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A Note on the Core of TU-cooperative Games with Multiple Membership Externalities

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Abstract: A generalization of transferable utility cooperative games from the functional forms introduced by von Neumann and Morgenstern (1944, *Theory of Games and Economic Behavior*) and Lucas and Thrall (1963, *Naval Research Logistics Quarterly*, 10, 281–298) is proposed to allow for multiple membership. The definition of the core is adapted analogously and the possibilities for the cross-cutting of contractual arrangements are illustrated and discussed.

1. Introduction

The coalitional game as defined by [1] associates a unique worth with each coalition. Such a characterization is restrictive for many applications as it may be reasonable to allow the worth of one coalition to depend on the formation of other coalitions. Consequently, in [2]'s definition of a cooperative game, the worth of coalitions depends on the partitions of the rest of society, thus allowing different worth to be associated with each possible coalition depending on what coalitions are formed in the rest of society ("externalities"). This representation is still restrictive in the sense that it "presumes that coalitions are mutually exclusive, but in reality, a player might belong to multiple coalitions that interact with one another (e.g., a country might belong to both the United Nations and the European Union)" [3]. (See [4–6] for an international relations perspective on issue linkage through multilateral agreements.)

This note introduces a functional representation of a cooperative game where coalitions can form in multiple spheres of interaction simultaneously such that each coalition in each sphere is associated

with a worth that depends on the overall coalition structure. Inherent to the model, therefore, is a new type of "cross externality": the effect of forming coalitions across spheres. Such a formulation is relevant for many applications because, with multiple membership in the underlying application, a compartmentalized approach to the study of each sphere in isolation may lead to wrong conclusions concerning the stability of coalitional agreements. In a multiple membership setting, different layers may imbalance or balance each other depending on the structure of total spillovers (within and across spheres). Coalitions that seemed stable (or unstable) from the compartmentalized single-sphere viewpoint may turn out to be destabilized (stabilized) by the multi-sphere game. To assess the stability of candidate agreements, we adapt definitions of the core [7,8] of the von Neumann–Morgenstern game [1] as done for the Lucas–Thrall game [2] in [9], using an analogous "conjecture/ expectation formation approach" [10] to recover the Bondareva–Shapley theorem [11,12]. To achieve this, the set of feasible deviations is restricted to a specific class. Further inspection of the resulting non-emptiness constraints reveals that different types of cross externalities create further opportunities for the cross-cutting of contractual arrangements. Our analysis builds on the work of [13] who identify conditions for when non-emptiness of the core is facilitated through combining additively separable von Neumann–Morgenstern games (the single-sphere and no externalities case). (Not our lead example but some of our later examples are borrowed and generalized from theirs. See also [16] on the additivity of the core.) Our work also complements [14]'s generalization of [15] value in an environment like ours.

The rest of this note is structured as follows. Next, the model is motivated by means of a multimarket competition game. In Sections 3 and 4, we introduce the general game, define its core, and illustrate the core characteristics at hand of examples and observations. We conclude with some remarks.

2. A Worked Example

To motivate our model, we consider a multimarket Cournot economy with mergers and spillovers. (See, for example, [10,17,18] for single-market Cournot competition games in this spirit.)

Example 1: *A population of firms, $N = \{f_1, ..., f_n\}$, competes in a multimarket industry, $K = \{1, ..., m\}$, by setting production quantities. Each firm f is described by a vector of specializations, $s_f = \{s_f^1, ..., s_f^m\}$, where each s_f^k is a real number representing firm f's constant marginal costs in market k when no merger occurs.*

In any market k, coalitions of firms $S \subseteq N$ may merge and form a new firm. The resulting industry configuration, $\mathcal{M}$, describes the partitions in each market, $\{\rho_1, ..., \rho_m\}$. Given $\mathcal{M}$, any firm $S \in \rho^k$ produces quantity q_S^k in market $k \in K$ at cost

$$C_S^k(q_S^k; \mathcal{M}) = c_S^k(\mathcal{M}) \times q_S^k + x_S^k(\mathcal{M}).$$

Fixed costs of merger. $x_S^k(\mathcal{M})$, *the fixed cost of merging S in market k, is a real-valued function that depends on $\mathcal{M}$ in the following way:*

$$x_S^k(\mathcal{M}) = \begin{cases} 0 & \text{if } |S| = 1 \\ \kappa & \text{if } |S| > 1 \text{ and there exists } k' \neq k \text{: } S \in \rho_{k'} \\ \lambda & \text{if } |S| > 1 \text{ and there does not exist } k' \neq k \text{: } S \in \rho_{k'} \end{cases}$$

Marginal costs of production. *Given any merger $S \subseteq N$ in market k, the firms in S select the lowest marginal cost firm to be the only active firm amongst them in market k. Hence, the marginal cost of production of S, c_S^k, as a result of the merger is given by*

$$c_S^k = \min\{s_f^k\}_{f \in S}.$$

Given merger $S \subseteq N$ in market k, the marginal cost of production of any coalition C, c_C^k, in any other market $k' \neq k$ is affected in the following way. For any $C \in \rho_{k'}$, we write $\underline{c}_C^{k'}$ for $\min\{s_f^{k'}\}_{f \in C}$, i.e., for the marginal cost of the lowest marginal cost firm amongst C in market k'. For all C such that $C \cap S = \emptyset$, $c_C^{k'} = \underline{c}_C^{k'}$. For all C such that $C \cap S \neq \emptyset$, given some $\alpha \in (0, 1)$,

$$c_C^{k'} = \min\{\underline{c}_C^{k'};\ \alpha \times c_S^{k'} + (1 - \alpha) \times \underline{c}_C^{k'}\}.$$

The motivation for this marginal cost effect across markets is that firms connected by merger in one market may learn something about each others' production technologies and thus also improve (to some extent) their respective production technologies even in markets where they remain unmerged.

Demands. *The demand of any product is the same in all markets (normalized to be equal-sized). Products are neither substitutes nor complements, meaning that all markets can be described by identical and independent linear demands. (These markets could be countries for example.) For any market k, therefore,*

$$p^k = 1 - Q^k \text{ where } Q^k \equiv \sum_{f \in N} q_f^k.$$

2.1. Oligopoly Externalities

A merger in a multimarket Cournot situation as introduced here has three different externality effects on the other firms in the same market and across markets. First, due to market consolidation, if merger occurs, the resulting quantity and price competition will change in that market, since the merged firms will be represented by the firm with the lowest marginal cost amongst them. Second, due to technology/knowledge spillovers across markets, the resulting quantity and price competition will also change in the markets where the merger did not occur because of the potential reduction in marginal costs (by how much is described by parameter α). Third, due to sharing of fixed costs merger, if the same merger were to occur in more than one market, the fixed costs of merger per market would decline.

Due to the independence of the demand markets, the firms' optimization problems, given any industry configuration, can be solved for each market separately. The adjustments of equilibrium quantities and prices following mergers in any given market, however, have an effect in not only that same market because both the technology spillovers and the fixed cost effects may additionally influence the optimization problems in the other markets. A traditional representation of a cooperative game could not make these effects explicit. We shall illustrate these effects in more detail with a numerical illustration.

2.2. Two-firm, Two-market Numerical Illustration

Take a symmetric two-firm, two-market case with $s_{f_1}^1 = 4/9$, $s_{f_1}^2 = 5/9$, $s_{f_2}^1 = 5/9$, $s_{f_2}^2 = 4/9$ (firm f_1 is specialized in market 1 and firm f_2 is specialized in market 2). Merger costs are

$\lambda = 2/108 > \kappa = 1/81$. Choosing equilibrium outputs given the decision to merge in none, one, or both of the markets yields four cases: the no merger case, two one merger cases, and the full merger case. The equilibrium profits of these four cases are obtained by solving for the firms' profit-maximization problems. Table 1 summarizes the competition. (Writing $(1), (2)$ means "no merger" in the underlying market and writing $(1, 2)$ means "merger".)

Table 1. *Numerical illustration.*

	Industry configuration		Profits (scaled $\times 81$)	
	market 1	**market 2**	**market 1**	**market 2**
mergers:				
none	(1),(2)	(1),(2)	4, 1	1, 4
market 1	(1,2)	(1),(2)	4.75	$(1 + \alpha \times 1.\overline{7}), (4 - \alpha \times 1.\overline{2})$
market 2	(1),(2)	(1,2)	$(4 - \alpha \times 1.\overline{2}), (1 + \alpha \times 1.\overline{7})$	4.75
full merger	(1,2)	(1,2)	5.25	5.25

The direct effect of merger in market 1 is negative: profits fall from $1 + 4 = 5$ to 4.75 if market 2 is not merged and from $2.\overline{7} + 2.\overline{7} = 5.\overline{5}$ to 5.25 if market 2 is merged. If market 2 is merged, the overall cross effect on market 2 is positive: total profits in market 2 rise from 4.75 to 5.25. If market 2 is not merged, the profits in market 2 change from $1 + 4 = 5$ to $5 + \alpha \times 0.\overline{5}$. If market 2 is not merged, the individual cross effect is negative on the strong firm in market 2 (profits fall from 4 to $4 - \alpha \times 1.\overline{2}$), and positive on the weak firm (profits increase from 1 to $1 + \alpha \times 1.\overline{7}$). The net total of the merger effects is therefore always positive if $\alpha \times 0.\overline{5} > 0.25$, *i.e.*, when $\alpha > 0.45$.

Since, ceteris paribus, mergers always decrease the worth of the merging market due to the high direct costs of merger, a partial view of one market suggests that merger is not in the firms' interests. When both markets are analyzed simultaneously, however, the cross-market effects of mergers are internalized. Since the cross effects are net positive if $\alpha > 0.45$, these effects would already render a single merger worthwhile overall.

When no merger takes place, each firm's profits from both markets are $4 + 1 = 5$ and the total profits are 10. When one merger takes place, the firms can agree on sharing the total payoffs of $(5 + \alpha \times 0.\overline{5}) + 4.75 = 9.75 + \alpha \times 0.\overline{5}$. Under full merger, contracts can share the total profits of $5.25 + 5.25 = 10.5$. Therefore, no contracts can be written that Pareto-improve on contracts that result in full merger and share the total profits efficiently, paying each player at least 5 (which are the profits that each firm can guarantee itself from no merger). Whether a single merger already has a net-positive effect depends on whether $\alpha > 0.45$ or not.

3. The Model

This section generalizes the example to a representation of a cooperative game. Let $N = \{f_1, f_2, ..., f_n\}$ be the fixed population of agents. Write ρ for a partition of N and $\rho(S)$ for the partition of some $S \subset N$. Let $\mathbf{P}(N)$ be the set of partitions of N and $\mathbf{P}(S)$ the set of partitions of $S \subset N$. Let $K = \{1, ..., m\}$ be the set of cooperative layers, that is, different spheres over which

cooperation amongst $S \subseteq N$ may ensue. Write $\mathcal{M}$ for a society consisting of a partition of each layer, $\mathcal{M} = \{\rho_1, ..., \rho_m\}$, and $\mathcal{M}_S = \{\rho_1(S), ..., \rho_m(S)\}$ for a subsociety consisting of a partition of each layer of some $S \subset N$ (in which case $\mathcal{M}_S$ and $\mathcal{M}_{N \setminus S}$ are "separable" subsocieties; *i.e.*, there is no coalition that includes members from both subsocieties).

Now, $G(v, K, N)$ is a multiple membership game (MMG), defined by N, K and v. v is the characteristic multiple membership function that assigns, for every layer $k \in K$, a worth in terms of transferable utility of v_k to each $C \in \rho_k$ given $\mathcal{M}$: for any $k \in K$, $v_k(\cdot \,; \mathcal{M}) : \rho_k \to \mathbf{R}$ for all $\rho_k \in \mathbf{P}(N)$. Naturally, an MMG is a partition function game (PFG) as in [2] if K consists of only one layer (when no multiple membership exists). With only one layer, the MMG/PFG further reduces to a characteristic function game (CFG) as in [1] if, for any $C \subseteq N$, $v(C; \rho)$ is constant for all $\rho \in \mathbf{P}(N)$ with $C \in \rho$.

3.1. Externalities

When multiple membership exists, externalities come in various kinds. In these notes, an externality is said to be present if one instance of its effect is present so that a game may exhibit different kinds of externalities over different parts of the game. This allows to model interesting situations like the above Cournot model: merger in one market has both positive and negative effects on the other firms and on the other markets.

The externalities will be defined using the notion of embedded coalitions. Given partition ρ of N, C is an embedded coalition if $C \in \rho$. Partition ρ embeds ρ' if, for all $C' \in \rho'$, there is some $C \in \rho$ such that $C' \subseteq C$. One externality is the "partition" externality, which is the externality known from PFGs: the intra-layer externality of an $n(\geq 3)$-player Cournot game, for example, where one firm's payoff varies with the remaining firms' decisions on whether to merge or not, is such an externality.

Partition externality. $G(v, k, N)$ exhibits a positive (or negative) partition externality if there exist $\mathcal{M}$, $\mathcal{M}'$ such that $\mathcal{M} \setminus \rho_k = \mathcal{M}' \setminus \rho_k'$, ρ_k embeds ρ_k' with $C \in \rho_k$, $C \in \rho_k'$, and

$$v_k(C; \mathcal{M}) > (\text{or} <) \, v_k(C; \mathcal{M}').$$

The other "cross" externality stems from the effects of the formation of coalitions in one layer on the payoffs of some coalition in another. This inter-layer effect is new and peculiar to multiple membership and cannot be expressed through existing cooperative game representations. In the multimarket Cournot example, the cross externality was the effect of merger in one product market on the firms' profits in the other.

Cross externality. $G(v, k, N)$ exhibits a positive (or negative) cross externality if there exist $\mathcal{M}$, $\mathcal{M}'$ such that $\mathcal{M} \setminus \rho_k = \mathcal{M}' \setminus \rho_k'$ with $C \in \rho_{k'}$, $C \in \rho_{k'}'$ for some $k' \neq k$, ρ_k embeds ρ_k', and

$$v_{k'}(C; \mathcal{M}) > (\text{or} <) \, v_{k'}(C; \mathcal{M}').$$

A subclass of cross externalities are "partition-cross" ("partition externalities across layers"). They have elements of cross and of partition externalities: coalition formation of one set of players $S_1 \subseteq N$ in one layer affects the worth of coalitions of another $S_2 \subseteq N$ in another layer with $S_1 \cap S_2 = \emptyset$. This occurs when, for example, a merger of firms one and two in one market affects the profits of firm three in another.

Partition-cross externality. $G(v, k, N)$ exhibits a positive (or negative) partition-cross externality if there exist $\mathcal{M}$, $\mathcal{M}'$ such that $\mathcal{M} \setminus \rho_k = \mathcal{M}' \setminus \rho'_k$ with $C \in \rho_{k'}$, $C \in \rho'_{k'}$ for some $k' \neq k$, ρ_k embeds ρ'_k while being identical w.r.t. the coalitions that all members of C join (*i.e.*, for all f such that $f \in C \in \rho_{k'}$, $(f \in S \in \rho_k) \Leftrightarrow (f \in S \in \rho'_k)$ with the same S in both), and

$$v_{k'}(C; \mathcal{M}) > (\text{or} <) \, v_{k'}(C; \mathcal{M}').$$

A partition-cross externality is a partition externality where partitions ρ_{k_i} and ρ'_{k_i} are identical w.r.t. the coalitions that all members of C join: for all f such that $f \in C \in \rho_{k_j}$, $(f \in S \in \rho_{k_i}) \Leftrightarrow (f \in S \in \rho'_{k_i})$ with the same S in both.

3.2. Feasible Deviations

In the absence of externalities and multiple membership (*i.e.*, in characteristic function games, CFGs), a deviation by some $S \subset N$ when forming a coalition has a one-to-one association with a unique worth of S [1]. In the presence of externalities, however, further *expectation conjectures* (assumptions about how the rest of society, $N \setminus S$, reacts to a coalitional deviation by S) are needed [1,19]. For partition function games (PFGs), that is, in the presence of externalities within a single sphere (no multiple membership), [20–23] propose definitions of the core dependent on different conjectures to evaluate the profitability of coalitional deviations. [10] provides an excellent discussion of these, also analyzing their axiomatic foundations.([9] provides additional results on the externality structure relevant for the corresponding non-emptiness results for several of these cores.) Suppose the partition was ρ before $S \subset N$ deviated and reorganized itself to form $\rho(S)$, then these are the existing conjecture rules that have been proposed in PFG environments (see [10] for a detailed classification and an axiomatic analysis):

1. *Max rule [10]:* $(N \setminus S)$, taking $\rho(S)$ as given, organizes itself to $\rho(N \setminus S)$ in order to maximize $(N \setminus S)$'s total worth
2. *Pessimistic [19,20]:* $(N \setminus S)$ organizes and forms $\rho(N \setminus S)$ in order to minimize S's total worth
3. *Optimistic [23]:* $(N \setminus S)$ organizes and forms $\rho(N \setminus S)$ in order to maximize S's total worth
4. *Singleton [21,22]:* $(N \setminus S)$ breaks down into singletons
5. *Collective [10]:* $(N \setminus S)$ forms one joint coalition ([10] call this rule $\overline{N}$-exogenous)
6. *Disintegrative [1,20]:* all $C \in \rho$ such that $C \cap S = \emptyset$ remain organized in the same way, all other coalitions C' from which members in S deviated break up into singletons
7. *Projective [20]:* all $C \in \rho$ such that $C \cap S = \emptyset$ remain organized in the same way, all other coalitions C' from which members in S deviated form coalitions amongst the remaining $(C' \setminus S)$

Note that conjecture rules 1–3 depend on $\rho(S)$ and on the underlying PFG, but not on the original partition ρ. Rules 4–5 depend only on S. Rules 6–7 depend on S and on the original partition ρ.

With multiple membership, in addition to the need of specifying a conjecture, we must specify what kinds of deviations are deemed feasible. The feasibility of deviations needs to be interpreted here because, for example, starting with the grand coalition in some layer, each $S \subset N$ may deviate in many ways: in some or all of the layers, forming different coalitions in each layer or the same coalition in all layers. If cooperation is compartmentalized without cross externalities in between the layers,

players may deviate in one layer but continue to form the grand coalition in another layer. When cross externalities are present, however, the worth of coalitions vary with the coalition constellations across layers and deviators need to endogenize the cross external effects of their deviations. If $S \subset N$ deviates and forms S in layer 1, for example, it cannot expect to form N in another layer because S's members need to cooperate with $(N \setminus S)$ to form this constellation. Therefore, this note only considers the following deviations:

Feasible deviations. Any $S \subseteq N$ can form any subsociety $\mathcal{M}_S \in \mathbf{P}(S)^m$ (a partition of S in every layer). $\mathcal{M}_S$ and $\mathcal{M}_{N \setminus S}$ must be separable.

From the feasible set of subsocieties available to S, it aims to form subsociety $\widehat{\mathcal{M}_S} \in \mathbf{P}(S)^m$ that maximizes its total payoffs. For that, each $S \subset N$ needs to conjecture how the rest of the population responds to its deviation. The reason for restricting deviations in this way is to guarantee that society $\mathcal{M}$ after deviation by some $S \subset N$ occurs is separable into subsocieties $\mathcal{M}_S$ and $\mathcal{M}_{N \setminus S}$. If this is the case, then the above list of conjectures can be adapted directly.

Suppose Z represents any of the above conjectures so that Z, for every $\mathcal{M}_S$ deviating from $\mathcal{M}$, specifies a resulting subsociety $\mathcal{Z}((N \setminus S); \mathcal{M}_S) \in \mathbf{P}(N \setminus S)^m$ of $(N \setminus S)$ (a partition of $(N \setminus S)$ in every layer, but not necessarily the same one in all layers). Write $\widehat{\mathcal{M}_S}(\mathcal{N})$ for the resulting overall society $\{\widehat{\mathcal{M}_S}, \mathcal{Z}((N \setminus S); \widehat{\mathcal{M}_S})\}$. Hence, S forms the optimal subsociety $\widehat{\mathcal{M}_S}$ such that, given conjecture Z,

$$\sum_{k \in K} \sum_{C \in \rho_k(S)} v_k(C; \widehat{\mathcal{M}_S}(\mathcal{N})) = \max_{\mathcal{M}_S \in \mathbf{P}(S)^m} \sum_{k \in K} \sum_{C \in \rho_k(S)} v_k(C; \{\mathcal{M}_S, \mathcal{Z}((N \setminus S); \mathcal{M}_S)\}).$$

The finiteness of possible coalition structures guarantees the existence of such a (not necessarily unique) subsociety for any $S \subseteq N$. We will now define a function summarizing their worth.

Conjectured worth function. The conjectured worth function (CWF), z, summarizes the conjectured worth for all coalitions: given Z, for all $C \subseteq N$, $z(\cdot): C \to \mathbf{R}$. For any $S \subseteq N$, $z(S)$ is the largest feasible sum of payoffs for S under conjecture Z:

$$z(S) = \sum_{k \in K} \sum_{C \in \widehat{\rho_k(S)}} v_k(C; \widehat{\mathcal{M}_S}(\mathcal{N}))$$

Note that z filters the information in the MMG to obtain a CFG view of deviating demands.

3.3. Superadditivity

When externalities exist, a detailed analysis of the effects of coalition formation may be needed to evaluate the global benefits of cooperation and a superadditivity assumption may be difficult to uphold. When one agent is able to take free ride on the coalition formed by others, for example, the grand coalition may no longer be the efficient coalition structure and it may indeed be insightful to work with a given coalition structure to analyze the effects of free ride.

In the presence of multiple membership and externalities, coalition formation may be mutually beneficial in some layer but not necessarily globally as negative cross externalities may exist. Suitably defined, MMGs may conversely be globally superadditive if the overall effect of coalition formation,

which takes into account all external and direct effects, is positive for those that come together to cooperate even if coalition formation itself is not mutually beneficial in the separate layers.

The numerical illustration of the multimarket Cournot game for the cases when $\alpha > 0.45$, for instance, is superadditive because the total profits of the firms rise with every further merger: the no merger case has total payoffs of 10, compared with the $10 + (\alpha - 0.45) \times 0.\overline{5}$ of both one merger cases, and compared with the 10.5 of full merger. The below definition of MMG superadditivity embeds definitions of superadditivity for CFGs and PFGs and implies efficiency of forming the grand coalition in all layers.

Superadditivity: An MMG is superadditive if, for all $\mathcal{M}, \mathcal{M}'$ such that $\mathcal{M} \setminus \rho_k = \mathcal{M}' \setminus \rho_k'$, and ρ_k embeds ρ_k' in layer k,
$$\sum_{k \in K} \sum_{C \in \rho_k} v_k(C; \mathcal{M}) \geq \sum_{k \in K} \sum_{C \in \rho_k'} v_k(C; \mathcal{M}').$$

Superadditivity implies the efficiency of the "grand coalition" by which we mean society $\{N\}$ (the grand coalition) forms in all layers.

When the game consists of a single layer without externalities (described by a CFG), the above definition implies the simple pairwise superadditivity that $v(C \cup C') \geq v(C) + v(C')$ is to be satisfied for all $(C, C') \subset N : C \cap C' = \emptyset$. (Note that the implied sense of superadditivity when there is only one layer has also been defined as *full cohesiveness* ([9], section 2.2 "Convexity") in the contexts of PFGs, as opposed to a pairwise view of superadditivity ([9], section 2.1 "Superadditivity"). [9]'s pairwise view of superadditivity does not imply the efficiency of the grand coalition.) Note that the optimization problem underlying z, which is a CFG, entails that z is superadditive by definition, even if the MMG is not superadditive: for any $S, S' \subseteq N$ with $S \cap S' = \emptyset$, $z(S) + z(S') \leq z(S \cup S')$.

4. Coalitional Stability and the Core

We now turn to the stability of an outcome. By outcome we mean $(\mathcal{M}, x)$; a coalition structure together with an allocation of the common gains. To assess its stability, we will use the conjectured worth function. For allocation x, we write $x = \{x_{f_1}, ..., x_{f_n}\}$ such that each allocated player payoff $x_f = \sum_{k \in K} x_f^k$ summarizes the payoffs to each $f \in N$ obtained in all layers. Consequently, for some $S \subseteq N$, $x(S)$ is a vector of all-layer payoffs for the players in S. Naturally, an allocation must be feasible: given any $\mathcal{M}$, $\sum_{f \in N} x_f \leq \sum_{k \in K} \sum_{C \in \rho_k} v(C; \mathcal{M})$.

Recall our numerical illustration of the multimarket Cournot game. Independent of $\alpha = (0, 1)$, one unique conjectured worth function is derived, *i.e.*, z is such that $z(f_1) = z(f_2) = 1 + 4 = 5$ and $z(f_1, f_2) = \max\{(5.25 + 5.25); (9.75 + \alpha \times 0.\overline{5})\} = 10.5$. Note that no conjecture is needed for this assessment. It is easy to verify in this particular example that $G(v, K, N)$ has a nonempty core: for an example of a core outcome, consider full merger with contract $x = (5.25, 5.25)$, paying both firms 5.25. This outcome is in the core because no firm can do better by deviating. In fact, any split of full merger paying each firm at least his individually rational payoff of 5 (what he gets from no merger) and the other the residual to 10.5 is a stable core allocation.

4.1. Core Stability

Assume $G(v, K, N)$ is superadditive such that the grand coalition is efficient. Whether there exists a core-stable allocation supported by the grand coalition depends on v and on the conjecture. We now provide definitions for any given conjecture. The Z-core (based on conjecture Z) can be defined using the conjectured worth function.

Z-core: Given Z, the Z-core of forming the efficient society of $G(v, K, N)$ with total payoff allocation x is

$$\zeta(G(v, K, N); Z) = \{x \in \mathbf{R}^n; \sum_{f \in N} x_f \leq z(N) \text{ and } \sum_{f \in S} x_f \geq z(S) \ \forall \ (S \subseteq N)\}.$$

Theorem. *The Z-core of $G(v, K, N)$ is nonempty if, and only if, its conjectured worth function z is balanced.*

The theorem is a (straightforward) recovery of the Bondareva–Shapley theorem via the conjectured worth function in our setup (see [11] and [12] for independent proofs). What is interesting is that several characteristics can be identified to describe the core structure, which turns out to be very complex.

Characteristic 1: *If the cores of a superadditive MMG layer-by-layer separately are nonempty, the Z-core of the whole MMGs is also nonempty.*

While z is always additive over coalitions and layers, v does not need to be additive when externalities are present. In every layer, superadditivity implies that it is beneficial for members of any $S \subseteq N$ to form the largest possible coalition $\{S\}$. Hence, whenever x is in a Z-core, $\sum_{f \in N} x_f = z(N)$. Now, z_k describes the game described by the conjectured worth function of layer k, *i.e.*, the conjectured CFG view of layer k. Given any z_k, a core stable allocation of forming the grand coalition in that layer exists if, and only if, every z_k is balanced. Since the sum of balanced games is balanced, the Z-core of $G(v, k, N)$ is, therefore forcedly, nonempty when all z_ks are balanced.

Characteristic 2: *In the presence of cross externalities but without partition and partition-cross externalities, the core is unambiguously defined (independent of conjecture).*

In the absence of partition and partition-cross externalities, in a society $\mathcal{M}$ that is separable into $\mathcal{M}_S$ and $\mathcal{M}_{N \setminus S}$, the worth of any $C \subseteq S$ is independent of $\mathcal{M}_{N \setminus S}$ in all layers: $v_k(C; \mathcal{M}) = v_k(C; \mathcal{M}')$ for all coalitions, layers and societies provided $\mathcal{M}_S = \mathcal{M}'_S$, $(C \in \rho_k \in \mathcal{M})$ and $(C \in \rho'_k \in \mathcal{M}')$. Therefore, one unique game described by a characteristic worth function is derived, which implies one unambiguous definition of the core. This unambiguity is independent of the existence of cross externalities that are not partition-cross because deviators endogenize all other cross external variations that may still exist and affect them. The need to conjecture is therefore inherent to the presence of PFG-type (partition and partition-cross) externalities. The core of example 1, for instance, is unambiguously defined.

Characteristic 3: *In the presence of positive cross externalities, the core of the MMG may be nonempty even if coalition formation in any of the layers is, ceteris paribus, never beneficial.*

Example 1 as described by Table 1 illustrates this.

Characteristic 4: *In the presence of negative cross externalities, the core of forming the grand coalition in any layer of the MMG may be empty even if coalition formation in all layers is, ceteris paribus, always beneficial.*

Example 2: *Let $n = k = 2$ and v be described by Table 2.*

Holding the coalition structure of one layer fixed, any coalition formation in the other layer is beneficial. However, due to the negative cross externality of coalition formation in one layer on the other, the total worth of all coalitions is reduced as coalitions form. The core of forming the grand coalition in one or both of the layers of example 2 is empty: $z(1) + z(2) = z(N) = (v_1(1) + v_2(1)) + (v_1(2) + v_2(2)) = 4 \times 1 = 4 > 3 = 0 + 0 + 3 = (v_1(1) + v_1(2)) + v_2(N) > 2 = 1 + 1 = v_1(N) + v_2(N)$.

Table 2. *Example 2.*

Society		Coalition worth	
layer 1	layer 2	layer 1	layer 2
(1),(2)	(1),(2)	1, 1	1, 1
(1,2)	(1),(2)	3	0, 0
(1),(2)	(1,2)	0, 0	3
(1,2)	(1,2)	1	1

Characteristic 5: *Multiple membership may facilitate cooperation not because of cross external effects but because the layers "balance each other": Even in the complete absence of externalities when all layers have empty cores, the core of an MMG may be nonempty.(See [13] "Examples 1 and 2" for a 4- and related 5-player examples.)*

Example 3: *Let $n = 5$, $k = 2$ and let there be no externalities so that the MMG is described by two 5-player CFGs, v_1 and v_2. Let $v_1(N) = 1$, $v_1(C) = 4/5 + \varepsilon$ (where ε is small) if $|C| = 4$ and $v_1(C) = 0$ otherwise. Let $v_2(N) = 1$, $v_2(C) = 3/5 + \varepsilon$ if $|C| = 3, 4$ and $v_2(C) = 0$ otherwise.*

v_1 is unbalanced: for the balanced collection of the 5 coalitions of size 4, $\zeta_{|4|} = \{(1, 2, 3, 4), ..., (2, 3, 4, 5)\}$, with balancing weights $\lambda_{|4|} = (1/4, ..., 1/4)$, $5 \times 1/4 \times v_1(i, j, k, l) = 5 \times 1/4 \times (4/5 + \varepsilon) = 1 + 5/4 \times \varepsilon > 1 = v_1(N)$. v_2 is unbalanced: for the balanced collection of the 10 coalitions of size 3, $\zeta'_{|3|} = \{(1, 2, 3), ..., (3, 4, 5)\}$, with balancing weights $\lambda'_{|3|} = (1/6, ..., 1/6)$, $10 \times 1/6 \times v_2(i, j, k) = 10 \times 1/6 \times (3/5 + \varepsilon) = 1 + 5/3 \times \varepsilon > 1 = v_2(N)$. However, it is easy to verify that $x = (2/5, 2/5, 2/5, 2/5, 2/5)$ is a core allocation of v: z associates $z(N) = 2$, $z(C) = 7/5 + 2 \times \varepsilon$ if $|C| = 4$, $z(C) = 3/5 + \varepsilon$ if $|C| = 3$ and $z(C) = 0$ otherwise.

Characteristic 6: *The presence of positive (or negative) cross and/or partition externalities may lead to inefficient herding.*

Example 4: *Let $n = 3$, $k = 2$ and $v_k(N; \{\{N\}, \{N\}\}) = 1$ for all k, $v_k(1; \{\rho_1, \rho_2\}) = 2$ $\forall i$ if $\rho_1 = \rho_2 = \{(1), (2, 3)\}$ and $v_k(C; M) = 0$ otherwise.*

The *Pessimistic*-core of forming the inefficient grand coalition in both layers is nonempty because player 1 expects to receive 0 from being the singleton in both layers, e.g., $x = (2/3, 2/3, 2/3)$ is such a *Pessimistic*-core allocation. Inefficient herding results from the positive externality: the formation of the coalition of (2,3) in both layers creates worth for player 1, but player 1 is too pessimistic to agree to stay separate. The same effect may be due to negative externalities as a simple variation of v illustrates: consider, for example, v' with $v'_k(N; \{\{N\}, \{N\}\}) = 1$ for all k, $v'_k(1; \{\rho_1, \rho_2\}) = 2 \; \forall k$ if $\rho_1 = \rho_2 = \{(1), (2), (3)\}$ and $v'_k(C; M) = 0$ otherwise.

5. Concluding Remarks

This paper sets out to define the core of coalitional games with multiple membership externalities. The point of departure is the representation in partition function form as introduced by [2]. Inherent to our multiple membership game are two types of externalities; those from within a given layer of cooperation where a coalitional decision of one set of agents has payoff consequences for another set of agents, and those from across different layers of cooperation where coalitional decisions in one sphere of cooperation influence payoffs in another sphere. Recent contributions explore the consequences for core existence of the first externality type [9] and of the second [13]. Work that is complementary to ours concerns extensions of the Shapley value to multiple membership externality environments [14]. Our work illustrates how the two externality types may interact with coalitional incentives to deviate. Moreover, our model highlights one crucial issue with defining the core in the presence of multiple membership externalities, namely that of feasibility of deviations. In this note, we take a somewhat extreme stance and allow deviations by some subsociety only if they do not expect to form coalitions in any of the layers with any of the players outside of their subsociety. This assumption drives the analysis in this note, and we aim to relax this assumption in future work, likely in conjunction with an axiomatic approach.

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Conflicts of Interest

The author declares no conflict of interest.

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