-Scale Equilibrium

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Abstract: Homogeneous turbulence and turbulence in scale-by-scale equilibrium, played a leading role in the turbulence research of the second half of the twentieth century, and Jack Herring was an important contributor to these developments. The research activity which has followed these developments over the past ten to fifteen years concerns turbulence, which is out of scale-by-scale equilibrium either because it is non-stationary or because it is non-homogeneous or both. This paper is a short review of recent progress in this relatively new direction of turbulence research.

Keywords: turbulence; turbulent flows; non-equilibrium; non-stationarity; non-homogeneity

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1. Introduction

The last time I met Jack Herring was in April 2019 at NCAR in Boulder, Colorado. I was there to give a seminar and was very touched by the fact that he took the time to come to my seminar and even contributed to the discussion at the end. I had already met Jack Herring a few times over the years between 1999 and 2010 at various turbulence events in the UK and in France. I had come out of every single one of my interactions with him very much struck by his gentle manner, kindness and quiet sense of humor. I remember him, in particular, smiling when hearing mention of the Craya–Herring frame and expressing bemusement at his name being attached to a frame of reference. However, his work, which spanned five decades from the 1960s to the 2010s, was wide enough and deep enough to definitely deserve some such way of lasting recognition. One remembers, for example, his self-consistent field theory of turbulence [1,2] his theory of quasi-geostrophic turbulence [3], his contribution to confirming the inverse cascade in two-dimensional turbulence [4] (arguably one of the most important results of homogeneous turbulence research in the second half of the twentieth century), and his observations of three-dimensional small-scale structure in strongly stratified turbulence [5].

Most if not all of Jack Herring's work (that I know of, at least) is concerned with homogeneous turbulence. Statistical homogeneity is a necessary prerequisite for scale-by-scale equilibrium between the average turbulence dissipation rate and average interscale transfer rate at scales within an inertial scale range. The past ten to fifteen years have seen significant research activity on turbulence and turbulent flows beyond such equilibrium (e.g., [6–36]) In particular, in the four years since I last saw Jack Herring, there has been progress on scale-by-scale non-equilibrium arising from non-homogeneity. In this paper, I summarize some of these recent developments, as they are a natural next step after the turbulence research era of Jack Herring, which was, to a large extent, dominated by scale-by-scale equilibrium and homogeneity.

2. The Turbulence Problem

The turbulence problem is to reliably reduce the number of degrees of freedom as much as possible in as general/universal of a way as possible or within well-defined universality classes. This requires to understand the physics of turbulence in a variety of turbulent flows. In general, the rate of change of the turbulent kinetic energy (following a mean flow if it is meaningful to define one) is balanced by the turbulence production rate, turbulence transport rate, work performed by pressure gradient–velocity correlations, viscous diffusion rate and turbulence dissipation rate. Of these, only turbulence production and viscous diffusion are linear mechanisms. Research since the 1950s on the linearized Navier–Stokes equation, starting perhaps with [37], effectively addresses these two terms. However, the other three mechanisms, turbulence transport, pressure gradient–velocity correlations and turbulence dissipation, are fully non-linear. Turbulence dissipation requires fully non-linear interscale transfer physics (e.g., cascade) to be understood, modeled and scaled.

3. Turbulence Dissipation and Equilibrium Cascade

The Kolmogorov equilibrium turbulence cascade (see [38–41]) has been an essential cornerstone of turbulence theory and modeling. It leads to the turbulence dissipation scaling $\varepsilon \sim K^{3/2}/L$ (where ε stands for average turbulence dissipation rate, K for average turbulent kinetic energy and L for a measure of the average size of the largest turbulent eddies), which is, in turn, central for the following:

(i) Estimating the Reynolds number (Re) scaling of the number of turbulent degrees of freedom $(L/\eta)^3 \sim Re^{9/4}$ (where η is the Kolmogorov length scale) (e.g., [39,42,43]);
(ii) Establishing a formula for the turbulent viscosity in one-point turbulence models, such as the $K - \epsilon$ model (e.g., [43,44]);
(iii) Predicting the power law decay of homogeneous isotropic turbulence (e.g., [39]);
(iv) Establishing the spreading rates and mean flow evolutions of self-preserving boundary-free turbulent shear flows, such as wakes and jets (e.g., [45–48]);
(v) Estimating the entrainement and the motion of turbulent/non-turbulent interfaces in self-preserving turbulent shear flows (e.g., [48,49]);
(vi) Providing, with the related Kolmogorov form of the turbulent energy spectrum, a physical justification of subgrid scale models for large eddy simulations (e.g., [42,43]).

The Kolmogorov equilibrium turbulence cascade has its foundation on a two-point turbulent kinetic energy equation (see [50,51]), where the rate of change of the turbulent kinetic energy at scales smaller than r (integrated over all two-point directions) is, in general, balanced by the two-point turbulence production rate, two-point interspace turbulence transport rate, interscale turbulent transfer rate at scale r, two-point pressure gradient–velocity correlations, two-point viscous diffusion rate and the sum of turbulence dissipation rates at two points in the flow. The Kolmogorov scale-by-scale equilibrium in an inertial range of scales r is achieved contingent on the existence of the following:

(a) Local homogeneity and therefore zero average two-point turbulence production rate, zero average two-point interspace turbulence transport rate, and zero average two-point pressure gradient-velocity correlations at scale r;
(b) Local stationarity and, therefore, zero average rate of change of the turbulent kinetic energy at scales smaller than r.

In that case, the average interscale transfer rate balances the average turbulence dissipation rate at all scales small enough for (a) and (b) to hold and large enough to be able to neglect a two-point viscous diffusion rate (see Appendix B of [52] and pages 86–87 in [39]). This balance is the Kolmogorov scale-by-scale equilibrium, from which one may deduce the scaling $\varepsilon = C_\varepsilon K^{3/2}/L$ with C_ε being a dimensionless constant independent of Reynolds number if the inertial range can be assumed to extend all the way to scales comparable to L. The scaling $\varepsilon \sim K^{3/2}/L$ is the Taylor–Kolmogorov equilibrium dissipation law first proposed by [53] and then given a theoretical underpinning by Kolmogorov in 1941 (Kolmogorov's three turbulence papers of 1941 can all be found in [40]).

4. Non-Equilibrium Turbulence Dissipation Laws

A different turbulence dissipation law has been observed in a variety of turbulent flows, including decaying and forced periodic turbulence [12,13,54], turbulence generated by fractal and regular grids [9], turbulent bluff body wakes [14], turbulent jets [48], atmospheric turbulence [34,35], and even wall turbulence [20,55]. This is a non-equilibrium dissipation scaling such that $C_\varepsilon \sim \sqrt{Re_G}/Re_\lambda$, where Re_G is some global Reynolds number characterizing the inlet/initial/boundary conditions and Re_λ is a local Taylor length-based Reynolds number. Given its clear difference from the Taylor–Kolmogorov equilibrium scaling $C_\varepsilon = Const$, the underlying interscale transfer and cascade must be out of scale-by-scale equilibrium. There are currently two different attempts at explaining this non-equilibrium dissipation scaling: one by [56], who obtain it by setting to zero all turbulent kinetic energy at scales larger than L and whose calculation leads to different dissipation scalings if different energy spectral shapes are assumed at wavenumbers below L^{-1}; and one by [18], who obtain it by making the hypothesis that the turbulence dissipation rate at scales larger than L evolves in proportion to the turbulence dissipation rate at scales smaller than L in the presence of large scale coherent structures. Even if validated in the near wake of a square prism by [57], this hypothesis of [18] is as puzzling as the non-equilibrium turbulence dissipation scaling that it attempts to explain. If correct, this hypothesis could perhaps imply that qualitatively different large-scale coherent structures may lead to different turbulence dissipation laws. Indeed, Ref. [30] found a different non-equilibrium dissipation scaling in the high Reynolds number wake of a slender (rather than bluff) body, namely $C_\varepsilon \sim (\sqrt{Re_G}/Re_\lambda)^{4/3}$, and they made the point that the large scale coherent structures are helical in the wake of a slender body rather than dominated by vortex shedding as in bluff body wakes. While their observation supports the hypothesis of [18] in some qualitative sense, it invalidates it quantitatively because the argument of [18] leads to $C_\varepsilon \sim \sqrt{Re_G}/Re_\lambda$ but does not lead to $C_\varepsilon \sim (\sqrt{Re_G}/Re_\lambda)^{4/3}$.

A satisfactory theoretical explanation of the new non-equilibrium turbulence dissipation laws discovered in various turbulent flows over the past 15 years remains elusive. Nevertheless, these new laws make it clear that non-equilibrium interscale transfer and cascade physics common to various classes of turbulent flows must exist. In the following section, we present a classification of scale-by-scale non-equilibria and some of their properties as they seem to currently appear from research over the past few years.

5. Classification of Scale-by-Scale Non-Equilibria

Given that stationarity and homogeneity are the pillars of Kolmogorov scale-by-scale equilibrium, the attempt at a classification of scale-by-scale non-equilibria summarized here is in terms of different cases of non-stationarity and non-homogeneity. This section is no more than a synthesizing summary intended to help interested readers direct themselves toward the cited papers, where detailed expositions of each category of turbulent flow can be found, and also help them see the broader picture, which results from all of these papers together but is in none of them individually. Note that the classification into categories A to F below is not finalized yet. It is still evolving, hopefully toward a complete classification of physically meaningful universality classes.

5.1. Category A: Statistically Stationary and Homogeneous Turbulence

This type of turbulence is the case where, on average, Kolmogorov scale-by-scale equilibrium holds. However, there is fluctuating non-stationarity and non-homogeneity, and resulting non-zero fluctuations of the rate of change of the turbulent kinetic energy at scales smaller than r, two-point interspace turbulence transport rate and the two-point pressure gradient–velocity term. Even though these processes average to zero in statistically stationary homogeneous turbulence, their fluctuations are so intense that they dwarf the non-zero average interscale transfer and turbulent dissipation rates. Even though these fluctuations are not within the scale-by-scale equilibrium, they nevertheless obey certain rules. The interscale transfer rate can be decomposed into a solenoidal part resulting from the integrated vorticity equation and an irrotational part resulting from the integrated Poisson equation for pressure. Only the solenoidal part is involved in the average scale-by-scale Kolmogorov equilibrium. The irrotational part of the interscale transfer rate, the irrotational part of the two-point interspace transport rate and half the two-point pressure gradient–velocity term average to zero but are exactly equal to each other everywhere in space and at any time. The fluctuating rate of change of the turbulent kinetic energy at scales smaller than r is typically balanced by the solenoidal fluctuations of the interscale transfer rate and the two-point interspace transport rate. At intense interscale transfer rate events, the interscale transfer rate and the two-point interspace transport rate tend to have opposite signs and partially cancel each other. Further aspects of a detailed picture of fluctuating scale-by-scale non-equilibrium in statistically stationary homogeneous turbulence can be found in [58].

5.2. Category B: Non-Stationary Statistically Homogeneous Turbulence

This is the case of freely decaying statistically homogeneous and isotropic turbulence. Ref. [59] by mathematical treatment of the Navier–Stokes equations, Ref. [60] by analysis of experimental data from grid-generated turbulence, and [61] by EDQNM simulations showed that average (over space or realizations) scale-by-scale equilibrium is achieved asymptotically, as the Reynolds number tends to infinity only in the vicinity of the Taylor length. They also showed that there are systematic increasing departures from average scale-by-scale equilibrium both as the scale increases toward L because of increasing non-stationarity and as the scale decreases toward η because of increasing viscous diffusion. Given that the Taylor length λ depends on kinematic viscosity and on the kinetic energy of the turbulence, the asympotic balance between average interscale transfer rate and turbulence dissipation rate at the vicinity of λ is not an inertial range balance.

5.3. Category C: Non-Homogeneous Statistically Stationary Turbulence with Two-Point Turbulence Production and without Two-Point Interspace Transport

This is the case of the intermediate layer in fully developed turbulent channel flow (FD TCF), which is known to be in approximate one-point equilibrium between the one-point turbulence production rate and turbulence dissipation rate locally in distance y from the wall. Ref. [62] showed theoretically and confirmed by using direct numerical simulation (DNS) data that scale-by-scale equilibrium between the interscale transfer rate and turbulence dissipation rate (both averaged over a sphere of radius r) occurs only at the vicinity of $r = \lambda$ in the limit of the infinite global Reynolds number δ/δ_v and very large local Reynolds number y/δ_v (where δ_v is the wall unit length, δ is the channel's half width, and $y \ll \delta$). As the scale r grows above λ, there is a systematic departure from scale-by-scale equilibrium because of the increasing two-point turbulence production rate (averaged over a sphere of radius r), which is negligible only at scales close to η in the intermediate layer $\delta_v \ll y \ll \delta$.

The DNS data analysis of [62] has also shown that the two-point turbulence production rate is positive (scales smaller than r gain energy) mainly because one-point turbulence production is positive. However, two-point correlations conditioned on more or less aligned fluctuating velocities act to reduce this positivity. The positivity of two-point turbulence production is enhanced by two-point correlations conditioned on more or less anti-aligned fluctuating velocities, particularly at larger two-point separations r.

Finally, their DNS data analysis revealed that the approach to scale-by-scale equilibrium is at length scales (around λ), where aligned fluctuating velocities (which reduce two-point production) are stretching with their difference maximally aligned with the separation vector between the two points and where anti-aligned fluctuations (which enhance two-point production) are maximally skewed toward fast compression. The word "maximally" is used to mean that as the separation distance r grows above λ and the departure from scale-by-scale equilibrium increases, the alignment between fluctuating velocities and the skewness of anti-aligned fluctuations progressively diminish.

5.4. Category D: Non-Homogeneous Statistically Stationary Turbulence with Negligible Two-Point Turbulence Production

A case of turbulent flow, which falls under this category, is the turbulent flow under the rotating blades in a baffled container (mixer), where the baffles break the rotation in the flow to enhance mixing. In the absence of turbulence production, Refs. [50,51]

make the hypothesis that every term in the non-homogeneous but statistically stationary scale-by-scale (two-point) energy balance is self-similar in the sense that it has the same dependence on two-point separation r at different positions in space if rescaled by local (in space) velocity and length scales. The resulting intermediate asymptotic theory concludes that the interscale transfer rate, the two-point interspace turbulence transport rate and the two-point pressure gradient velocity correlation term are all proportional to the turbulence dissipation rate over a wide range of length scales r larger than λ. Two-dimensional two-component highly space-resolved particle image velocimetry (PIV) confirms two of these three conclusions [52], the one involving the pressure field remaining inaccessible to the measurement technique. The experimental measurements also confirm the absence of two-point turbulence production over the range of scales concerned. Category D is different from categories B and C in one essential way: there is no balance between interscale transfer rate and turbulence dissipation rate at any scale, not even asymptotically at scale λ, just a proportionality between them over an inertial range of scales r. The accompanying proportionality confirmed by the PIV between two-point interspace turbulence transport rate and turbulence dissipation rate over the same range of scales reflects the non-homogeneity and related absence of scale-by-scale equilibrium.

5.5. Category E: Streamwise-Decaying Non-Homogeneous Turbulence with Negligible Two-Point Turbulence Production

The term "streamwise decaying" here reflects the presence of non-zero streamwise mean advection, which, in the absence of sufficient turbulence production, can result in a streamwise decay of the turbulence. This is the case of boundary-free turbulent shear flows such as wakes and jets along their streamwise centerline. Ref. [50] developed a theory along the lines described under category D (see above) and applied it to the centerline of the turbulent wake of two side-by-side square prisms. By varying the distance between the two prisms, the flow can change qualitatively, which gives the possibility to easily and meaningfully test generality of results in the wind tunnel. Ref. [50] considered the variation along the cross-stream direction of second order structure functions and their dependence on two-point separation r and showed both theoretically and experimentally that they collapse as $K(r/L)^{2/3}$ rather than $(\varepsilon r)^{2/3}$ over a cross-stream area that is small enough for two-point turbulence production to be negligible. The difference comes from the fact that ε does not scale as $K^{3/2}/L$ in the cross-stream direction of such flows as demonstrated experimentally by ref. [63]. The $K(r/L)^{2/3}$ rather than $(\varepsilon r)^{2/3}$ scaling reflects non-equilibrium emanating from non-homogeneity via non-homogeneity's presence in the scale-by-scale energy balance in terms of the non-zero two-point interspace turbulence transport rate and/or non-zero two-point pressure gradient velocity correlation.

5.6. Category F: External Intermittency: A Case of Extreme Fluctuating Non-Homogeneity and Non-Stationarity

The turbulent/non-turbulent (TNTI) interface is an extreme case of fluctuating non-homogeneity and non-stationarity. Ref. [64] reported non-equilibrium turbulence dissipation scalings at the vicinity of the TNTI. Refs. [65,66] developed tools to study scale-by-scale energy transfers at and near the TNTI. These three studies used DNS of boundary-free turbulent shear flows. A significant conclusion concerns interscale transfers at the TNTI. In the direction more or less aligned with the local normal to the TNTI, the interscale transfer is down-scale (from large to small scales) and dominated by compressive motions, whereas in the direction more or less aligned with the local tangent to the TNTI, the interscale transfer is up-scale (from small to large scales) and dominated by stretching motions. There is an in-between region in terms of orientations relative to the TNTI where stretching motions dominate on average, but the interscale transfer is nevertheless down-scale because of extreme fluctuating compression events, which can be significantly more likely than extreme fluctuating stretching events. Furthermore, interscale energy transfer rates are typically half the magnitude of two-point interspace energy transport rates, and the two types of transfer/transport are anti-correlated. This anti-correlation is reminiscent of an anti-correlation in the fluctuations of interscale and two-point interspace turbulent energy transfer/transport in statistically stationary and homogeneous turbulence (category A). It must, of course, be noted that the interscale and interspace transfer/transport mechanisms are related by the obvious fact that they are both direct consequences of the turbulence nonlinearity.

6. Conclusions

Jack Herring was a leading contributor to the post-war era of turbulence research, where homogeneity and scale-by-scale equilibrium were central reference concepts. The Taylor–Kolmogorov turbulence dissipation scaling, which results from considerations of scale-by-scale equilibrium, was used in those years to obtain theoretical predictions even for non-homogeneous turbulent flows, for example, mean flow properties of boundary-free turbulent shear flows [45,46].

Following this era, turbulence research over the past ten to fifteen years includes a strand concerned with scale-by-scale non-equilibrium and the different ways that such non-equilibrium may result from different types of non-stationarity and non-homogeneity. A starting attempt at a classification of different non-equilibrium categories is summarized in the previous section. For example, the non-homogeneity of the producing but non-transporting type, as in the intermediate layer of FD TCF, leads to asymptotic scale-by-scale equilibrium only around the Taylor length with increasing deviations caused by two-point turbulence production in the inertial range. This is similar to decaying homogeneous turbulence, where the increasing deviations in the inertial range are caused by non-stationarity.

On the other hand, the non-homogeneity of the transporting but non-producing type, as in a baffled mixer under the rotating blades, leads to inertial range scale-independent interscale transfer rate, two-point interspace transport rate and two-point pressure gradient velocity correlation term, all of which scale with turbulence dissipation.

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