Games **2014**, *5*, 97-115; doi:10.3390/g5020097

OPEN ACCESS
games
ISSN 2073-4336
www.mdpi.com/journal/games

Article

Characterizing the Incentive Compatible and Pareto Optimal Efficiency Space for Two Players, k Items, Public Budget and Quasilinear Utilities

Anat Lerner [1] and Rica Gonen [2,*]

[1] Department of Mathematics and Computer Science, The Open University of Israel, 1 University Road, Raanana 4353701, Israel; E-Mail:anat@cs.openu.ac.il

[2] Department of Management and Economics, The Open University of Israel, 1 University Road, Raanana 4353701, Israel

* Author to whom correspondence should be addressed; E-Mail: gonenr@openu.ac.il; Tel.: +972-54-7278563; Fax: +972-9-7780668.

Received: 9 December 2013; in revised form: 1 April 2014 / Accepted: 17 April 2014 / Published: 30 April 2014

Abstract: We characterize the efficiency space of deterministic, dominant-strategy incentive compatible, individually rational and Pareto-optimal combinatorial auctions in a model with two players and k nonidentical items. We examine a model with multidimensional types, private values and quasilinear preferences for the players with one relaxation: one of the players is subject to a publicly known budget constraint. We show that if it is publicly known that the valuation for the largest bundle is less than the budget for at least one of the players, then Vickrey-Clarke-Groves (VCG) uniquely fulfills the basic properties of being deterministic, dominant-strategy incentive compatible, individually rational and Pareto optimal. Our characterization of the efficient space for deterministic budget constrained combinatorial auctions is similar in spirit to that of Maskin 2000 for Bayesian single-item constrained efficiency auctions and comparable with Ausubel and Milgrom 2002 for non-constrained combinatorial auctions.

Keywords: budget constraints; Pareto efficiency; incentive compatibility

---

1. Introduction

We characterize the efficiency space of deterministic, dominant-strategy incentive compatible, individually rational and Pareto-optimal combinatorial auctions in a model with two players and k nonidentical items (2^k outcomes). Our model has multidimensional types [1], private values, nonnegative prices and quasilinear preferences for the players with one relaxation: one of the players is subject to a publicly-known budget constraint. This setting is somewhat more complex than that of common auction literature, as it adds budgets and heterogeneity, which more accurately describe mechanisms used in practice.

The investigated space better characterizes many real world problems, such as commonly studied bandwidth (combinatorial) auctions. Consider the German and British 3G radio spectrum auctions in early 2000, where telecom companies bid so high as to have jeopardized their financial viability and consequently considerably slowed down capital investment in 3G equipment. Another contemporary example arises from globalized supply chains. Globalization has substantially increased competition among suppliers. As such, there are many suppliers who are trying to win business, while incapable of delivering the contracted quantity/quality of procured goods.

The phenomena in the German and British 3G radio spectrum auctions, as well as the present day proliferation of suppliers highlights the potential gap between willingness to pay and ability to pay and the potential of better understanding how budget constraints affect auction design. Further consider that most goods are not sold in uniform bundles or used as single goods. Though blocks of radio bandwidth are apparently uniform they are not identical, as well can be said for goods in supply chain auctions, which are bundled to fulfill diverse bills of materials. The addition of the seemingly minor dimension of heterogeneity profoundly affects auction design complexity.

Our result characterizes the space of efficient outcomes and prices in this context. For instance, the characterization answers whether and under which conditions it is possible for a small telecommunications company to compete meaningfully with a better financed company in a bandwidth auction.

The authors of [1] showed that in a Bayesian setting where one indivisible item is for sale, there exists a threshold value, such that if at least one of the players has a valuation that is less than the threshold, then the solution is efficient, Bayesian incentive compatible and individually rational in expectation. Recently, it was shown by [2] that there exists a unique family of dictatorial solutions in deterministic combinatorial auctions that are dominant-strategy incentive compatible, individually rational and Pareto-optimal and has publicly-known budget-constrained players. Moreover, it was shown [3] that if dictatorial mechanisms are ruled out by a natural anonymity property, then an impossibility of design is revealed and there is no deterministic combinatorial auction that is dominant-strategy incentive compatible, individually rational and Pareto optimal where players have publicly-known budget constraints and the all-item bundle is nonarbitrarily allocated (a property termed nonarbitrary hoarding), in a model with multiple nonidentical items and nonnegative prices. Therefore, some additional public knowledge needs to be assumed to allow one to characterize the space of possible mechanisms.

---

[1] Multidimensional types, meaning that a player may have a separate arbitrary value for each of the 2^k possible outcomes.

More specifically, we prove that the combinatorial efficiency space is dependent upon the players' preference value for the k-item bundle. We show that if it is publicly known that at least one of the players value the k-item bundle less than the constrained player's budget and the allocation of the k-item bundle is nonarbitrary, then Vickrey-Clarke-Groves (VCG) [4–6] is the unique mechanism that is dominant-strategy incentive compatible, individually rational and Pareto optimal.

The authors of [7] observed a dependency between the combinatorial efficiency space and the players' preferences in the non-constrained combinatorial auctions model. They [7] investigated the effect of the submodularity of players' coalitional preferences, i.e., nondecreasing marginal coalitional social welfare, on coalition deviation-resilient mechanisms, which in a two player model is equivalent to dominant-strategy incentive compatible mechanisms. When no budgets exist and two players are assumed, [7] proved that when players' coalitional preferences are submodular, then the VCG mechanism is among the efficient [2] and dominant-strategy incentive compatible mechanisms. Moreover, it is the only Pareto-optimal mechanism among the efficient and dominant-strategy incentive compatible mechanisms.

It is well known that in quasilinear environments with a complete preference domain over at least three outcomes and non-constrained players, only VCG mechanisms satisfy the dominant-strategy incentive compatible property [8] [3]. Nevertheless, when preferences are subject to free disposal and no externalities are assumed, as is common in combinatorial auctions, then the possibility space of dominant-strategy incentive compatible combinatorial auctions in the multidimensional type model is not yet defined. Several papers investigate computationally feasible dominant-strategy incentive compatible (but inefficient) auctions with one-dimensional private-values [11,12], as well as with multidimensional private-value settings with some additional restrictions on the preference space [13,14].

Throughout the paper we assume deterministic mechanisms. To understand the role of determinism in our result one must look into the literature of nondeterministic constrained auctions, such as [1]. [1]'s work defines the properties of constrained-efficient auctions, i.e., maximizing the expected social welfare under Bayesian incentive compatibility and budget-constrained players. It [1] states that the domains of efficiency and inefficiency are determined by a threshold value. The computation of the threshold value makes use of expectation and allows for allocations with negative utility for the players. Therefore [1]'s threshold cannot be used in an individually rational deterministic setting. The domains of efficiency that can be concluded from our analysis are determined by the budget. The immediate implication of the budget as the threshold of efficiency is that the budget cannot be a privately known value, but must be publicly known. As such, in our deterministic setting the budget is publicly known, much like [3,15–17]; while in [1] and [18]'s nondeterministic setting the budgets are privately known.

---

2 Under the utility quasilinearity assumption.

3 In quasilinear environments, only Groves mechanisms satisfy the dominant-strategy incentive compatible and Pareto optimal properties ([9,10])

1.1. Our Contribution

Our contribution consists of three aspects. First, we show that combinatorial auctions with a multidimensional type model in a rich preference space of 2^k outcomes require only VCG in order to fulfill the basic properties of deterministic, dominant-strategy incentive compatible, individually rational and Pareto optimal when both players are not constrained by budget and the allocation of the all-item bundle is nonarbitrary. This aspect of the contribution takes a step toward a Roberts-like result [8] for the combinatorial auction domain. Second, we show that [1]'s characterization of the efficient outcome space in a Bayesian single-item model carries to more complex settings, such as our deterministic multidimensional-type model. Third, we show that combinatorial auctions with a multidimensional-type model in a rich preference space of 2^k outcomes require only VCG in order to fulfill the basic properties above when one of the players might be constrained by a budget and the allocation of the all-item bundle is nonarbitrary. This aspect of the contribution shows that [7]'s characterization of the submodularity effect on efficient non-constrained combinatorial auctions with quasilinear utility and a two player model carries to budgeted settings as our two player model trivially holds the coalitional submodularity property.

1.2. Prior Literature

In recent years, several papers studied budget-constrained combinatorial auctions. The authors of [16] showed that there does not exist a deterministic auction that is individually rational, dominant-strategy incentive compatible and Pareto optimal with potentially negative prices and privately known budgets, even when players are one-dimensional types. The authors of [17] showed that the same impossibility holds for one-dimensional types with different items and publicly known multi-item demand. The authors of [15] also showed the same impossibility with publicly known budgets if multidimensional types (two identical items with three outcomes) are considered.

The authors of [15–17] allow negative prices to exist. This means that some players are paid for participation in the auction, either by the mechanism or by the other players. Practical auction implementations usually cannot afford or are unwilling to consider paying bidders for their participation, nor are they interested in encouraging side payments among the participants. Therefore, similar to [1]'s and [3]'s model we chose to assume that all prices are nonnegative. This assumption narrows down the domain of possible allocations in comparison with the potential negative prices model with multidimensional types. Nevertheless, some of the mechanisms that fulfill the three properties of dominant-strategy incentive compatible, individually rational and Pareto optimal in the nonnegative price model are not included in the mechanism space that fulfills the same properties in the negative price model. The reason for the above derives from the property of Pareto optimality. Since the model with nonnegative prices has a smaller set of possible allocations, there exist situations where a mechanism does not fulfill the Pareto optimal property in the model with negative prices, but does fulfill the Pareto optimal property in the model with nonnegative prices.

The authors of [16] also characterize the possibility space of dominant-strategy incentive compatibility and Pareto optimal budget-constrained combinatorial auction mechanisms. Their [16] characterization is restricted to one-dimensional types and therefore, their possibility space characterization does not imply the possibility space in our model with multidimensional types. More specifically, [16] showed that for multi-unit demand and identical items, Ausubel's clinching auction, which assumes public budgets and additive valuations, uniquely satisfies the properties described above. In a similar model with small randomized modification, [18] showed that [16]'s result can be obtained with private budgets. Similarly, Ausubel's clinching auction was concluded by [17] for one-dimensional types with different items and publicly known multi-item demand. For unit-demand players with private values and budget constraints in the one-dimensional-type model, there are several deterministic mechanisms that fulfill the properties of incentive compatible and Pareto optimality (see [19,20]). In nondeterministic mechanisms with a one-dimensional-type model (one indivisible unit), [1] characterizes constrained-efficiency mechanisms, which are mechanisms that maximize the expected social welfare under Bayesian incentive compatibility and budget constraints in a nonnegative price model.

There are few other works that focus on revenue maximization under budget constraints. The authors of [21,22] analyze how budgets change the classic results on "standard" auction formats, showing, for example, that first-price auctions raise more revenue than second-price auctions when bidders are budget-constrained. These results also show that the revenue of a sequential auction is higher than the revenue of a simultaneous ascending auction. The authors of [23,24] construct single-item auctions that maximize the seller's revenue.

A related result in the area of non-constrained combinatorial auctions is [7]. The authors of [7] characterize non-constrained combinatorial auctions that are resilient to coalition defection and false-name bidding phenomena. Under our assumption of quasilinear utility and the two player model, [7]'s result reduces to efficient dominant-strategy incentive compatible mechanism characterization. They [7] define a restriction on the valuations (named bidder-submodularity) that trivially holds in the two player model. Under the two player model, bidder-submodularity implies that the set of efficient and dominant-strategy incentive compatible mechanisms is not empty, as it includes the VCG mechanism. The latter is also shown to be the unique Pareto optimal solution that is efficient and dominant-strategy incentive compatible. [7]'s results differ from ours in several ways. First and foremost, [7]'s model does not assume the existence of budgets. Second, under the quasilinear utility assumption, [7] requires efficiency and then proves that under submodularity VCG is the unique Pareto-optimal solution. Our work does not require efficiency but rather concludes efficiency from the requirements of the four properties discussed above (and under some valuation public knowledge restriction). Nevertheless the characterization of VCG as the unique solution that is dominant-strategy incentive compatible and Pareto optimal is consistent with [7].

1.3. Take Home Point and Example

The take home point from our research is that the general space of dominant-strategy incentive compatible (and Pareto optimal, anonymous) combinatorial auctions with budgets most likely includes only VCG mechanisms when none of the players are budget constrained. It also appears that even if one player is constrained by his budget, then there are still domains under which VCG is the unique solution.

We conclude this section with an example. Two telecommunications companies, T_1 and T_2, are competing over bandwidth in a region with k distinct broadcast bands. Company T_1 has a limited budget, b, to spend on bandwidth. On the other hand, company T_2 is a large telecommunications company with practically unlimited funds to spend on the discussed region. If company T_1 does not value taking control of all the bandwidth available in the region more than b, then T_1's and T_2's bandwidth allocation will be determined and priced by VCG. If company T_2 does not value taking control over all of the bandwidth available in the region more than b, then despite the fact that company T_1 might be interested in taking control of all of the available bandwidth for more money than its budget, T_1 and T_2's bandwidth allocation will be determined and priced by VCG.

1.4. Organization

The paper is organized as follows. Notation and definitions are presented in Section 2. Section 3 shows the implications of the aforementioned properties on the k-item's price. Section 4 defines and proves our main result: the efficient mechanisms' space.

2. Notation and Definitions

We consider combinatorial auction mechanisms with k different types of items and two players. Let $N = \{1, 2\}$ be the set of players and $C = \{c_1, \cdots, c_k\}$ be the set of items. Let $\mathcal{B}$ be the set of all subsets of items, $\mathcal{B} = 2^C$.

Each player, i, has a private value $v_i(B)$ for every bundle, $B \in \mathcal{B}$, drawn from a valid valuation space, $\mathcal{V}_i$. This means that players are multi-minded and have different private values for different bundles of items. We denote player i's private values by a 2^C-tuple:

$$V_i = \{v_i(B) | B \in \mathcal{B}\}$$

Definition 1. *We say that $\mathcal{V}_i$ is a valid valuation space if for every $V_i \in \mathcal{V}_i$, the following two conditions hold:*

- *The valuation of the empty bundle is zero, i.e., $v_i(\emptyset) = 0$.*
- *Free disposal, meaning for both players, the allocation of an extra item cannot reduce their valuation (the usual assumption in combinatorial auctions);*
  i.e., $\forall B, B' \in \mathcal{B},\ B \subset B' \Rightarrow v_i(B) \leq v_i(B')$.

In some cases in this paper we further restrict the valuation space to a subset of all the valid valuation spaces.

As players are multi-minded and have different private values for different bundles of the items, a player, i, may have a separate arbitrary value for each of the 2^k possible outcomes, meaning that our valuation space is a multidimensional valuation space and the players have multidimensional-type valuations.

We assume that player 1 has a limited budget, b_1, while player 2 has an unlimited budget for acquiring the items. We also assume that these budgets are publicly known information.

We denote the auction mechanism $F(V_1, V_2, b_1) = (B_1, B_2, p(B_1), p(B_2))$, where B_i is the bundle allocated to Player i and $p(B_i)$ is the price of bundle B_i. We assume that all the prices are nonnegative, *i.e.*, $p(B_i) \geq 0$ for $i = \{1, 2\}$. We also assume that the auction mechanism, $F(V_1, V_2, b_1)$, can produce at least $2^k - 1$ outcomes, *i.e.*, there exist at least $2^k - 1$ inputs with different outputs.

Definition 2. *Player 1, player 2 and the auctioneer's utilities are defined as follows:*
Player 1's utility is:

$$u_1(F(V_1, V_2, b_1)) = \begin{cases} v_1(B_1) - p(B_1) & if\ p(B_1) \leq b_1 \\ -\infty & otherwise \end{cases}$$

Player 2's utility is $u_2(F(V_1, V_2, , b_1)) = v_2(B_2) - p(B_2)$.
The auctioneer's utility is $u_a(F(V_1, V_2, b_1)) = p(B_1) + p(B_2)$.

For simplicity of notation, whenever F, V_1, V_2 and b_1 will be clear from the context, we will denote $u_i(F(V_1, V_2, b_1))$ by u_i.

Definition 3. Determinism

An auction mechanism, $F(V_1, V_2, b_1)$*, is called deterministic if for every given input, it outputs a single outcome.*

Definition 4. Dictatorship

An auction mechanism, $F(V_1, V_2, b_1)$*, is called dictatorial if there exists a player,* $i \in \{1, 2\}$ *(the dictator), such that for every* V_i*,* $V_{\hat{i}}$*,* $V'_{\hat{i}}$*,* $u_i(F(V_i, V_{\hat{i}}, b_1)) = u_i(F(V_i, V'_{\hat{i}}, b_1))$.

Intuitively, a mechanism is called a dictatorship if there is a player, i, such that the valuations of the other player, $\hat{i}$, cannot affect his utility. Note that if the dictator, i, is indifferent to whether he receives one allocation or another, then Player $\hat{i}$'s valuations can affect the output.

Definition 5. Trivial Pricing Mechanism

We say that a mechanism, F*, is a trivial pricing mechanism if there is an input and there is a player, such that this player is allocated a nonempty bundle for free. Formally,*

$$\exists(V_1, V_2),\ and\ \exists i \in \{1, 2\},\ s.t.$$

$$F(V_1, V_2, b_1) = (B_1, B_2, p(B_1), p(B_2))\ and\ B_i \neq \emptyset\ and\ p(B_i) = 0$$

We next define the four properties under which [3]'s impossibility holds: individual rationality (IR), nonarbitrary hoarding, Pareto optimality and truthfulness.

Definition 6. Property 1: Individual Rationality (IR)

An auction mechanism, $F(V_1, V_2, b_1)$*, is called individually rational if for every player* i*,* $u_i(F(V_1, V_2, b_1)) \geq 0$*. Specifically the following must hold:*

1. $v_1(B_1) - p(B_1) \geq 0$ *and* $p(B_1) \leq b_1$ *(IR of player 1)*

2. $v_2(B_2) - p(B_2) \geq 0$ *(IR of player 2)*

Note that the auctioneer's utility is nonnegative from our assumption that all the prices are nonnegative.

Definition 7. Property 2: Nonarbitrary Hoarding

An auction mechanism, F, upholds nonarbitrary hoarding if the following two conditions hold:

1. *If $B_2 = \{c_1, \cdots, c_k\}$, then $v_2(c_1, \cdots, c_k) \geq \min\{b_1, v_1(c_1, \cdots, c_k)\}$.*
2. *If $B_1 = \{c_1, \cdots, c_k\}$, then $\min\{b_1, v_1(c_1, \cdots, c_k)\} \geq v_2(c_1, \cdots, c_k)$.*

Intuitively, a mechanism fulfills nonarbitrary hoarding if whenever all k items are allocated to a single player, the player is chosen nonarbitrarily, i.e., in accordance with the valuations and budget of the two players. Furthermore, the player chosen must be able to afford the k-item bundle more than the other player. Note that the property of nonarbitrary hoarding does not require if and only if. That is, a mechanism can be considered nonarbitrary hoarding even when $v_2(c_1, \cdots, c_k) \geq \min\{b_1, v_1(c_1, \cdots, c_k)\}$ and player 2 is not allocated the k-item bundle or when $\min\{b_1, v_1(c_1, \cdots, c_k)\} \geq v_2(c_1, \cdots, c_k)$ and player 1 is not allocated the k-item bundle.

Definition 8. Property 3: Pareto Optimality

An auction mechanism, F, is called Pareto optimal if, for every input (V_1, V_2, b_1), where (V_1, V_2) in $(\mathcal{V}_1 \times \mathcal{V}_2)$, there is no allocation $(B'_1, B'_2, p(B'_1), p(B'_2))$, such that all the following inequalities hold, with at least one strong inequality:

- $u_1(B'_1, p(B'_1), B'_2, p(B'_2)) \geq u_1(F(V_1, V_2, b_1))$
- $u_2(B'_1, p(B'_1), B'_2, p(B'_2)) \geq u_2(F(V_1, V_2, b_1))$
- $u_a(B'_1, p(B'_1), B'_2, p(B'_2)) \geq u_a(F(V_1, V_2, b_1))$

Definition 9. Property 4: Truthfulness

An auction mechanism, $F(V_1, V_2, b_1)$, is called truthful if neither of the two players can increase his own utility by reporting false valuations. That is, given the true valuations, $(V_1, V_2) \in (\mathcal{V}_1 \times \mathcal{V}_2)$, for every $(V'_1, V'_2) \in (\mathcal{V}_1 \times \mathcal{V}_2)$, the following hold:

- $u_1(F(V_1, V'_2, b_1)) \geq u_1(F(V'_1, V'_2, b_1))$
- $u_2(F(V'_1, V_2, b_1)) \geq u_2(F(V'_1, V'_2, b_1))$

3. The Implication of the Properties on the k-item Bundle's Price

In this section, we derive the k-item bundle's price as implied by the properties of IR, nonarbitrary hoarding, Pareto optimality and truthfulness in any nontrivial pricing mechanism.

Lemma 1. *In any nontrivial pricing mechanism that satisfies IR, Pareto optimality, nonarbitrary hoarding and truthfulness, the k-item bundle's price must be:*

1. *If $B_1 = \emptyset$ and $B_2 = \{c_1, \cdots, c_k\}$, then $p(B_1) = 0$ and $p(B_2) = \min\{b_1, v_1(c_1, \cdots, c_k)\}$.*
2. *If $B_1 = \{c_1, \cdots, c_k\}$ and $B_2 = \emptyset$, then $p(B_1) = v_2(c_1, \cdots, c_k)$ and $p(B_2) = 0$.*

Proof of Lemma 1. *Consider the two cases:*

1. $B_2 = \{c_1, \cdots, c_k\}$ *and* $B_1 = \emptyset$

   (a) *Suppose, to the contrary, that* $p(B_2) < \min\{b_1, v_1(c_1, \cdots, c_k)\}$.

   Let $p(B_2) = \min\{b_1, v_1(c_1, \cdots, c_k)\} - 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$.
   Consider the same valuations for player 1 ($V_1' = V_1$*) and the following valuations for player 2:*

   - $v_2'(c_1, \cdots, c_k) = \min\{b_1, v_1(c_1, \cdots, c_k)\} - \varepsilon$ *and*
   - $v_2'(B) = 0$ *for any other bundle,* $B \neq \{c_1, \cdots, c_k\}$.

   Then, from nonarbitrary hoarding (Definition 7.), player 2 will not be allocated the k*-item bundle. However, player 2's utility from any other bundle is zero. Thus, player 2 is better off deviating and stating* V_2 *instead of* V_2' *and being allocated the* k*-item bundle for a positive utility.*

   (b) *Suppose to the contrary that* $v_2(c_1, \cdots, c_k) \geq p(B_2) > \min\{b_1, v_1(c_1, \cdots, c_k)\}$.
   Let $p(B_2) = \min\{b_1, v_1(c_1, \cdots, c_k)\} + 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$. *We show that such a price cannot be truthful for player 2. Consider the following deviation for player 2:*

   - $v_2'(c_1, \cdots, c_k) = \min\{b_1, v_1(c_1, \cdots, c_k)\} + \varepsilon$ *and*
   - $v_2'(B) = 0$ *for any bundle* $B \neq \{c_1, \cdots, c_k\}$.

   Then, from nonarbitrary hoarding, player 1 will not be allocated the k*-item bundle.*
   Suppose that each player is allocated a non-empty bundle, B_1' *and* B_2'*, respectively. Then from player 2's IR, it must be the case that* $p(B_2') = 0$. *However, this is a trivial pricing mechanism, i.e., a player is allocated a non-empty bundle for free. Therefore, player 2 must be allocated the* k*-item bundle for a lower price. Thus, whenever the true valuations are* V_1, V_2*, player 2 can deviate and be allocated the same bundle for a lower price.*

2. $B_1 = \{c_1, \cdots, c_k\}$ *and* $B_2 = \emptyset$

   (a) *The proof that* $p(B_1) < v_2(c_1, \cdots, c_k)$ *is not possible is similar to 1.(a).*

   (b) *Suppose to the contrary that* $p(B_1) > v_2(c_1, \cdots, c_k)$.
   Let $p(B_1) = v_2(c_1, \cdots, c_k) + 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$. *We show that such a price cannot be truthful for player 1. Consider the following deviation for player 1:*

   - $v_1'(c_1, \cdots, c_k) = v_2(c_1, \cdots, c_k) + \varepsilon$ *and*
   - $v_1'(B) = 0$ *for any bundle* $B \neq \{c_1, \cdots, c_k\}$

   From player 1's IR, we know that $p(B_1) = v_2(c_1, \cdots, c_k) + 2 \cdot \varepsilon \leq b_1$*, and therefore,* $v_2(c_1, \cdots, c_k) < b_1$. *We then conclude from nonarbitrary hoarding that player 2 will not be allocated the* k*-item bundle. Suppose that each player is allocated a non-empty bundle,* B_1' *and* B_2'*, respectively. Then, from player 1's IR, it must be the case that* $p(B_1') = 0$. *However, this is a trivial pricing mechanism, i.e., a player is allocated a non-empty bundle for free. Therefore, player 1 must be allocated the* k*-item bundle for a lower price. Thus, whenever* *the true valuations are V_1, V_2, player 1 can deviate and be allocated the same bundle for a lower price.*

4. Mapping the VCG Space

In this section, we derive the price structure of any nontrivial pricing mechanism that satisfies the properties IR—truthfulness, Pareto optimality and nonarbitrary hoarding—when it is publicly known that certain restrictions on the private valuations hold. We use the following notation to describe the restrictions on valuations:

[R1] $v_1(c_1, \cdots, c_k) \leq b_1$
[R2] $v_2(c_1, \cdots, c_k) < b_1$

Throughout this section we assume that at least one of the restrictions, R1 or R2, is publicly known. In the previous section, we derive the k-item bundle's price as implied by the properties of IR, nonarbitrary hoarding, Pareto optimality and truthfulness in any nontrivial pricing mechanism. In the following lemma, we derive the price structure of any partial bundle given that it is publicly known that certain restrictions on the private valuations hold.

Lemma 2. *Given that at least one of the restrictions (R1 or R2) is publicly known, the prices must be as follows in any nontrivial pricing mechanism that satisfies IR, Pareto optimality, nonarbitrary hoarding and truthfulness:*

1. $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2)$
2. $p(B_2) = v_1(c_1, \cdots, c_k) - v_1(B_1)$

Proof of Lemma 2. *We consider the two prices:*

1. $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2)$
   - *Suppose to the contrary that $p(B_1) < v_2(c_1, \cdots, c_k) - v_2(B_2)$.*
     Let $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2) - 2 \cdot \varepsilon$ for some $\varepsilon > 0$.
     Consider the same valuations for player 2 ($V_2' = V_2$) and the following valuations for player 1:
     - $v_1'(B) = 0$ *for any bundle* $B \not\supseteq B_1$
     - $v_1'(B') = v_2(c_1, \cdots, c_k) - v_2(B_2) - \varepsilon$ *for any bundle* $B' \supseteq B_1$

     Allocating the k-item bundle to player 1 contradicts nonarbitrary hoarding as $v_1'(c_1, \cdots, c_k) < v_2'(c_1, \cdots, c_k)$. All allocations where $B_1' \not\supseteq B_1$ are not truthful for player 1, as player 1's utility from such an allocation is zero and player 1 can deviate and state V_1 instead of V_1'. In such a case, player 1 is allocated $B_1' = B_1$ for $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2) - 2 \cdot \varepsilon$, and his utility is strictly positive, $u_1 = \varepsilon > 0$

     To complete the argument, we now show that allocating $B_1'' \supseteq B_1$ and $B_2'' = C - B_1'' \neq \{c_1, \cdots, c_k\}$ is not feasible for player 2 in any nontrivial pricing mechanism. As we *assume that all prices are nonnegative, we thus contradict the assumption that* $p(B_1) < v_2(c_1, \cdots, c_k) - v_2(B_2)$.

Consider the following valuations:

- $V_1'' = V_1'$
- $v_2''(B) = 0$ *for any bundle* $B \neq \{c_1, \cdots, c_k\}$
- $v_2''(c_1, \cdots, c_k) = v_2'(c_1, \cdots, c_k) = v_2(c_1, \cdots, c_k)$.

The allocation $B_1'' = \{c_1, \cdots, c_k\}$ *contradicts nonarbitrary hoarding as* $v_1''(c_1, \cdots, c_k) = v_1'(c_1, \cdots, c_k) < v_2(c_1, \cdots, c_k) = v_2''(c_1, \cdots, c_k)$;

Now, consider any non-empty bundle B_2'', *such that* $B_2'' \neq \{c_1, \cdots, c_k\}$. *As* $v_2''(B_2'') = 0$, *player 2's IR implies that* $p(B_2'') = 0$. *However,* $B_2'' \neq \emptyset$, *and therefore this mechanism is a trivial pricing mechanism.*

- *Suppose to the contrary that* $p(B_1) > v_2(c_1, \cdots, c_k) - v_2(B_2)$.

  Let $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2) + 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$.

  We claim that the truthfulness property does not hold for player 1. Consider player 1's deviation:

  - $v_1'(B) = v_2(c_1, \cdots, c_k) - v_2(B_2) + \varepsilon$ *for any bundle,* $B \supseteq B_1$.
  - $v_1'(B') = 0$ *for any bundle,* $B' \not\supseteq B_1$.

  We show that when player 1 declares V_1', *no other allocation but* $B_1' = B_1$ *can maintain the properties of IR, Pareto optimality and nonarbitrary hoarding, and therefore player 1 is allocated* B_1 *for a lower price. We first show that the allocation* $B_2' = \{c_1, \cdots, c_k\}$ *contradicts either Pareto optimality or nonarbitrary hoarding. We showed in Lemma 1. that* $p(B_2' = \{c_1, \cdots, c_k\}) = \min\{b_1, v_1'(c_1, \cdots, c_k)\}$.

  - *If* $v_1'(c_1, \cdots, c_k) \leq b_1$, *that means that* $p(B_2') = v_1'(c_1, \cdots, c_k) = v_2(c_1, \cdots, c_k) - v_2(B_2) + \varepsilon$. *We claim that this allocation is not Pareto optimal, as the allocation* $B_1'' = B_1$, $p(B_1'') = v_2(c_1, \cdots, c_k) - v_2(B_2) + \varepsilon$, $B_2'' = B_2$ *and* $p(B_2'') = 0$ *is strictly better for player 2, while the auctioneer and player 1 are indifferent to the two allocations:*
    * $u_1'' = v_1'(B_1'') - p(B_1'') = v_2(c_1, \cdots, c_k) - v_2(B_2) + \varepsilon - v_2(c_1, \cdots, c_k) + v_2(B_2) - \varepsilon = 0 = u_1'$
    * $u_2'' = v_2(B_2'') - p(B_2'') = v_2(B_2)$. *However,* $v_1'(c_1, \cdots, c_k) = v_2(c_1, \cdots, c_k) - v_2(B_2) + \varepsilon$, *and therefore,* $v_2(B_2) > v_2(c_1, \cdots, c_k) - v_1'(c_1, \cdots, c_k) = u_2'$
    * $u_a'' = u_a'$.
  - *If* $v_1'(c_1, \cdots, c_k) > b_1$ *and since at least one of the restrictions, R1 or R2, is publicly known, then it must be the case that R2 is public knowledge and* $v_2(c_1, \cdots, c_k) < b_1$. *Therefore, the allocation* $B_2' = \{c_1, \cdots, c_k\}$ *contradicts nonarbitrary hoarding.*

  Following player 1's IR, any allocation, B_1', B_2', $p(B_2')$, $p(B_1')$, *such that* B_1' *does not contain* B_1 *and* $B_1' \neq \emptyset$, *implies that* $p(B_1') = 0$ *and, therefore is trivial pricing. Thus, player 1 is allocated* B_1 *for a lower price, which contradicts the assumption that* $p(B_1) > v_2(c_1, \cdots, c_k) - v_2(B_2)$.

2. $p(B_2) = v_1(c_1, \cdots, c_k) - v_1(B_1)$

   - *Suppose to the contrary that* $p(B_2) < v_1(c_1, \cdots, c_k) - v_1(B_1)$.
     Let $p(B_2) = v_1(c_1, \cdots, c_k) - v_1(B_1) - 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$. *Consider the following valuations:*
     - $V_1' = V_1$
     - $v_2'(B) = 0$ *for any bundle* $B \not\supseteq B_2$.
     - $v_2'(B') = v_1(c_1, \cdots, c_k) - v_1(B_1) - \varepsilon$ *for any bundle* $B' \supseteq B_2$.

     If R1 is publicly known, then $v_2'(c_1, \cdots, c_k) = v_1(c_1, \cdots, c_k) - v_1(B_1) - \varepsilon < b_1$. *If R2 is publicly know, then* $v_2'(c_1, \cdots, c_k) < b_1$. *Therefore, we conclude that* $v_2'(c_1, \cdots, c_k) < min\{b_1, v_1'(c_1, \cdots, c_k)\}$, *and allocating the* k*-item bundle to player 2 contradicts nonarbitrary hoarding. Any allocation,* B_2', *such that* $B_2' \not\supseteq B_2$, *is not truthful for player 2, as player 2's utility is zero. Player 2 can deviate and state* V_2 *instead of* V_2'. *With this deviation he will be allocated* B_2 *for a positive utility, as:*
     $u_2' = v_2'(B_2) - p(B_2) = v_2'(B_2) - v_1(c_1, \cdots, c_k) + v_1(B_1) + 2 \cdot \varepsilon = v_1(c_1, \cdots, c_k) - v_1(B_1) - \varepsilon - v_1(c_1, \cdots, c_k) + v_1(B_1) + 2 \cdot \varepsilon = \varepsilon > 0$.
     To complete the argument, we now claim that the truthfulness property does not hold for player 1 when allocating B_1 *and* B_2 *to the two players, respectively. Consider player 1's deviation:*
     - $v_1''(B) = 0$ *for any bundle* $B \neq \{c_1, \cdots, c_k\}$
     - $v_1''(c_1, \cdots, c_k) = v_1'(c_1, \cdots, c_k) = v_1(c_1, \cdots, c_k)$

     We show that when player 1 declares V_1'', *no other allocation, but* B_1'', *can maintain the properties of IR, Pareto optimality and nonarbitrary hoarding, and therefore, player 1 is allocated* B_1'' *for a higher utility.*
     We first show that allocating the k*-item bundle to player 2 contradicts nonarbitrary hoarding as* $v_2''(c_1, \cdots, c_k) = v_2'(c_1, \cdots, c_k) < v_1(c_1, \cdots, c_k) = v_1''(c_1, \cdots, c_k)$ *and* $v_2''(c_1, \cdots, c_k) = v_2'(c_1, \cdots, c_k) < b_1$. *Allocating any other allocation, such that* $B_1'' \notin \{\emptyset, \{c_1, \cdots, c_k\}\}$, *must be trivial pricing, as player 1 is allocated a non-empty bundle for free.*
     We proved in Lemma 1. that in any nontrivial pricing mechanism that satisfies the four properties, if player 1 is allocated the k*-item bundle, then* $p(c_1, \cdots, c_k) = \min\{b_1, v_2''(c_1, \cdots, c_k)\}$.
     We now show that player 1 can gain a higher utility when allocated the k*-item bundle.*
     $u_1'' = v_1(c_1, \cdots, c_k) - p(c_1, \cdots, c_k) \geq v_1(c_1, \cdots, c_k) - v_2''(c_1, \cdots, c_k) = v_1(B_1) + \varepsilon$.
     That is, player 1's utility from the k-item bundle is higher than his utility from allocation B_1, *even if* B_1 *is allocated for free.*

   - *Suppose to the contrary that* $p(B_2) > v_1(c_1, \cdots, c_k) - v_1(B_1)$
     Let $p(B_2) = v_1(c_1, \cdots, c_k) - v_1(B_1) + 2 \cdot \varepsilon$ *for some* $\varepsilon > 0$.
     We claim that the truthfulness property does not hold for player 2. Consider the following deviation for player 2:

– $v'_2(B) = v_1(c_1, \cdots, c_k) - v_1(B_1) + \varepsilon$ *for any bundle,* $B \supseteq B_2$.
– $v'_2(B') = 0$ *for any bundle,* $B' \not\supseteq B_2$.

We show that when player 2 declares V'_2*, no other allocation, but* $B'_2 = B_2$*, can maintain the properties of IR and Pareto optimality, and therefore player 2 is allocated* B_2 *for a lower price. We first show that the allocation* $B'_1 = \{c_1, \cdots, c_k\}$ *contradicts Pareto optimality. We showed in Lemma 1. that* $p(B'_1 = \{c_1, \cdots, c_k\}) = v'_2(c_1, \cdots, c_k) = v_1(c_1, \cdots, c_k) - v_1(B_1) + \varepsilon$*. We claim that this allocation is not Pareto optimal, as the allocation* $B''_1 = B_1$*,* $B''_2 = B_2$ *and* $p(B''_2) = v_1(c_1, \cdots, c_k) - v_1(B_1) + \varepsilon$*,* $p(B''_1) = 0$ *is strictly better for player 1, while the auctioneer and player 2 are indifferent to the two allocations:*

– $u''_1 = v_1(B''_1) - p(B''_1) = v_1(B''_1) = v_1(B_1)$*. However,* $u'_1 = v_1(c_1, \cdots, c_k) - p(B'_1) = v_1(B_1) - \varepsilon$*. Thus,* $u''_1 > u'_1$*.*
– $u''_2 = v'_2(B''_2) - p(B''_2) = 0 = u'_2$*.*
– $u''_a = u'_a$*.*

Following player 2's IR, any allocation, B'_1*,* B'_2*,* $p(B'_1)$*,* $p(B'_2)$*, such that* B'_2 *does not contain* B_2*, and* $B'_2 \neq \emptyset$ *implies that* $p(B'_2) = 0$ *and, therefore is trivial pricing. Thus, player 2 is allocated* B_2 *for a lower price, which contradicts the assumption that* $p(B_2) > v_1(c_1, \cdots, c_k) - v_1(B_1)$*.*

4.1. VCG Mechanism Possibility Space

In this section, we show that at least one of the restrictions, R1 or R2, is publicly known, then an efficient mechanism that applies VCG prices is the unique mechanism that satisfies the four properties. The efficiency definition follows.

Definition 10. *Let* $F(V_1, V_2, b_1) = (B_1, B_2, p(B_1), p(B_2))$ *be an auction mechanism. We say that* F *is efficient if* $B_1 \in \operatorname{argmax}_{B \in \mathcal{B}}(v_1(B) + v_2(C - B))$

Theorem 1. *If at least one of the restrictions (R1 or R2) is publicly known, then the unique mechanism that satisfies the four properties is an efficient mechanism that applies VCG prices.*

We divide the proof of Theorem 1. into two parts. One part proves sufficient and the other part proves necessary. The sufficient part of the proof shows that if it is publicly known that at least one of restrictions, R1 or R2 hold then an efficient mechanism with VCG prices satisfies the four properties. The necessary part of the proof shows that if it is publicly known that at least one of restrictions, R1 or R2, hold, then any mechanism that satisfies the four properties must be efficient with VCG prices.

Proof of Theorem 1. *We start by proving sufficiency.*

Part I: Any efficient mechanism with VCG prices satisfies the four properties.

1. *IR: We divide our proof into two allocation cases. In the first case, player 1 is allocated the empty bundle, and in the second case, player 1 is allocated a nonempty bundle.*
   - $B_1 = \emptyset$ *and* $B_2 = \{c_1, \cdots, c_k\}$

The assumed VCG price for player 1 is $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2) = 0$*. Thus, the efficient mechanism with VCG prices satisfies the IR property for player 1.*
The assumed VCG price for player 2 is $p(B_2) = v_1(c_1, \cdots, c_k)$*. As we assume that the mechanism is efficient, it must be the case that for any bundle,* B*,* $v_2(c_1, \cdots, c_k) \geq v_1(B) + v_2(C - B)$*. In particular,* $v_2(c_1, \cdots, c_k) \geq v_1(c_1, \cdots, c_k)$*. Thus, the efficient mechanism with VCG prices satisfies the IR property for player 2.*

- $B_1 \neq \emptyset$
  - *IR of player 1: Suppose to the contrary that player 1's IR is not satisfied, i.e.,* $\min\{b_1, v_1(B_1)\} < v_2(c_1, \cdots, c_k) - v_2(B_2)$*.*
    If restriction R1 holds, then from free disposal, it means that $\min\{b_1, v_1(B_1)\} = v_1(B_1)$*, and therefore, we conclude that* $v_1(B_1) < v_2(c_1, \cdots, c_k) - v_2(B_2)$*. The last inequality implies that* $v_1(B_1) + v_2(B_2) < v_2(c_1, \cdots, c_k)$*, which contradicts the assumption that the allocation is efficient.*
    If restriction R2 holds, then:
    * *If, on the one hand,* $b_1 < v_1(B_1)$*, meaning that* $b_1 < v_2(c_1, \cdots, c_k) - v_2(B_2)$*, so together with restriction R2, we have* $v_2(c_1, \cdots, c_k) > v_2(B_2) + b_1 > v_2(B_2) + v_2(c_1, \cdots, c_k)$*. However, the last inequality implies that* $v_2(B_2) < 0$*, which contradicts our valuations' definition as* $v_2(\emptyset) = 0$*, and we assume free disposal.*
    * *If, on the other hand,* $b_1 > v_1(B_1)$*, meaning that* $v_1(B_1) < v_2(c_1, \cdots, c_k) - v_2(B_2)$*, so* $v_1(B_1) + v_2(B_2) < v_2(c_1, \cdots, c_k)$*, which contradicts efficiency.*
  - *IR of player 2: Suppose to the contrary that player 2's IR is not satisfied, i.e.,* $v_2(B_2) < v_1(c_1, \cdots, c_k) - v_1(B_1)$*. If* $B_2 = \emptyset$*, then the inequality given by the contrary assumption implies that* $v_2(B_2) < 0$*, which contradicts our valuations' definition. If* $B_2 \neq \emptyset$*, then the inequality given by the contrary assumption implies that* $v_1(B_1) + v_2(B_2) < v_1(c_1, \cdots, c_k)$*, which contradicts the assumption that the allocation is efficient, as* $B_2 = \emptyset$ *should have been chosen.*

2. *Pareto Optimality: Suppose to the contrary that the allocation,* B_1*, and* $B_2 = C - B_1$*, with VCG prices is efficient, but not Pareto optimal. That is, there is another allocation* $B_1' \neq B_1$ *and* $B_2' = C - B_1'$ *with prices* $p(B_1') \leq b_1$ *and* $p(B_2')$*, such that all of the following conditions hold and at least one of conditions 2b–2d is strictly better.*
   (a) $v_1(B_1') + v_2(B_2') \leq v_1(B_1) + v_2(B_2)$*, meaning allocation* B_1', B_2' *does not maximize the sum of valuations.*
   (b) $p(B_1') + p(B_2') \geq p(B_1) + p(B_2)$*, meaning the auctioneer is better off with or indifferent to allocation* B_1', B_2'*.*
   (c) $v_1(B_1') - p(B_1') \geq v_1(B_1) - p(B_1)$*, meaning player 1 is better off with or indifferent to allocation* B_1', B_2'*.*
   (d) $v_2(B_2') - p(B_2') \geq v_2(B_2) - p(B_2)$*, meaning player 2 is better off with or indifferent to allocation* B_1', B_2'*.*

   However, from conditions 2c and 2d, we conclude that
   $v_1(B_1') - v_1(B_1) + v_2(B_2') - v_2(B_2) \geq p(B_1') + p(B_2') - p(B_1) - p(B_2)$.

From condition 2b, we conclude that the right-hand side must be positive, and therefore, with one strict inequality, it must be the case that $v_1(B_1') - v_1(B_1) + v_2(B_2') - v_2(B_2) > 0$ *in contradiction to condition 2a. Therefore, allocation* B_1, B_2 *is Pareto optimal.*

3. *Nonarbitrary hoarding: If restriction R1 is public knowledge, then we need to show that:*
   - *if player 1 is allocated the* k*-item bundle, then* $v_1(c_1, \cdots, c_k) \geq v_2(c_1, \cdots, c_k)$
   - *if player 2 is allocated the* k*-item bundle, then* $v_2(c_1, \cdots, c_k) \geq v_1(c_1, \cdots, c_k)$

   Suppose to the contrary that $v_1(c_1, \cdots, c_k) < v_2(c_1, \cdots, c_k)$*, then* $B_1 = \{c_1, \cdots, c_k\}$ *does not maximize the sum of the valuations. The same argument is valid for the case where player 2 is allocated the* k*-item bundle.*
   If restriction R2 is public knowledge, then the allocation $B_1 = \emptyset$ *cannot be efficient and, therefore, is not feasible. Therefore, to prove nonarbitrary hoarding, we only have to show that if player 1 is allocated the* k*-item bundle, then* $\min\{b_1, v_1(c_1, \cdots, c_k)\} \geq v_2(c_1, \cdots, c_k)$*. However, this inequality is implied directly from restriction R2.*

4. *Truthfulness of player 1: Suppose that* $F(V_1, V_2, b_1) = (B_1, B_2, p(B_1), p(B_2))$ *and that player 1 can gain utility by deviating and stating other valuations,* V_1'*. Denote the outcome of the false valuations by* $F(V_1', V_2, b_1) = (B_1', B_2', p(B_1'), p(B_2'))$*. From the fact that the mechanism is efficient and applies VCG prices, we know that:*
   (a) $v_1(B_1) + v_2(B_2) \geq v_1(B_1') + v_2(B_2')$
   (b) $p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2)$
   (c) $p(B_1') = v_2(c_1, \cdots, c_k) - v_2(B_2')$
   (d) $v_1(B_1) - p(B_1) < v_1(B_1') - p(B_1')$*, meaning player 1 strictly increased his own utility by deviating.*

   We conclude from 4b to 4d that $v_1(B_1) - v_2(c_1, \cdots, c_k) + v_2(B_2) < v_1(B_1') - v_2(c_1, \cdots, c_k) + v_2(B_2')$*, that is* $v_1(B_1) + v_2(B_2) < v_1(B_1') + v_2(B_2')$*. However, the last inequality contradicts 4a.*
   Truthfulness of player 2: Similar arguments hold for proving player 2's truthfulness.

We continue by proving necessity.

Part II: Any mechanism that satisfies the four properties must be efficient with VCG prices.
In Lemma 2. we showed that if at least one of the restrictions (R1 or R2) is publicly known, then any nontrivial pricing mechanism that satisfies the four properties must use VCG prices. Suppose to the contrary that there is a nontrivial pricing mechanism, F*, that satisfies the four properties but is not efficient. We divide our proof into two cases. In the first case, player 1 is allocated an empty set and in the second case player 1 is allocated a non-empty set.*

- *Suppose that* $F(V_1, V_2, b_1) = (B_1 = \emptyset, B_2 = \{c_1, \cdots, c_k\}, p(B_1) = 0, p(B_2) = v_1(c_1, \cdots, c_k))$*. It follows from player 2's IR that restriction R2 cannot hold. Suppose that restriction R1 holds, i.e.,* $v_1(c_1, \cdots, c_k) \leq b_1$*. As we assumed that efficiency does not hold, there exists a bundle* $B_1' \neq \emptyset$*, such that* $v_1(B_1') + v_2(B_2') > v_1(\emptyset) + v_2(c_1, \cdots, c_k)$*. We show that the above inequality implies that* F *is not Pareto optimal. The allocation* $(B_1', B_2' = C - B_1', p(B_1') = v_1(B_1'), p(B_2') = v_1(c_1, \cdots, c_k) - v_1(B_1'))$ *is strictly better for player 2, while player 1 and the auctioneer are indifferent to the two alternatives.*

- *Player 1's utility is zero in both cases;* $u_1' = u_1 = 0$. *R1 and free disposal lead us to conclude that* $p(B_1') = v_1(B_1') < b_1$, *and therefore,* $p(B_1')$ *is within player 1's budget limitations.*
- *The auctioneer's utility is the same;* $u_a = v_1(c_1, \cdots, c_k) = p(B_1') + p(B_2') = u_a'$
- *Player 2's utility is strictly increased;*
  $u_2' = v_2(B_2') - v_1(c_1, \cdots, c_k) + v_1(B_1') > v_2(c_1, \cdots, c_k) - v_1(c_1, \cdots, c_k) = u_2$.

- *Suppose that* $F(V_1, V_2, b_1) = (B_1 \neq \emptyset, B_2, p(B_1) = v_2(c_1, \cdots, c_k) - v_2(B_2), p(B_2) = v_1(c_1, \cdots, c_k) - v_1(B_1))$. *As we assumed that efficiency does not hold, there exists a bundle* $B_1' \neq B_1$, *such that* $v_1(B_1') + v_2(B_2') > v_1(B_1) + v_2(B_2)$. *We show that the above inequality implies that* F *is not Pareto optimal.*
  Consider the following cases for player 2's valuations:

  - $v_2(B_2') > v_2(B_2)$ *and* $v_1(B_1') + v_2(B_2') > v_1(B_1) + v_2(B_2)$.
    Consider the following allocation:

    $$(B_1', B_2', p(B_1') = v_2(c_1, \cdots, c_k) - v_2(B_2'), p(B_2') = v_1(c_1, \cdots, c_k) - v_1(B_1) + v_2(B_2') - v_2(B_2)).$$

    We first show that $0 \leq p(B_1') \leq b_1$ *and* $0 \leq p(B_2')$.
    $p(B_1') = v_2(c_1, \cdots, c_k) - v_2(B_2') < v_2(c_1, \cdots, c_k) - v_2(B_2) = p(B_1)$. *As* F *satisfies IR,* $p(B_1) \leq b_1$, *and therefore,* $p(B_1') < b_1$.
    $p(B_1') = v_2(c_1, \cdots, c_k) - v_2(B_2') \geq 0$ *follows from free disposal.*
    $p(B_2') = v_1(c_1, \cdots, c_k) - v_1(B_1) + v_2(B_2') - v_2(B_2) > v_1(c_1, \cdots, c_k) - v_1(B_1) \geq 0$ *follows from the assumption that* $v_2(B_2') > v_2(B_2)$ *and free disposal.*

    We now show that player 1's utility is strictly better while player 2 and the auctioneer are indifferent to the two alternatives:

    * *Player 1's utility is* $u_1' = v_1(B_1') - p(B_1') = v_1(B_1') - v_2(c_1, \cdots, c_k) + v_2(B_2') > v_1(B_1) - v_2(c_1, \cdots, c_k) + v_2(B_2) = u_1$.
    * *The auctioneer's utility is unchanged;*
      $u_a' = p(B_1') + p(B_2') = p(B_1) + p(B_2) = u_a$
    * *Player 2's utility is is unchanged;*
      $u_2' = v_2(B_2') - p(B_2') = v_2(B_2') - v_1(c_1, \cdots, c_k) + v_1(B_1) - v_2(B_2') + v_2(B_2) = u_2$.

  - $v_2(B_2') \leq v_2(B_2)$ *and* $v_1(B_1') + v_2(B_2') > v_1(B_1) + v_2(B_2)$, *meaning that* $v_1(B_1') > v_1(B_1)$.
    Consider the following allocation:

    $$(B_1', B_2', p(B_1') = p(B_1) + v_2(B_2) - v_2(B_2') = v_2(c_1, \cdots, c_k) - v_2(B_2'), p(B_2') = p(B_2) - v_2(B_2) + v_2(B_2'))$$

    We first show that $0 \leq p(B_1') \leq b_1$ *and* $0 \leq p(B_2')$.

    * *If R2 is publicly known, then it follows that* $p(B_1') \leq v_2(c_1, \cdots, c_k) < b_1$.

    * *If R1 is publicly known, then* $p(B_1') = p(B_1) + v_2(B_2) - v_2(B_2') < p(B_1) + v_1(B_1') - v_1(B_1)$. *It follows from IR that* $p(B_1) \leq v_1(B_1)$. *We therefore*

conclude that $p(B_1') < v_1(B_1')$*. We then conclude from R1 that* $p(B_1') < b_1$*.*

$p(B_1') = v_2(c_1, \cdots, c_k) - v_2(B_2') \geq 0$ *follows from free disposal.*
$p(B_2') = v_1(c_1, \cdots, c_k) - v_1(B_1) + v_2(B_2') - v_2(B_2) > v_1(c_1, \cdots, c_k) - v_1(B_1') \geq 0$ *follows from the assumption that* $v_2(B_2') - v_2(B_2) > v_1(B_1) - v_1(B_1')$ *and free disposal.*

Player 1's utility is strictly better while player 2 and the auctioneer are indifferent to the two alternatives:

* *Player 1's utility is* $u_1' = v_1(B_1') - p(B_1') = v_1(B_1') - p(B_1) - v_2(B_2) + v_2(B_2') > v_1(B_1) - p(B_1) = u_1$.
* *The auctioneer's utility is unchanged;*
  $u_a' = p(B_1') + p(B_2') = p(B_1) + p(B_2) = u_a$
* *Player 2's utility is as before;*
  $u_2' = v_2(B_2') - p(B_2') = v_2(B_2') - p(B_2) + v_2(B_2) - v_2(B_2') = v_2(B_2) - p(B_2) = u_2$.

5. Conclusions

We characterize the efficiency space of deterministic, dominant-strategy incentive compatible, individually rational and Pareto-optimal combinatorial auctions in a model with two players and k nonidentical items, where the all-item bundle is nonarbitrarily allocated. Our model has multidimensional types, private values and quasilinear preferences for the players with one relaxation: one of the players is subject to a publicly known budget constraint. We show that there exists a unique VCG mechanism made possible by the public knowledge of the relationship between the players' valuation for the k-item bundle and the constrained player's budget. The authors of [2] indicate that the existence of inefficient solutions is the space of deterministic, dominant-strategy incentive compatible, individually rational and Pareto-optimal combinatorial auctions. The authors of [3] provide insights for the existence of valuation domains that enforce inefficient solutions, even with the assumption of nonarbitrary hoarding. An interesting outcome of our result would be to characterize the remainder of the possibility space and determine if and which valuation domains enforce inefficient solutions in the deterministic, dominant-strategy incentive compatible, individually rational and Pareto-optimal combinatorial auctions, where the all-item bundle is nonarbitrarily allocated.

Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

The authors contribution is equal in all aspects of the reported research and writing of the paper.

References

1. Maskin, E.S. Auctions, development and privatization: Efficient auctions with liquidity-constrained buyers. *Eur. Econ. Rev.* **2000**, *44*, 667–681.
2. Lerner, A.; Gonen, R. Dictatorial mechanisms in constrained combinatorial auctions. *B.E. J. Theor. Econ.* **2013**, *13*, doi:10.1515/bejte-2013-0006.
3. Gonen, R.; Lerner, A. The Incompatibility of Pareto Optimality and Incentive Compatibility in quasilinear Settings with Anonymous and Budget Constrained players. *Games* **2013**, *4*, 690–710, doi:10.3390/g4040690.
4. Vickrey, W. Counterspeculation, Auctions and Competitive Sealed Tenders. *J. Financ.* **1961**, *16*, 8–37.
5. Clarke, E.H. Multipart Pricing of Public Goods. *Public Choice* **1971**, *2*, 17–33.
6. Groves, T. Incentives in teams. *Econometrica* **1973**, *41*, 617–631.
7. Ausubel, L.M.; Milgrom, P. Ascending Auctions with PackageBidding. *Front. Theor. Econ.* **2002**, *1*, 1–42.
8. Roberts, K. The Characterization of Implementable Choice Rules. In *Aggregation and Revelation of Preferences*; Laffont, J.J., Ed.; North-Holland Publishing Company: North-Holland, The Netherlands, 1979; pp. 321–348.
9. Green, J.; Laffont, J. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica* **1977**, *45*, 427–438.
10. Holmstrom, B. Groves' scheme on restricted domains. *Econometrica* **1979**, *47*, 1137–1144.
11. Nisan, N.; Ronen, A. Computationally Feasible VCG-based Mechanisms. *J. Artif. Intell. Res.* **2007**, *29*, 19–47.
12. Lehmann, D.; O'Callaghan, L.I.; Shoham, Y. Truth Revelation in Approximately Efficient Combinatorial Auctions. *J. ACM* **2002**, *49*, 577–602.
13. Bartal, Y.; Gonen, R.; Nisan, N. Incentive Compatible Multi-Unit Combinatorial Auctions. In Proceedings of the 9th Theoretical Aspects of Rationality and KnowledgeTARK, Bloomington, IN, USA, 20–22 June , 2003; pp. 72–87.
14. Dobzinski, S.; Nisan, N. Multi-Unit Auctions: Beyond Roberts. In Proceedings of the 12th ACM Conference on Electronic Commerce EC, San Jose, CA, USA, 5–9 June , 2011; pp. 233–242.
15. Lavi, R.; May, M. A Note on the Incompatibility of Strategy-proofness and Pareto-optimality in Quasi-linear Settings with Public Budgets. *Econ. Lett.* **2012**, *115*, 100–103.
16. Dobzinski, S.; Lavi, R.; Nisan, N. Multi-Unit Auctions with Budget Limits. *Games Econ. Behav.* **2008**, *74*, 486–503.
17. Fiat, A.; Leonardi, S.; Saia, J.; Sankowski, P. Single Valued Combinatorial Auctions with Budgets. In Proceedings of the 12th ACM Conference on Electronic Commerce EC, San Jose, CA, USA, 5–9 June , 2011; pp. 223–232.
18. Bhattacharya, S.; Conitzer, V.; Munagala, K.; Xiax, L. Incentive Compatible Budget Elicitation in Multi-Unit Auctions. In Proceedings of the 21st Annual ACM-SIAM SODA, Austin, TX, USA, 17–19 January , 2010; pp. 554–572.

19. Aggarwal, G.;Muthukrishnan, S.; Pa'l, D.; Pa'l, M. General Auction Mechanism for Search Advertising. In Proceedings of the 18th WWW '09, New York, NY, USA, 20–24 April , 2009; pp. 241–250.
20. Ashlagi, I.; Braverman, M.; Hassidim, A.; Lavi, R.; Tennenholtz, M. Position auctions with budgets: Existence and uniqueness. *B.E. J. Theor. Econ. (Adv.)* **2010**, *10 (1)*
21. Che, Y.K.; Gale, I. Standard auctions with financially constrained bidders. *Rev. Econ. Stud.* **1998**, *65*, 1–21.
22. Benoit, J.P.; Krishna, V. Multiple object auctions with budget constrained bidders. *Rev. Econ. Stud.* **2001**, *68*, 155–179.
23. Laffont, J.J.; Robert, J. Optimal auction with financially constrained buyers. *Econ. Lett.* **1996**, *52*, 181–186.
24. Optimal auctions with financially constrained bidders. Available online: http://ideas.repec.org/p/nwu/cmsems/1471.html (accessed on 29 April 2014).