sustainability

MDPI

Equation (7): The detailed derivation of the Bayesian updating

The detailed derivation of the Bayesian updating for the normal distribution is presented in this supplementary material. Recall Equation (3) of the text.

$$f(\mu|x) = cf(\mu)f(x|\mu) = \left[\int_{-\infty}^{\infty} f(\mu)f(x|\mu)\,d\mu\right]^{-1} f(\mu)f(x|\mu) \tag{S.1}$$

In Equation (S.1), an unknown parameter μ for which the prior beliefs can be express in terms of a normal distribution, so that

$$f(\mu)\sim N(\mu_0,\sigma_0^2) = \frac{1}{\sqrt{2\pi\sigma_0^2}} exp\left\{\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \tag{S.2}$$

Where μ_0 and σ_0^2 are known, and, if the likelihood uncertainty (σ) is known, Equation (S.3) can be defined as follow.

$$f(x|\mu)\sim N(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} \tag{S.3}$$

In this case, the derivation of the posterior distribution when μ given that we have on observation data x can be calculated as follows.

$$\begin{aligned}
f(\mu|x) &= c \times \frac{1}{\sqrt{2\pi\sigma_0^2}} exp\left\{\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \times \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} \\
&= c \times \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\mu^2+2\mu\mu_0-\mu_0^2}{2\sigma_0^2} - \frac{x^2-2\mu x+\mu^2}{2\sigma^2}\right\} \\
&= c \times \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\mu^2\sigma^2+2\mu\mu_0\sigma^2-\mu_0^2\sigma^2-x^2\sigma_0^2+2\mu x\sigma_0^2-\mu^2\sigma_0^2}{2\sigma_0^2\sigma^2}\right\} \\
&= c \times \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\mu^2(\sigma^2+\sigma_0^2)+2\mu(\mu_0\sigma^2+\sigma_0^2x)-(\mu_0^2\sigma^2+\sigma_0^2x^2)}{2\sigma_0^2\sigma^2}\right\} \\
&= c \times \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\mu^2+2\mu\frac{(\mu_0\sigma^2+\sigma_0^2x)}{(\sigma^2+\sigma_0^2)} - \frac{(\mu_0\sigma^2+\sigma_0^2x)^2}{(\sigma^2+\sigma_0^2)^2} + \frac{(\mu_0\sigma^2+\sigma_0^2x)^2}{(\sigma^2+\sigma_0^2)^2} - \frac{\mu_0^2\sigma^2+\sigma_0^2x^2}{(\sigma^2+\sigma_0^2)}}{\frac{2\sigma_0^2\sigma^2}{(\sigma^2+\sigma_0^2)}}\right\} \\
&= c \times \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\left(\mu-\frac{\mu_0\sigma^2+\sigma_0^2x}{\sigma^2+\sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{(\sigma^2+\sigma_0^2)}}\right\} \times exp\left\{\frac{(\mu_0\sigma^2+\sigma_0^2x)^2}{2\sigma_0^2\sigma^2(\sigma^2+\sigma_0^2)}\right\} \times exp\left\{\frac{-(\mu_0^2\sigma^2+\sigma_0^2x^2)}{2\sigma_0^2\sigma^2}\right\}
\end{aligned} \tag{S.4}$$

In this step, a normalizing constant c is defined by

$$\begin{aligned}
c &= \left[\int_{-\infty}^{\infty}\left[\frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{-\left(\mu-\frac{\mu_0\sigma^2+\sigma_0^2x}{\sigma^2+\sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{(\sigma^2+\sigma_0^2)}}\right\} exp\left\{\frac{(\mu_0\sigma^2+\sigma_0^2x)^2}{2\sigma_0^2\sigma^2(\sigma^2+\sigma_0^2)}\right\} exp\left\{\frac{-(\mu_0^2\sigma^2+\sigma_0^2x^2)}{2\sigma_0^2\sigma^2}\right\}\right] d\mu\right]^{-1} \\
&= \left[\frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} exp\left\{\frac{(\mu_0\sigma^2+\sigma_0^2x)^2}{2\sigma_0^2\sigma^2(\sigma^2+\sigma_0^2)}\right\} exp\left\{\frac{-(\mu_0^2\sigma^2+\sigma_0^2x^2)}{2\sigma_0^2\sigma^2}\right\} \int_{-\infty}^{\infty}\left[exp\left\{\frac{-\left(\mu-\frac{\mu_0\sigma^2+\sigma_0^2x}{\sigma^2+\sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{(\sigma^2+\sigma_0^2)}}\right\}\right] d\mu\right]^{-1}
\end{aligned} \tag{S.5}$$

To calculate $f(\mu|x)$, Equation (S.5) are inserted into c of Equation (S.4) to obtain Equation (S.6):

$$f(\mu|x) = \left[\int_{-\infty}^{\infty}\left[exp\left\{\frac{-\left(\mu - \frac{\mu_0\sigma^2 + \sigma_0^2 x}{\sigma^2 + \sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{(\sigma^2 + \sigma_0^2)}}\right\}\right]d\mu\right]^{-1} exp\left\{\frac{-\left(\mu - \frac{\mu_0\sigma^2 + \sigma_0^2 x}{\sigma^2 + \sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{(\sigma^2 + \sigma_0^2)}}\right\} \tag{S.6}$$

Letting

$$\mu_1 = \frac{\mu_0\sigma^2 + \sigma_0^2 x}{\sigma^2 + \sigma_0^2} \tag{S.7}$$

$$\sigma_1^2 = \frac{\sigma_0^2\sigma^2}{(\sigma^2 + \sigma_0^2)} \tag{S.8}$$

The Equation (S.6) can be redefined with Equations (S.7) and (S.8):

$$\begin{aligned} f(\mu|x) &= \left[\int_{-\infty}^{\infty}\left[exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\}\right]d\mu\right]^{-1} \times exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\} \\ &= \left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_1^2}}\left[exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\}\right]d\mu\right]^{-1} \times \frac{1}{\sqrt{2\pi\sigma_1^2}} exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\} \\ &= [1]^{-1} \times \frac{1}{\sqrt{2\pi\sigma_1^2}} exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\} = \frac{1}{\sqrt{2\pi\sigma_1^2}} exp\left\{\frac{-(\mu-\mu_1)^2}{2\sigma_1^2}\right\} = N(\mu_1, \sigma_1^2) \end{aligned} \tag{S.9}$$

Considering the inspection or monitoring time (T) of a bridge, prior ($f(\mu)$), likelihood ($f(x|\mu)$), and posterior ($f(\mu|x)$) can be redefined as follows.

$$f(\mu) \sim N\left(\mu_T^{exi}, (\sigma_T^{exi})^2\right) = \frac{1}{\sqrt{2\pi\,(\sigma_T^{exi})^2}} exp\left\{\frac{-\left(\mu - \mu_T^{exi}\right)^2}{2\,(\sigma_T^{exi})^2}\right\} \tag{S.10}$$

$$f(x|\mu) \sim N\left(\mu_T^{ins}, (\sigma_T^{ins})^2\right) = \frac{1}{\sqrt{2\pi\,(\sigma_T^{ins})^2}} exp\left\{\frac{-\left(x - \mu_T^{ins}\right)^2}{2\,(\sigma_T^{ins})^2}\right\} \tag{S.11}$$

$$f(\mu|x) \sim N\left(\mu_T^{bayes}, (\sigma_T^{bayes})^2\right) = \frac{1}{\sqrt{2\pi\,(\sigma_T^{bayes})^2}} exp\left\{\frac{-\left(x - \mu_T^{bayes}\right)^2}{2\,(\sigma_T^{bayes})^2}\right\} \tag{S.11}$$

Also, these are equal to Equations (4) to (6) of the text.

Equation (8): The detailed derivation of the characteristic for the posterior variance

The variance of prior and likelihood function of the text has the characteristic as follows.

$$\sigma_T^{exi} \geq 0 \Leftrightarrow (\sigma_T^{exi})^2 \geq 0 \tag{S.12}$$

$$\sigma_T^{ins} \geq 0 \Leftrightarrow (\sigma_T^{ins})^2 \geq 0 \tag{S.13}$$

According to the characteristic in Equations (S.12) and (S.13), Equations (S.14) and (S.15) can be derived.

$$(\sigma_T^{exi})^2 \times \left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right) \geq (\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2 \Leftrightarrow (\sigma_T^{exi})^2 \geq \frac{(\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2}{\left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right)} \tag{S.14}$$

$$(\sigma_T^{ins})^2 \times \left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right) \geq (\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2 \Leftrightarrow (\sigma_T^{ins})^2 \geq \frac{(\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2}{\left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right)} \tag{S.15}$$

Also, these Equations (S.14) and (S.15) can be expressed as follows.

$$(\sigma_T^{exi})^2 \geq \frac{(\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2}{\left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right)} \Leftrightarrow (\sigma_T^{exi})^2 \geq (\sigma_T^{bayes})^2 \Leftrightarrow \sigma_T^{exi} \geq \sigma_T^{bayes} \tag{S.16}$$

$$(\sigma_T^{ins})^2 \geq \frac{(\sigma_T^{exi})^2 \times (\sigma_T^{ins})^2}{\left((\sigma_T^{exi})^2 + (\sigma_T^{ins})^2\right)} \Leftrightarrow (\sigma_T^{ins})^2 \geq (\sigma_T^{bayes})^2 \Leftrightarrow \sigma_T^{ins} \geq \sigma_T^{bayes} \tag{S.17}$$

So, these Equations (S.16) and (S.17) are equal to Equation (8).