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Review

Matrix-Based Method for the Analysis and Control of Networked Evolutionary Games: A Survey

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Abstract: In this paper, a detailed survey is presented for the analysis and control of networked evolutionary games via the matrix method. The algebraic form of networked evolutionary games is firstly recalled. Then, some existing results on networked evolutionary games are summarized. Furthermore, several generalized forms of networked evolutionary games are reviewed, including networked evolutionary games with time delay, networked evolutionary games with bankruptcy mechanism, networked evolutionary games with time-varying networks, and random evolutionary Boolean games. The computational complexity of general networked evolutionary games is still challenging, which limits the application of the matrix method to large-scale networked evolutionary games. Future works are finally presented in the conclusion.

Keywords: networked evolutionary games; time delay; bankruptcy mechanism; random evolutionary Boolean game; semi-tensor product of matrices

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1. Introduction

Evolutionary game theory was introduced by biologists to investigate the evolution of species [1–4]. The last few decades have witnessed the wide applications of evolutionary game theory in communications, networking, and social physics [5–8]. In the evolutionary game, an important concept is the evolutionarily stable strategy, which was proposed by Smith and Price in [1]. Different from the classical Nash equilibrium, which is immune to the strategy deviation of one player, the evolutionarily stable strategy prevents several players choosing alternative strategies, which is an equilibrium refinement of the classical Nash equilibrium [9]. Another essential concept of evolutionary game theory is replicator dynamics, which is one of the most important continuous evolutionary dynamics [10]. The replicator dynamics indicate whether or not the evolution will converge to a certain profile, which provides an elegant and powerful means to investigate the evolutionarily stable strategy [11]. In the last few years, both the evolutionarily stable strategy and replicator dynamics have been widely discussed [12–14]. Particularly, the relation between the evolutionarily stable strategy and the dynamical equilibria of replicator dynamics was explored in [15].

Classic evolutionary games are based on uniformly mixed forms, i.e., they assume each player plays with all others or randomly chosen ones. Due to the influence of a complex environment, each player in the evolutionary game may only capture the information from a part of the players. Accordingly, the topological structure between players plays a significant role in the evolutionary games, which can be described in terms of a network [16,17]. The evolutionary games played on the networks are called networked evolutionary games, which has attracted extensive attention from physicists [17], mathematical biologists [18,19], and so on [20–23]. In the network, nodes represent players, and edges describe the interaction relationship among players. The existing results on networked evolutionary games are mainly based on statistical approximation [24] and simulation analysis [18], which are not convenient for the theoretical analysis and optimization of general networked evolutionary games [25].

Recently, a new mathematical tool, the semi-tensor product (STP) of matrices, has been proposed and successfully applied to the analysis and control of logical dynamic systems [26,27]. Using the STP method, the topological structure and stability of logical networks were deeply investigated [28,29]. Later, several basic control problems of logical control networks were studied, such as stabilization [30–35], controllability [36,37], optimal control [38–40], and observability [41]. For some generalized forms of logical dynamic systems, the analysis and control problems were also extensively discussed, including delayed logical networks [42–44], probabilistic logical networks [45,46], switched logical networks [40,47], and perturbed logical networks [48–51].

The STP method was also used for the modeling, analysis, and control of networked evolutionary games [52–54]. As soon as the strategy updating rule is assigned, the fundamental evolutionary equation can be determined as a k-valued logical dynamic system via the local information. Accordingly, the fundamental evolutionary equation can be expressed in an algebraic form by using STP. Then, from the fundamental evolutionary equations of all the players, the strategy profile dynamics of networked evolutionary games can be constructed [52]. The fundamental evolutionary equation and strategy profile dynamics are crucial to analyzing the dynamic behaviors of networked evolutionary games. Under the matrix-based framework, several fundamental problems of networked evolutionary games were considered via the existing results of logical networks. For example, based on the topological structure analysis of logical networks, a criterion was proposed for verifying the existence of stationary stable profiles [52,55]. Since then, several other meaningful problems of networked evolutionary games have been well investigated, including Nash equilibrium [56,57], potential equation [58], stable degree of strategy profiles [59], strategy consensus [60], and strategy optimization [61,62]. Particularly, the evolutionarily stable strategy of both homogeneous and heterogeneous networked evolutionary games was investigated via the STP method [63]. However, there exist fewer results discussing the relation between the evolutionarily stable strategy and the replicator dynamics for networked evolutionary games [64].

In some recent studies, the matrix-based framework has been extended to networked evolutionary games with various generalized forms; for example, networked evolutionary games with time delay [65–68], networked evolutionary games with bankruptcy mechanism [69–72], networked evolutionary games with time-varying networks [73–76], and random evolutionary Boolean games [77–81]. In this paper, we present a detailed survey of the recent development of networked evolutionary games and their generalized forms via the matrix-based method.

The remainder of this paper is organized as follows. Section 2 presents the definition and mathematical model of networked evolutionary games. Then, the model and recent developments corresponding to several generalized forms of networked evolutionary games are introduced in Section 3. Section 4 is a brief conclusion.

2. Preliminaries

2.1. Networked Evolutionary Games

In this subsection, we review the model of networked evolutionary games [52]. Before we give the definition of networked evolutionary games, we briefly recall some basic concepts, including a network graph, fundamental network game, and strategy updating rule.

Definition 1. *A network graph is defined by (N, E), where $N = \{1, \cdots, n\}$ and*

$$E = \{(\alpha, \beta) \ : \ \alpha, \beta \in N, \textit{ and there exists an edge from } \alpha \textit{ to } \beta\}$$

represent the set of nodes and the set of edges, respectively. (N, E) is called an undirected graph if $(\alpha, \beta) \in E$ implies that $(\beta, \alpha) \in E$. Otherwise, it is called a directed graph. Furthermore, $\alpha \in N$ is called the l-neighbor of $\beta \in N$, denoted by $\alpha \in U_l(\beta)$, if there exists a path between α and β with length $0 \leq l' \leq l$. In particular, $U_0(\beta) := \{\beta\}$.

A network graph (N, E) is said to be homogeneous if either it is directed and all nodes have the same in-degree and out-degree or it is undirected and $|U_1(\alpha)| = |U_1(\beta)|$, $\forall\, \alpha, \beta \in N$. Otherwise, it is said to be heterogeneous.

A normal form game consists of three fundamental ingredients: (i) the set of n players $N = \{1, \cdots, n\}$; (ii) the set of strategies for each player $S_\alpha = \{1, \cdots, k_\alpha\}$, $\forall\, \alpha \in N$, where we denote $S := \prod_{\alpha=1}^{n} S_\alpha$ as the set of profiles; (iii) the payoff function $c_\alpha : S \to \mathbb{R}$, $\forall\, \alpha \in N$.

Definition 2. *A normal form game is called a fundamental network game if* $N = \{1, 2\}$ *and* $S_1 = S_2 := S = \{1, \cdots, k\}$.

Definition 3. *A strategy updating rule of a networked evolutionary game, denoted by* Π*, is expressed as*

$$x_\alpha(t+1) = f_\alpha(x_\beta(t), c_\beta(t); \beta \in U_1(\alpha)),\ \forall\, \alpha \in N, \tag{1}$$

where $x_\alpha(t)$ *represents the strategy of player* α *at time* t*,* $c_\alpha(t)$ *represents the payoff of player* α *at time* t *which is usually calculated by*

$$c_\alpha(t) = \frac{1}{|U_1(\alpha) - 1|} \sum_{\beta \in U_1(\alpha) \setminus \{\alpha\}} c_{\alpha,\beta}(t),\ \forall\, \alpha \in N \tag{2}$$

or

$$c_\alpha(t) = \sum_{\beta \in U_1(\alpha) \setminus \{\alpha\}} c_{\alpha,\beta}(t),\ \forall\, \alpha \in N, \tag{3}$$

$c_{\alpha,\beta}(t)$ *represents the payoff of player* α *in the fundamental network game with player* β *at time* t*, and* f_α *represents a mapping deciding the strategy of player* α *at the next time.*

There are several common strategy updating rules, such as unconditional imitation with fixed priority [18], unconditional imitation with equal probability [52], myopic best response adjustment [82], and the simplified Fermi rule [83]. We briefly introduce these strategy updating rules below. We denote $x(t) := (x_1(t), \cdots, x_n(t)) \in S$.

- Unconditional imitation with fixed priority: If $\beta^* = \arg\max_{\beta \in U_1(\alpha)} c_\beta(t)$, then $x_\alpha(t+1) = x_{\beta^*}(t)$. If the arg max set is not a singleton, that is, $\arg\max_{\beta \in U_1(\alpha)} c_\beta(t) = \{\beta_1^*, \cdots, \beta_r^*\}$ and $r \geq 2$, then

$$x_\alpha(t+1) = x_{\beta^*}(t),\ \beta^* = \min\{\beta_1^*, \cdots, \beta_r^*\}. \tag{4}$$

- Unconditional imitation with equal probability: If $\beta^* = \arg\max_{\beta \in U_1(\alpha)} c_\beta(t)$, then $x_\alpha(t+1) = x_{\beta^*}(t)$. Otherwise, $\arg\max_{\beta \in U_1(\alpha)} c_\beta(t) = \{\beta_1^*, \cdots, \beta_r^*\}$ and $r \geq 2$ is satisfied. In this case, let

$$\mathbb{P}\{x_\alpha(t+1) = x_{\beta_\mu^*}(t)\} = \frac{1}{r},\ \mu \in \{1, \cdots, r\}. \tag{5}$$

- Myopic best response adjustment: Denote $O_\alpha(t) = \arg\max_{x_\alpha \in S_\alpha} c_\alpha(x_\alpha, x_{-\alpha}(t))$, where $x_{-\alpha} \in S_{-\alpha} = \prod_{\beta \neq \alpha} S_\beta$. If $x_\alpha(t) \in O_\alpha(t)$, then $x_\alpha(t+1) = x_\alpha(t)$. If $x_\alpha(t) \notin O_\alpha(t)$ and $|O_\alpha(t)| = 1$, then $x_\alpha(t+1) = O_\alpha(t)$. Otherwise, assume $O_\alpha(t) = \{\beta_1^*, \cdots, \beta_r^*\}$ and $r \geq 2$. Then, (4) or (5) can be used.
- Simplified Fermi rule: Randomly choose a neighbor $\beta \in U_1(\alpha) \setminus \{\alpha\}$. Let

$$x_\alpha(t+1) = \begin{cases} x_\alpha(t), \text{ if } c_\alpha(t) \geq c_\beta(t), \\ x_\beta(t), \text{ otherwise.} \end{cases} \tag{6}$$

Next, we give the concept of a networked evolutionary game as follows [52].

Definition 4. *A networked evolutionary game, denoted by* $((N, E), G, \Pi)$*, consists of three fundamental ingredients as follows:*

(i) *A network graph* (N, E)*;*
(ii) *A fundamental network game G. If* $(\alpha, \beta) \in E$*, then players* α *and* β *play the fundamental network game G;*
(iii) *A strategy updating rule* Π.

The networked evolutionary game is a special kind of game on graphs [84] where the payoff and strategy updating rule of each player only depend on the actions and payoffs of their 1-neighbor in the graph. Furthermore, a networked evolutionary game $((N, E), G, \Pi)$ is said to be homogeneous if the network graph (N, E) is homogeneous. Otherwise, the networked evolutionary game is said to be heterogeneous.

Consider a networked evolutionary game $((N, E), G, \Pi)$. A profile $s^* = (s_1^*, \cdots, s_n^*) \in S$ is called a Nash equilibrium if

$$c_\alpha(s_\alpha, s_{-\alpha}^*) \leq c_\alpha(s_\alpha^*, s_{-\alpha}^*),\ \forall\, s_\alpha \in S_\alpha,\ \forall\, \alpha \in N.$$

2.2. Mathematical Modeling of Networked Evolutionary Games

In this subsection, we review the fundamental evolutionary equation of networked evolutionary games. Furthermore, based on the STP method, we establish the algebraic forms of the fundamental evolutionary equation and strategy profile dynamics.

Note that $c_\beta(t)$ in the strategy updating rule (1) depends on the strategies of its 2-neighbor. Thus, the strategy updating rule (1) can be further expressed as

$$x_\alpha(t+1) = g_\alpha(x_\beta(t); \beta \in U_2(\alpha)),\ \forall\, \alpha \in N. \tag{7}$$

In the following, we call (7) the fundamental evolutionary equation of player α, $\forall\, \alpha \in N$.

In fact, the fundamental evolutionary Equation (7) is a k-valued logical dynamic system. Then, letting $i \sim \delta_k^i$ and $x = \ltimes_{\alpha=1}^n x_\alpha$ and using the properties of STP, one can convert the fundamental evolutionary Equation (7) into an equivalent algebraic form:

$$x_\alpha(t+1) = L_\alpha x(t), \tag{8}$$

where $L_\alpha \in \mathcal{L}_{k\times k^n}$ or $L_\alpha \in \Upsilon_{k\times k^n}$ is satisfied, which is determined by the specific strategy updating rule. Based on the fundamental evolutionary Equation (7), the strategy profile dynamics can be defined as

$$x(t+1) = Lx(t), \tag{9}$$

where $L = L_1 * \cdots * L_n$ is called the profile transition matrix.

The following example is used to illustrate the procedure of obtaining the profile transition matrix.

Example 1. *Given a networked evolutionary game, the network graph and payoff bi-matrix are shown in Figure 1 and Table 1, respectively.*

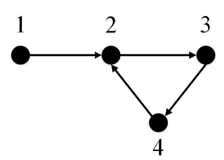

Figure 1. The network graph.

Table 1. Payoff bi-matrix.

$P_\alpha \backslash P_\beta$	**1**	**2**
1	$(2,1)$	$(0,0)$
2	$(0,0)$	$(1,2)$

(i) Assume that the strategy updating rule is an unconditional imitation with fixed priority. Then, f_α, $\alpha = 1,2,3,4$ can be figured out as in Table 2. Letting $1 \sim \delta_2^1$ and $2 \sim \delta_2^2$, one has the algebraic form of fundamental evolutionary equation as $x_\alpha(t+1) = L_\alpha x(t)$, $\alpha = 1,2,3,4$ with $L_1 = \delta_2[1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2]$, $L_2 = \delta_2[1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2]$, $L_3 = \delta_2[1\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 2]$ and $L_4 = \delta_2[1\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 2]$. Then, the strategy profile dynamics is derived below:

$$x(t+1) = Lx(t),$$

where $L = L_1 * L_2 * L_3 * L_4 = \delta_{16}[\ 1\ 1\ 1\ 4\ 1\ 16\ 16\ 16\ 1\ 1\ 1\ 16\ 13\ 16\ 16\ 16]$.

(ii) Consider the myopic best response strategy with equal probability. In this situation, the algebraic form of fundamental evolutionary equation is $x_\alpha(t+1) = \hat{L}_\alpha x(t)$, $\alpha = 1,2,3,4$ with $\hat{L}_1 = \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2]$, $\hat{L}_2 = \delta_2[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2]$,

$$\hat{L}_3 = \begin{bmatrix} 1 & 1/2 & 1 & 1/2 & 1/2 & 0 & 1/2 & 0 & 1 & 1/2 & 1 & 1/2 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 & 1 & 0 & 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 & 1 \end{bmatrix}$$

and

$$\hat{L}_4 = \begin{bmatrix} 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 & 1 \end{bmatrix}$$

Accordingly, the strategy profile dynamics is derived below:

$$x(t+1) = \hat{L}x(t),$$

where $\hat{L} =$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

Table 2. From payoffs to dynamics.

payoff	1111	1112	1121	1122	1211	1212	1221	1222
c_1	2	2	2	2	0	0	0	0
c_2	4/3	1	2/3	1/3	0	2/3	1/3	1
c_3	3/2	1/2	0	1/2	1	0	1	3/2
c_4	3/2	0	1	1	1/2	1/2	0	3/2
payoff	2111	2112	2121	2122	2211	2212	2221	2222
c_1	0	0	0	0	1	1	1	1
c_2	1	2/3	1/3	0	2/3	4/3	1	5/3
c_3	3/2	1/2	0	1/2	1	0	1	3/2
c_4	3/2	0	1	1	1/2	1/2	0	3/2

Recently, using the STP method, the networked evolutionary game has been widely studied, including the topological structure [57], Nash equilibrium [56,57], evolutionary stable strategy [63], stable degree of strategy profiles [59], and strategy consensus [60].

3. Networked Evolutionary Games with Generalized Forms

In this section, we recall some new developments for several generalized forms of networked evolutionary games, including networked evolutionary games with time delay, networked evolutionary games with bankruptcy mechanism, networked evolutionary games with time-varying networks, and random evolutionary Boolean games.

3.1. Networked Evolutionary Games with Time Delay

It is usually assumed that each player updates the strategy at the next time only based on the strategies of its neighbors at the last time, such as in (7). Actually, due to the existence of information channel congestion and human interference, the information may need a certain time to transfer, which means that time delay is inevitable in many practical process of information transmission. Furthermore, as was discussed in [67,85], the player can remember the strategies of its neighbors in the last finite steps. Therefore, it is reasonable to consider networked evolutionary games with time delay.

A networked evolutionary game with time delay is described as $((N,E),G,\Pi-I)$, where (N,E) and G are the same as in Definition 4, and the strategy updating rule $\Pi-I$ is expressed as

$$x_\alpha(t+1)=f_\alpha(x_\beta(t-\tau+1),\cdots,x_\beta(t),c_\beta(t-\tau+1),\cdots,c_\beta(t):\beta\in U_1(\alpha))\,,\alpha\in N. \quad (10)$$

Particularly, if the strategies of all players at the next time only depend on the strategies of their neighbors at time $t-\tau+1$, then strategy updating rule (10) can be simplified as

$$x_\alpha(t+1)=f_\alpha(x_\beta(t-\tau+1),c_\beta(t-\tau+1):\beta\in U_1(\alpha)),\ \alpha\in N. \quad (11)$$

Similar to the derivation of (7), the fundamental evolutionary equations of networked evolutionary games with strategy updating rules (10) and (11) can be expressed as

$$x_\alpha(t+1)=g_\alpha(x_\beta(t-\tau+1),\cdots,x_\beta(t):\beta\in U_2(\alpha)),\ \alpha\in N \quad (12)$$

and

$$x_\alpha(t+1)=g_\alpha(x_\beta(t-\tau+1):\beta\in U_2(\alpha)),\ \alpha\in N, \quad (13)$$

respectively. Since $x_\alpha\in S_\alpha$, $\forall\ \alpha\in N$, (12) and (13) are indeed logical dynamic systems with time delay, which can be converted into equivalent algebraic forms by using the STP method. In recent years, a large number of excellent results has been proposed for investigating the logical dynamic systems with time delay, including topological structure [86], stability [87,88], stabilization [44,89,90], controllability [91–95], and observability [95–97].

For networked evolutionary games with time delay, the profile trajectory was proposed to describe the strategy updating process, and then the Nash equilibrium and convergence of networked evolutionary games with time delay were investigated [65–68]. Specifically, Wang and Cheng [66] proved that a potential networked evolutionary game with time-invariant delay (11) converges to a pure Nash equilibrium under a kind of myopic best response adjustment rule. Zhao et al. [67] verified the existence of a Nash equilibrium for the networked evolutionary game with time-invariant delay (10) and designed a free-type strategy sequence to guarantee that the considered networked evolutionary game converges to the Nash equilibrium. Furthermore, the dynamics of networked evolutionary game with other kinds of time delay were also discussed, including different length delay [65] and switched time delay [68].

3.2. Networked Evolutionary Games with Bankruptcy Mechanism

It is universal that players exit from a game [69,72]. For example, due to the low profits or other reasons, several financial institutions may go bankrupt in a short time. Another example is the death of individuals in a practical ecosystem. In these economical or biological systems, any individual should maintain the lowest amount of profit to survive, and otherwise, individuals will disappear from the game. Therefore, it is meaningful to introduce a bankruptcy mechanism into the networked evolutionary games.

In the networked evolutionary games with a bankruptcy mechanism, a new strategy "bankruptcy" represents the situation that the player is bankrupt. Furthermore, given $0 \leqslant a_\alpha \leqslant 1$, we let $T_\alpha = a_\alpha P_\alpha^N$ be the survival payoff for player α, where P_α^N is the payoff of player α when all players choose cooperation.

Then, a networked evolutionary game with bankruptcy mechanism is described as $((N,E),G,\Pi - II)$, where G represents the snowdrift game or hawk-dove game. The payoff bi-matrix of fundamental network game G is shown in Table 3, and the strategy update rule $\Pi - II$ is expressed as

$$x_\alpha(t+1) = \begin{cases} \text{bankruptcy}, \text{ if } c_\alpha < T_\alpha, \\ f_\alpha\big(\{x_\beta(t), c_\beta(t) : \beta \in U_1(\alpha)\}\big), \text{ otherwise}, \end{cases} \tag{14}$$

c_α is defined in (3), and f_α is the unconditional imitation updating rule.

Table 3. Payoff bi-matrix for a game with bankruptcy.

$P_\alpha \backslash P_\beta$	Cooperate	Defect	Bankruptcy
cooperate	(R,R)	(S,T)	$(0,0)$
defect	(T,S)	(P,P)	$(0,0)$
bankruptcy	$(0,0)$	$(0,0)$	$(0,0)$

By incorporating a bankruptcy mechanism, Wang et al. [72] investigated the catastrophic behaviors in evolutionary games via the computer simulation method. Using STP method, Wang et al. [70] proposed an algebraic framework for the networked evolutionary games with a bankruptcy mechanism and studied the strategy optimization control problem. Furthermore, the strategy optimization of networked evolutionary games with memories under the bankruptcy mechanism was investigated in [69]. Recently, a state feedback controller has been designed to maximize the long-term average payoff of networked evolutionary games with a bankruptcy mechanism [71].

3.3. Networked Evolutionary Games with Time-Varying Networks

It is clear that many economic activities indicate an obvious fact that each player is unceasingly able to choose to abandon their opponents for more payoff. Correspondingly, the topology structure of a network graph is changed along with the evolutionary game. As was proved in [98], it is indeed possible for players to make some new permanent connections with neighbors who have not yet linked. Thus, it is reasonable to consider a type of NEG with time-varying networks.

A networked evolutionary game with time-varying networks consists of four ingredients: (i) m network graphs $\mathcal{M} := \{1, 2, \cdots, m\}$, where we denote (N, E_z) as the z-th network graph, where E_z is the set of edges in the z-th network graph, $z \in \mathcal{M}$; (ii) a fundamental network game G; (iii) a player's strategy updating rule, which is expressed as

$$x_\alpha(t+1) = f_{\alpha,z}\big(x_\beta(0), x_\beta(1), \ldots, x_\beta(t) : \beta \in U_{1,z}(\alpha)\big),\ \forall\, \alpha \in N, \tag{15}$$

where $U_{1,z}(\alpha)$ is the neighbors of player α in the z-th network graph; (iv) a network updating rule, which is expressed as

$$z(t) = h\big(x(0), x(1), \ldots, x(t)\big), \tag{16}$$

where $h : S^{t+1} \to \mathcal{M}$ determines the selection of a network graph at time t.

In the following, we particularly introduce the network updating rule.

Denote $c^z_\alpha(t) = \sum_{\beta \in U_{1,z}(\alpha)} c_{\alpha,\beta}(t)$ and $Q_{\alpha,z}(t) = \arg\max_{x_\alpha \in S_\alpha} c^z_\alpha(x_\alpha, x_{-\alpha}(t-1))$ as the payoff of player α and the strategy adopted by player α at time t in the z-th network, respectively. Then, the expected revenue function of player α at time t is defined as $ER_{\alpha,z}(x^*_\alpha, x_{-\alpha}(t-1)) = c^z_\alpha(x^*_\alpha, x_{-\alpha}(t-1))$, where $x^*_\alpha \in Q_{\alpha,z}(t)$.

For the player α, the network that maximizes the payoff at time t is

$$W_\alpha\big(x(t-1)\big) : \ = \arg\max_{z \in \mathcal{M}} ER_{\alpha,z}(x^*_\alpha, x_{-\alpha}(t-1)),\ x^*_\alpha \in Q_{\alpha,z}. \tag{17}$$

Then, the number of players who want to attend the z-th network at time t is

$$\delta_z\big(x(t-1)\big) = \Big|\big\{\alpha \mid \alpha \in N \text{ and } z \in W_\alpha\big(x(t-1)\big)\big\}\Big|, z \in \mathcal{M}. \tag{18}$$

The network updating rule can be described as follows: If $z^* = \arg\max_{z \in \mathcal{M}} \delta_z\ \big(x(t-1)\big)$, then $z(t) = z^*$; if $|\mathcal{P}_l > 1|$, then

$$z(t) = \ \max\big\{z^* \mid z^* \in \mathcal{M} \text{ and } z^* \in \arg\max_{z \in \mathcal{M}} \delta_z\big(x(t-1)\big)\big\}. \tag{19}$$

The algebraic form of networked evolutionary games with time-varying networks was established in [75]. Based on the algebraic form, Zhu et al. [76] and Yuan et al. [74] investigated the strategy optimization problem of networked evolutionary games with time-varying networks. A free-type strategy sequence was designed to guarantee the convergence of Nash equilibrium [75]. Furthermore, Fu et al. [73] investigated the networked evolutionary games with finite memories and time-varying networks and revealed the relationship between the strict Nash equilibriums and the fixed points of given networked evolutionary games.

3.4. Random Evolutionary Boolean Games

An n-person random evolutionary Boolean game with the fixed strategy updating rule can be shown as follows:

$$\begin{cases} x_\alpha(t+1) = f_\alpha\big(X(t), w_\alpha(t, p_\alpha), y(t)\big),\ \alpha = 1, \cdots, n; \\ y(t) = h\big(X(t)\big), \end{cases} \tag{20}$$

where $X(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big) \in D^n$ represents the strategy profile at time t, $w_\alpha(t, p_\alpha) \in D$ is a random variable satisfying $\mathbb{P}\{w_\alpha(t, p_\alpha) = 1\} = p_\alpha$ and $0 \leqslant p_\alpha \leqslant 1$ which represents the possibility for each player to make right choice, $y(t) \in D$ is the result of the game, $f_\alpha : D^{n+2} \to D, \alpha = 1, 2, \ldots, n$ and $h : D^n \to D$ are Boolean functions.

Adding pseudo-players as the control in an n-person random evolutionary Boolean game, the n-person random evolutionary Boolean game can be expressed as

$$\begin{cases} x_\alpha(t+1) = f_\alpha\big(X(t), U(t), w_\alpha(t, p_\alpha), y(t)\big), \ \alpha = 1, \cdots, n; \\ y(t) = h\big(X(t)\big), \end{cases} \tag{21}$$

where $X(t) \in D^n$, $y(t) \in D$, $w_\alpha(t, p_\alpha) \in D$ are the same to that in (20), $U(t) = \big(u_1(t), \ldots, u_m(t)\big) \in D^m$ represents pseudo-players' the strategy profile, $f_\alpha : D^{n+m+2} \to D, \alpha = 1, 2, \ldots, n$ and $h : D^n \to D$ are Boolean functions.

Using the STP method, the necessary and sufficient conditions were proposed for the set stabilization of n-person random evolutionary Boolean games [77,79]. Furthermore, the optimal control problem of n-person random evolutionary Boolean games was studied in [78].

As another type of random networked evolutionary games, random entrance was introduced to deal with the case that the number of new players attending the game at any time is a random variable [81]. In the network graph of networked evolutionary games with random entrance, the nodes consist of a major player and active minor players, and the edges exist only between the major player and the minor players. The network is determined by the random entrance. For the networked evolutionary games with random entrance, Zhao et al. [81] designed a state feedback controller to ensure the maximum payoffs of major player. After that, a class of networked evolutionary games with both random entrance and time delays was studied in [80].

3.5. Some Related Findings of STP Method

Several other methods are available for studying the networked evolutionary games, including simulation-based analysis and a statistical approach. The characteristics of these method were shown in Table 4. Under the simulation-based analysis, several evolutionary games on special networks were studied in [18,22]. In recent years, Martin Nowak's group has made several significant contributions to the analysis of networked evolutionary games by using statistical models [99–101].

Table 4. Comparative analysis of various methods for networked evolutionary games.

Method	Benefits	Limitations
Simulation-based analysis	Efficient for special networked evolutionary games	Not convenient for theoretical analysis
Statistical	Powerful when dealing with large-scale networked evolutionary games	Not convenient for theoretical analysis
STP	Convenient for the theoretical analysis of general networked evolutionary games	Computational complexity hampers its application to large-scale networks

Using the STP method to study networked evolutionary games has several unique advantages. On one hand, the dynamics of a networked evolutionary game can be transformed into an algebraic form, based on which the methods and tools in classical control theory can be used to analyze and control networked evolutionary games directly. On the other hand, the methods and results of Boolean games can be easily generalized to multi-strategy games.

It should be noted that the computational complexity of analyzing and controlling networked evolutionary games based on the STP method is exponential regarding the number of players, since it is required to handle matrices of size $k^n \times k^n$ or even larger. As discussed in [102], the STP method cannot be used to handle Boolean networks with more than approximately 30 nodes in practice. Thus, the STP method is hard to use in large-scale networked evolution games. However, practical social networks often have a large number of players [103]. Accordingly, it is a challenge and potential research gap to reduce the computational complexity of the STP method and make it more applicable to large-scale networked evolution games.

4. Conclusions

In this paper, we have recalled several new developments for the algebraic form of networked evolutionary games. Furthermore, we have reviewed some generalized forms of networked evolutionary games, including networked evolutionary games with time delay, networked evolutionary games with bankruptcy mechanism, networked evolutionary games with time-varying networks, and random evolutionary Boolean games. Then, we have briefly summarized the existing excellent results of networked evolutionary games with generalized forms. Finally, we have comparatively analyzed the various existing methods for networked evolutionary games and pointed out the benefits and limitations of the STP method.

From this survey, one can see that most of the existing works only pay attention to the theoretical investigation of modeling, Nash equilibrium, convergence, and the strategy optimization problem in networked evolutionary games. Actually, evolutionary game theory is widely used in communications, networking, and social physics. The theoretical results on the bankruptcy mechanism and time-varying networks can be further explored in some practical scenarios. In addition, the payoff matrix is fixed in the existing results, which can be assumed to be changed [104,105] in future works. Note that computational complexity is the main obstacle when using a matrix-based method to investigate the considered networked evolutionary games. Future work will consider the problem of reducing the computational complexity.

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