Nonsmooth Levenberg-Marquardt Type Method for Solving a Class of Stochastic Linear Complementarity Problems with Finitely Many Elements

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Abstract: Our purpose of this paper is to solve a class of stochastic linear complementarity problems (SLCP) with finitely many elements. Based on a new stochastic linear complementarity problem function, a new semi-smooth least squares reformulation of the stochastic linear complementarity problem is introduced. For solving the semi-smooth least squares reformulation, we propose a feasible nonsmooth Levenberg–Marquardt-type method. The global convergence properties of the nonsmooth Levenberg–Marquardt-type method are also presented. Finally, the related numerical results illustrate that the proposed method is efficient for the related refinery production problem and the large-scale stochastic linear complementarity problems.

Keywords: nonsmooth equations; stochastic linear complementarity problems; global convergence; Levenberg–Marquardt-type method

1. Introduction

Suppose \((Ω, F, P)\) is a probability space with \(Ω \subseteq \mathbb{R}^n\) and \(P\) is a known probability distribution. A stochastic linear complementarity problem takes the form:

\[
x \geq 0, M(\omega)x + q(\omega) \geq 0, x^T [M(\omega)x + q(\omega)] = 0 \quad \text{a.e.} \quad \omega \in Ω,
\]

where \(Ω \subseteq \mathbb{R}^n\) is the underlying sample space, for given probability distribution \(P\) and \(\forall \omega \in Ω, M(\omega) \in \mathbb{R}^{n \times n}\) and \(q(\omega) \in \mathbb{R}^n\). (1) is denoted as SLCP \((M(\cdot), q(\cdot))\) or SLCP; see [1–4]. If \(Ω\) has only one element, (1) is the standard linear complementarity problem (LCP), which has been studied in [5,6].

Generally, there is no \(x\) satisfying (1) for all \(\omega \in Ω\). In order to obtain a reasonable solution of Problem (1), several types of models have been proposed (one can see [1–4,7–14]). One of them is the expected value (EV) method (see [1]). The EV model is to find a vector \(x \in \mathbb{R}^n\), such that:

\[
0 \leq x \perp Mx + q \geq 0,
\]

where \(\bar{M} = E[M(\omega)]\), \(\bar{q} = E[q(\omega)]\), and \(E[.\] is the mathematical expectation. Another is the expected residual minimization (ERM) method (see [2]). The ERM model is to find a vector \(x \in \mathbb{R}^n\) that minimizes the expected residual function:

\[
\min_{x \geq 0} \sum_{i=1}^n E[\phi(x, M_i(\omega)x + q_i(\omega))]^2,
\]
where \( M_i(\omega) (i = 1, \cdots, n) \) is the \( i \)-th line of random matrix \( M(\omega) \) and \( \varphi \) satisfies:

\[
\varphi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.
\]

In addition to [1,2], Luo and Lin first considered a quasi-Monte Carlo method in [8,9]; they proved that the ERM method is convergent under mild conditions and studied the properties of the ERM problems. In [10], Chen, Zhang and Fukushima considered SLCP including the expectation of matrix is positive semi-definite. Under the condition of a new error bound, they use the ERM formulation to solve the SLCP. In [11], they also studied the ERM formulation, where the involved function is a stochastic \( R_0 \) function. In [12], Zhou and Caccetta put forward a new model of the stochastic linear complementarity problem with only finitely many elements. A feasible semi-smooth Newton method was also given. In [14], Ma also considered a feasible semi-smooth Gauss–Newton method for solving this new stochastic linear complementarity problem. The stochastic linear complementarity problem with only finitely many elements is to find a vector \( x \in \mathbb{R}^n \), such that:

\[
x \geq 0, M(\omega_i)x + q(\omega_i) \geq 0, x^T[M(\omega_i)x + q(\omega_i)] = 0 \quad i = 1, \cdots, m, \quad m > 1,
\]

where \( \Omega = \{\omega_1, \omega_2, \cdots, \omega_m\} \). Denote:

\[
\bar{M} = \sum_{i=1}^{m} p_i M(\omega_i), \bar{q} = \sum_{i=1}^{m} p_i q(\omega_i),
\]

where \( p_i = P(\omega_i \in \Omega) > 0, i = 1, \cdots, m \). Then, (3) is equivalent to:

\[
\begin{align*}
x & \geq 0, \bar{M}x + \bar{q} \geq 0, x^T(\bar{M}x + \bar{q}) = 0, \\
M(\omega_i)x + q(\omega_i) & \geq 0, \quad i = 1, \cdots, m.
\end{align*}
\]

where (4) is called the linear complementarity problem.

As we all know, in [15–29], many methods were given for solving the nonlinear complementarity problem (NCP) and linear complementarity problem (LCP), such as Kanzow and Petra, who presented a nonsmooth least squares reformulation of (4) in [15]. They defined \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^{2n} \) as:

\[
\Phi(x) = \begin{pmatrix}
\lambda \phi_{FB}(x_i, \bar{M}_i x + \bar{q}_i), i = 1 \cdots n \\
\vdots \\
(1 - \lambda) \phi_+(x_i, \bar{M}_i x + \bar{q}_i), i = 1 \cdots n
\end{pmatrix},
\]

where \( \lambda \in (0, 1) \), \( \phi_{FB}(a, b) = \| (a, b) \|_2 - (a + b), \phi_+(a, b) = a_+ b_+ z_+ = \max \{0, z\} \) for \( z \in \mathbb{R} \). This least squares formulation can give a faster reduction of the complementarity gap \( x^T(\bar{M}x + \bar{q}) \).

On the other hand, the authors of [16,18,19] studied the generalized Fischer–Burmeister function, i.e., \( \phi_p : \mathbb{R}^n \rightarrow \mathbb{R} \) given by \( \phi_p(a, b) = \| (a, b) \|_p - (a + b) \) \( (p \in (1, +\infty)) \). Additionally, their research work has shown that this class of functions enjoys some favorable properties as other NCP-functions. The given numerical results for the test problems from mixed complementarity problem library (MCPLIB) have shown that the descent method has better performance when \( p \) decreases in \( (1, +\infty) \).

The main motivation of this paper is to use the advantages of [12,15,16,18,19] to solve the stochastic linear complementarity problem denoted as (3). Firstly, we put forward a new robust reformulation of the complementarity Problem (3). Then, based on the new robust reformulation, we propose a feasible nonsmooth Levenberg–Marquardt-type method to solve (3). The numerical results in Section 4 showed that the given Method 1 is efficient for the related refinery production problem and the large-scale stochastic linear complementarity problems. We also make a comparison with Method 1 and the scaled trust region method (STRM) in [20]; preliminary numerical experiments showed...
that the numerical results of Method 1 are as good as the numerical results of the STRM method. Additionally, it generates less iterations in contrast to the STRM method. Additionally, we also make a comparison with Method 1 and the method in [13] for solving the related refinery production problem. The preliminary numerical experiments also indicate that Method 1 is quite robust. In other words, Method 1 has the following advantages.

- Faster reduction of the complementarity gap $x^T(Mx + q)$.
- Flexible NCP functions $\phi_p$.
- Randomly-chosen initial points and less calculation.

Now is the time to give a new reformulation of (4); the new reformulation of (4) is:

$$\phi_n(x) = \begin{pmatrix} 
\lambda \phi_p(x_i, \bar{M}_i x + \bar{q}_i), i = 1 \cdots n \\
(1 - \lambda) \phi_p(x_i, \bar{M}_i x + \bar{q}_i), i = 1 \cdots n
\end{pmatrix},$$

where $\phi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$, $\lambda \in (0, 1)$. Then, $x$ is a solution of $\phi_n(x) = 0$. Additionally, solving (4) is also equivalent to finding a solution of the unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} \Psi(x),$$

where:

$$\Psi(x) = \frac{1}{2} \| \phi_n(x) \|^2.$$

Then, (4) and (5) can be rewritten as:

$$F(x, y) = 0, y \geq 0,$$

where:

$$F(x, y) = \begin{pmatrix} 
\phi_n(x) \\
M(\omega_1)x + q(\omega_1) - y_1 \\
M(\omega_2)x + q(\omega_2) - y_2 \\
\vdots \\
M(\omega_m)x + q(\omega_m) - y_m
\end{pmatrix}.$$

Additionally, $y = [y_1^T, y_2^T, ..., y_m^T]^T \in \mathbb{R}^m$ is a slack variable with $y_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$. Then, we know that $F(x, y) = 0$ has $(m + 2)n$ equations with $(m + 1)n$ variables.

The remainder of this paper is organized as follows. In Section 2, we review some background definitions and some useful properties; some of them can be found in [6,14,15,19–23]; some of them are given for the first time.

2. Preliminaries

In this section, we give some related definitions and some properties; some of them can be found in [6,14,15,19–23]; some of them are given for the first time.

Let $G : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a locally-Lipschitzian function. The B-subdifferential of $G$ at $x$ is:

$$\partial_B G(x) = \{ V \in \mathbb{R}^{m \times n} | \exists \{x_k\} \subseteq D_G : \{x_k\} \rightarrow x, G'(x_k) \rightarrow V \},$$

where $D_G$ is the differentiable points set and $G'(x)$ is the Jacobian of $G$ at a point $x \in \mathbb{R}^n$. 
The Clarke generalized Jacobian of $G$ is defined as:

$$\partial G(x) = \text{conv}\{V \in \mathbb{R}^{m \times n} | \exists \{x_k\} \subseteq D_G : \{x_k\} \to x, G'(x_k) \to V\}.$$  

Furthermore,

$$\partial C G(x)^T = \partial G_1(x) \times \cdots \times \partial G_m(x)$$

denotes the C-subdifferential of $G$ at $x$.

If

$$\lim_{h \to 0} \frac{\|G(x + h^T) - G(x) - Hh\|}{\|h\|^2}$$

exists for any $h \in \mathbb{R}^m$, we call $G$ is semi-smooth at $x$.

**Definition 1.** (6) Matrix $M \in \mathbb{R}^{n \times n}$ is called a:

(a) $P_0$-matrix, if each of its principal minors is nonnegative.

(b) $P$-matrix, if each of its principal minors is positive.

**Proposition 1.** (6) $M \in \mathbb{R}^{n \times n}$ is a $P_0$-matrix $\iff \forall x \neq 0, x_i (Mx)_i \geq 0, x_i \neq 0$.

**Definition 2.** (23) Let $G : \mathbb{R}^n \to \mathbb{R}^n$; the following statements are given:

(a) We call $G$ monotone, if $(x - y)^T (G(x) - G(y)) \geq 0$, for $x, y \in \mathbb{R}^n$.

(b) $G$ is a $P_0$ function, if:

$$\max_{1 \leq i \leq n} (x_i - y_i) (G_i(x) - G_i(y)) \geq 0$$

for all $x, y \in \mathbb{R}^n, x \neq y$ and $x_i \neq y_i$.

**Proposition 2.** (21) Suppose $G$ is a locally-Lipschitz function and strongly semi-smooth at $x$. Additionally, it is also directionally differentiable in a neighborhood of $x$; we get:

$$\lim_{h \to 0, H \in \partial G(x+h)} \frac{\|G(x + h) - G(x) - Hh\|}{\|h\|^2} < \infty.$$

**Definition 3.** We call $x^*$ an $R$-regular solution of the complementarity problem $x \geq 0, G(x) \geq 0, x^T G(x) = 0$; if $G'(x^*)_{ab}$ is nonsingular and $G'(x^*)_{ab} G'(x^*)_{ba} G'(x^*)_{aa}^2$ is a $P$-matrix, where $a = \{i|x_i^* > 0, G_i(x^*) = 0\}, b = \{i|x_i^* = 0, G_i(x^*) = 0\}, \gamma = \{i|x_i^* = 0, G_i(x^*) > 0\}$.

**Proposition 3.** (15) The generalized gradient of $\phi_+$ at a point $(a, b)$ is defined as:

$$\partial \phi(a, b) = (\partial_a \phi, \partial_b \phi) = \left\{ \begin{array}{ll} \left(\frac{sgn(a)|a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1, \frac{sgn(b)|b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right), (a, b) \neq (0, 0) \\
(e - 1, \zeta), (a, b) = (0, 0) \end{array} \right.,$$

where $\|(e, \zeta)\|_p \leq 1$. The generalized gradient of $\phi_+$ at a point $(a, b)$ is defined as $\partial \phi_+(a, b) = \{(b_+, \partial_a a_+, a_+ \partial b_+)\}$, where:

$$\partial Z_+ = \left\{ \begin{array}{ll} 1, & Z > 0 \\
0, & Z < 0 \end{array} \right.,$$

$\partial Z_+ = [0, 1]$, if $Z = 0$.

**Definition 4.** When $x^* > 0, Mx^* + q > 0$, then (4) is called strictly feasible at $x^*$. 
Proposition 4. ([15]) All $H \in \partial_C \phi(x)$ can be defined as:

$$
\begin{pmatrix}
\lambda H_1 \\
(1 - \lambda) H_2
\end{pmatrix},
$$

where $H_1 \subseteq \text{diag}\{a_i(x)\} + \text{diag}\{b_i(x)\} M$, $H_2 \subseteq \text{diag}\{\hat{a}_i(x)\} + \text{diag}\{\hat{b}_i(x)\} \bar{M}$, $(a_i(x), b_i(x)) \in \partial \phi_p(x, (Mx + q)_i)$, $(\hat{a}_i(x), \hat{b}_i(x)) \in \partial \phi_p(x, (Mx + q)_i)$, and $\partial \phi_p(x, (Mx + q)_i)$ are given in Proposition 3.

Proposition 5. ([15]) Suppose that (4) is R-regular at $x^*$, then, all elements of $\partial_C \phi(x^*)$ have full rank.

For any $(x, y) \in \mathbb{R}^{(m+1)n}$, we know that:

$$
\partial_C F(x, y) = \left\{ \begin{array}{c}
V_{\phi_n} \\
M(\omega_1) - I & 0 \cdots 0 \\
M(\omega_2) & 0 \cdots 0 \\
\vdots \\
M(\omega_m) & 0 \cdots - I \\
\end{array} : V_{\phi_n} \in \partial_C \phi_n(x) \right\},
$$

where $I$ is the $n \times n$ identity matrix. Hence, by Proposition 5, we know that the following proposition is set up.

Proposition 6. Suppose (4) is R-regular at $x^*$ and $(x^*, y^*)$ is a solution of (9). Then, all $V \in \partial_C F(x^*, y^*)$ are nonsingular.

Proposition 7. If (4) is R-regular at a solution $x^*$, then, there exists $\alpha > 0, \beta > 0$, such that $\|H^T H\| \leq \beta$ for all $x^* \in \mathbb{R}^n$ with $\|x - x^*\| \leq \alpha$, where $H \in \partial_C \phi(x)$.

Proof of Proposition 7. The proof is similar to the ([15], Lemma 2.5) and therefore omitted here. □

3. The Feasible Nonsmooth Levenberg–Marquardt-Type Method and Its Convergence Analysis

In this section, we define a merit function $\theta(z) = \frac{1}{2} \| F(z) \|^2$ and give a feasible nonsmooth Levenberg–Marquardt-type method. At the same time, we also give some discussions about
this method.

Let \( z = (x, y) \in \mathbb{R}^{(m+1)n} \); define a merit function of (9) by:

\[
\theta(z) = \frac{1}{2} \| F(z) \|^2.
\]

If (3) has a solution, then solving (9) is equivalent to finding a global solution of the following constrained optimization problem:

\[
\begin{align*}
\min & \quad \theta(z) \\
\text{s.t.} & \quad z \geq 0, \quad s.t. \quad z \geq 0.
\end{align*}
\]

If \( z \) satisfies:

\[
P_Z[z - \nabla \theta(z)] - z = 0,
\]

where \( P_Z(\cdot) \) is an orthogonal projection operator onto \( Z = \{ z \in \mathbb{R}^{(m+1)n} | z \geq 0 \} \), then \( z \) is a stationary point of (10). Obviously, (11) is equivalent to the following problem:

\[
\nabla \theta(z) \geq 0, z \geq 0, z^T \nabla \theta(z) = 0.
\]

**Lemma 1.** \((14)\) Let \( P_Z(\cdot) \) be an orthogonal projection operator onto \( Z = \{ z \in \mathbb{R}^{(m+1)n} | z \geq 0 \} \). The following statements hold:

(a) \( \| P_Z(x) - P_Z(y) \| \leq \| x - y \| \) for all \( x, y \in \mathbb{R}^{(m+1)n} \).

(b) For any \( y \in Z \), \( P_Z(x) - x)^T(P_Z(x) - y) \leq 0 \) for all \( x \in \mathbb{R}^{(m+1)n} \).

**Proposition 9.** \((15)\) The merit function \( \theta \) has the following properties.

(a) \( \theta(z) \) is continuously differentiable on \( \mathbb{R}^{(m+1)n} \) with \( \nabla \theta(z) = H^T F(z) \) for any \( H \in \partial C F(z) \).

(b) Assume \( x \mapsto \bar{M}x + \bar{q} \) is monotone, if \( LCP(\bar{M}, \bar{q}) \) has a strictly feasible solution, then for all \( c > 0 \), we know that the level set:

\[
L(c) = \{ z \in \mathbb{R}^{(m+1)n} | \theta(z) \leq c \}
\]

is bounded.

For some monotone stochastic linear complementarity problems, the stationary points of (10) may not be a solution. Such as let \( n = 1, m = 2, \Omega = \{ \omega_1, \omega_2 \} = \{0, 1\}, M(\omega_1) = M(\omega_2) = 1, q(\omega_1) = 1, q(\omega_2) = -1 \), and \( p_1 = P(\omega_1 \in \Omega) = 0.5, i = 1, 2 \), (see \([12] \)).

By simple computation, we know that the above of problem is a monotone SLCP, obviously; all points \( x \geq 1 \) are feasible, but this example has no solution. By:

\[
F(x, y_1, y_2) = \begin{pmatrix}
\lambda(2x^q) - 2|x| \\
(1 - \lambda)(x^2) \\
x + 1 - y_1 \\
x - 1 - y_2
\end{pmatrix},
\]

and \((0, 1, 0)\) is a stationary point of the constraint optimization problem:

\[
\begin{align*}
\min & \quad \frac{1}{2} \| F(x, y_1, y_2) \|^2 \\
\text{s.t.} & \quad x \geq 0, y_1 \geq 0, y_2 \geq 0.
\end{align*}
\]

However, \( x = 0 \) is not a solution of this example.

Therefore, in the following proposition, we give some conditions for (3).
Proposition 10. For monotone Problem (3), let \( z^* = (x^*, y^*) \) be a stationary point of (10). If \( M(\omega) x^* + q(\omega) - y^* = 0, i = 1, 2, ..., m \), then \( x^* \) is a solution of (3).

Proof of Proposition 10. Assuming that \( z^* = (x^*, y^*) \) is a stationary point of (10), if \( M(\omega) x^* + q(\omega) - y^* = 0, i = 1, 2, ..., m \), by (12), we know that \( x^* \) is the stationary point of the following problem:

\[
\min \{ \Psi(x) | x \geq 0 \}.
\]

Similar to the proof of Theorem 3 in [24], it can be shown that \( x^* \) is a solution of \( \Psi(x) = 0 \). Thus, \( x^* \) is a solution of (3). \( \square \)

Now, we present the feasible nonsmooth Levenberg–Marquardt-type method for solving (3).

Method 1. Choose \( z_0 \in Z, \sigma \in (0, \frac{1}{2}), \epsilon \geq 0, \beta, \gamma \in (0, 1) \). Set \( k = 0 \).

Step 1. If \( |\theta(z_k)| \leq \epsilon \), stop.

Step 2. Choose \( H_k \in \partial C(z_k) \), \( v_k = \|F(z_k)\| > 0 \), and find the solution \( d_k \) of the equations:

\[
(H_k^T H_k + v_k I) d = -\nabla \theta(z_k).
\]

Step 3. If

\[
\|F(P_Z(z_k + d_k))\| \leq \gamma \|F(z_k)\|,
\]

then set \( z_{k+1} = P_Z(z_k + d_k), k = k + 1, \) and go to Step 1; otherwise, go to Step 4.

Step 4. Compute \( t_k = \max \{ \beta^l | l = 0, 1, 2, \ldots \} \), such that:

\[
\theta(z_k(t_k)) \leq \theta(z_k) + \sigma \nabla \theta(z_k)^T(z_k(t_k) - z_k),
\]

where \( z_k(t_k) = P_Z[z_k - t_k \nabla \theta(z_k)] \). Set \( z_{k+1} = z_k(t_k), k = k + 1, \) and go to Step 1.

We now investigate the convergence properties of Method 1. In the following sections, we assume that Method 1 generates an infinite sequence.

Theorem 1. Method 1 is well defined for a monotone SLCP (3). If Method 1 does not stop at a stationary point in finite steps, an infinite sequence \( \{z_k\} \) is generated with \( \{z_k\} \subset Z \), and any accumulation point of the sequence \( \{z_k\} \) is a stationary point of \( \theta \).

Proof of Theorem 1. Method 1 is well defined for the reason of \( v_k > 0 \), and \( d_k \) is always a descent direction for \( \theta \). Now, we consider the following two situations respectively.

(I) If the direction \( d_k \) is accepted by an infinite number of times in Step 3 of Method 1, we get:

\[
z_{k+1} = P_Z[z_k + d_k] \in Z.
\]

Since \( \nabla \theta(z_k) \neq 0 \) implies \( d_k \neq 0 \), we have:

\[
\nabla \theta(z_k)^T d_k = -((H_k^T H_k + v_k I) d_k)^T d_k < 0.
\]

From [17], we know that \( \{\theta(z_k)\} \) is monotonically decreasing. Obviously, this implies that the sequence \( \{\|F(z_k)\| \} \) is also monotonically decreasing. Since \( \|F(P_Z(z_k + d_k))\| \leq \gamma \|F(z_k)\| \) is accepted by an infinite number of times in view of our assumptions, therefore we get \( \|F(z_k)\| \to 0 \) for \( k \to \infty \) by \( \gamma \in (0, 1) \). This means that any accumulation point of \( \{z_k\} \) is the solution of (10); therefore, it is also a stationary point of \( \theta \).
(II) This case is the negation of Case (I); without loss of generality, we assume that the Levenberg–Marquardt direction is never accepted. If the direction \( P_z[z_k - t_k \nabla \theta(z_k)] - z_k \) is accepted by an infinite number of times in Step 4 of Method 1, we have:

\[
z_{k+1} = P_z[z_k - t_k \nabla \theta(z_k)] \in Z.
\]

By (b) in Lemma 1, taking \( x := z_k - t_k \nabla \theta(z_k) \), \( y := z_k \), we get:

\[
0 \geq [P_z(z_k - t_k \nabla \theta(z_k)) - (z_k - t_k \nabla \theta(z_k))]^T [P_z(z_k - t_k \nabla \theta(z_k)) - z_k] = [P_z(z_k - t_k \nabla \theta(z_k)) - z_k + t_k \nabla \theta(z_k)]^T [P_z(z_k - t_k \nabla \theta(z_k)) - z_k] = (P_z(z_k - t_k \nabla \theta(z_k)) - z_k)^2 + t_k \nabla \theta(z_k)^T [P_z(z_k - t_k \nabla \theta(z_k)) - z_k]
\]

that is,

\[
\nabla \theta(z_k)^T [P_z(z_k - t_k \nabla \theta(z_k)) - z_k] \leq -\frac{(P_z(z_k - t_k \nabla \theta(z_k)) - z_k)^2}{t_k} \leq 0,
\]

where \( t_k = \max\{\beta' | 0, 1, 2, \cdots \} \) with \( \beta \in (0, 1) \). By the Armijo line search properties, we know that any accumulation point of \( \{z_k\} \) is a stationary point of \( \theta \), and this completes the proof.

\( \square \)

**Theorem 2.** Let \( x^* \in \mathbb{R}^n \) be a \( R \)-regular solution; then the whole sequence generated by Method 1 converges to \( z^* \) \( Q \)-quadratically.

**Proof of Theorem 2.** By Proposition 6, there is a constant \( c_1 > 0 \), such that, for all \( z_k \in \bigcup (z^*, \delta_1) \), where \( \delta_1 \) is a sufficiently small positive constant, the matrices \( H_k^T H_k + v_k I \) are nonsingular, and \( \| (H_k^T H_k + v_k I)^{-1} \| \leq c_1 \) hold. Furthermore, by Proposition 2, there exists a constant \( c_2 > 0 \), such that:

\[
\| F(z_k) - F(z^*) - H_k(z_k - z^*) \| \leq c_2 \| z_k - z^* \|^2,
\]

for all \( z_k \in \bigcup (z^*, \delta_2) \), where \( \delta_2 \) is a sufficiently small positive constant. Moreover, in view of the upper semicontinuity of the C-subdifferential, we have:

\[
\| H_k^T \| \leq \zeta,
\]

where \( H_k \in \partial_c F(z_k) \), \( \zeta > 0, z_k \in \bigcup (z^*, \delta_3) \), and \( \delta_3 \) is a sufficiently small positive constant. Denote \( \delta = \min(\delta_1, \delta_2, \delta_3) \), for \( z_k \in \bigcup (z^*, \delta) \). Note that, from (13) and Lemma 1, we have:

\[
(H_k^T H_k + v_k I)(z_{k+1} - z^*) = (H_k^T H_k + v_k I)(P_z(z_k + d_k) - z^*) = (H_k^T H_k + v_k I)(z_k + d_k - z^* + P_z(z_k + d_k) - (z_k + d_k)) = (H_k^T H_k + v_k I)(z_k + d_k - z^*) + (H_k^T H_k + v_k I)(P_z(z_k + d_k) - (z_k + d_k)) = (H_k^T H_k + v_k I)(z_k - z^*) - H_k^T F(z_k) + (H_k^T H_k + v_k I)(P_z(z_k + d_k) - (z_k + d_k)) = H_k^T (F(z_k) - z^*) + v_k(z_k - z^*) - H_k^T F(z_k) + (H_k^T H_k + v_k I)(P_z(z_k + d_k) - (z_k + d_k)) = H_k^T (F(z_k) - z^*) + H_k(z_k - z^*) + v_k(z_k - z^*) + (H_k^T H_k + v_k I)(P_z(z_k + d_k) - (z_k + d_k))
\]
Since $F$ is a locally-Lipschitzian function and $v_k = \|F(z_k)\|$, by premultiplying this equation by $(H_k^T H_k + v_k I)^{-1}$ and taking norms both sides, we get:

\[
\|z_{k+1} - z^*\| \\
\leq c_1(\|H_k^T \| \|F(z^*) - F(z_k) + H_k(z_k - z^*)\| + \|F(z_k) - F(z^*)\| \|z_k - z^*\|) + \|P_z(z_k + d_k) - (z_k + d_k)\| \\
\leq c_1(\zeta c_2) \|z_k - z^*\|^2 + L \|z_k - z^*\|^2 + \|z_k + d_k - z^*\| \\
\leq c_1(\zeta c_2 + L) \|z_k - z^*\|^2 + \|z_k - z^* - (H_k^T H_k + v_k I)^{-1}H_k^T F_k\| \\
= c_1(\zeta c_2 + L) \|z_k - z^*\|^2 + \|(H_k^T H_k + v_k I)^{-1}(H_k^T (F(z^*) - F(z_k) + H_k(z_k - z^*)) + v_k(z_k - z^*))\| \\
\leq c_1(\zeta c_2 + L) \|z_k - z^*\|^2 + c_1(\|H_k^T \| \|F(z^*) - F(z_k) + H_k(z_k - z^*)\| + \|F(z_k) - F(z^*)\| \|z_k - z^*\|) \\
\leq c_1(\zeta c_2 + L) \|z_k - z^*\|^2 + c_1(\zeta c_2 + L) \|z_k - z^*\|^2 \\
\leq 2c_1(\zeta c_2 + L) \|z_k - z^*\|^2 \\
= \tau \|z_k - z^*\|^2,
\]

where $\tau = 2c_1(\zeta c_2 + L)$. Therefore, similar to the proof of ([20], Theorem 2.3), we know that the rate of convergence is $Q$-quadratic. This completes the proof. $\square$

4. Numerical Results

In this section, firstly, we make a numerical comparison between Method 1 and the scaled trust region method (STRM) in [20]. We apply Method 1 and the scaled trust region method to solve Examples 1 and 2. Secondly, we use Method 1 to solve the related refinery production problem, which also has been studied in [4,13]. Finally, numerical results about large-scale stochastic linear complementarity problems are also presented. We implement Method 1 in MATLAB and test the method on the given test problems using the reformulation from the previous section. Additionally, all of these problems were done on a PC (Acer) with i5-3210M and RAM of 2 GB. Throughout the computational experiments, the parameters in Method 1 are taken as:

$$\sigma = 0.3, \beta = 0.5, \gamma = 0.5.$$  

The stopping criteria for Method 1 are $\|\theta(z_k)\| \leq 10^{-15}$ or $k_{\text{max}} = 5000$. The parameters in the STRM method (see [20]) are taken as:

$$\Delta_0 = 10, \Delta_{\text{min}} = 10^{-6}, \rho_1 = 10^{-4}, \rho_2 = 0.75, \sigma_1 = 0.5, \sigma_2 = 2, \eta = 0.5.$$  

The stopping criteria for the STRM method are $\|Dq^k\| \leq 10^{-15}$ or $k_{\text{max}} = 5000$. In the tables of the numerical results, DIM denotes the dimension of the problem (the dimension of the variable $x$); $x^*$ denotes the solution of $\theta(x,y) = 0$; In the following part of this section, we give the detailed description of the given test problems.

**Example 1.** Consider SLCP $(M(\omega), q(\omega))$ with:

$$M(\omega) = \begin{pmatrix} 1 - 2\omega & -1 \\ 0 & -\omega \end{pmatrix}, q(\omega) = \begin{pmatrix} 1 \\ \omega + 1 \end{pmatrix},$$

where $\Omega = \{\omega_1, \omega_2\} = \{0, 1\}$, and $p_i = P(\omega_i \in \Omega) = 0.5, i = 1, 2$.

Numerical results of Example 1 are given in Table 1, Figures 1 and 2, respectively. $x_0$ are chosen randomly in $\mathbb{R}^2$; $y_0$ are chosen randomly in $\mathbb{R}^4$ and $\lambda = 0.1$. 

### Table 1. Numerical results for Example 1.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x^*$</th>
<th>Final Value</th>
<th>Iteration</th>
<th>$x^*$</th>
<th>Final Value</th>
<th>Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>$1.0 \times 10^{-8} \times (0.6159, 0.3060)$</td>
<td>$1.7595 \times 10^{-16}$</td>
<td>9</td>
<td>$(0, 0)$</td>
<td>$4.9304 \times 10^{-32}$</td>
<td>10</td>
</tr>
<tr>
<td>1.3</td>
<td>$1.0 \times 10^{-8} \times (0.8605, 0.3628)$</td>
<td>$2.1217 \times 10^{-16}$</td>
<td>9</td>
<td>$(0, 0)$</td>
<td>$2.9582 \times 10^{-31}$</td>
<td>10</td>
</tr>
<tr>
<td>1.5</td>
<td>$1.0 \times 10^{-7} \times (0.1079, 0.3994)$</td>
<td>$2.5396 \times 10^{-16}$</td>
<td>9</td>
<td>$(0, 0)$</td>
<td>$2.4652 \times 10^{-32}$</td>
<td>9</td>
</tr>
<tr>
<td>2.0</td>
<td>$1.0 \times 10^{-7} \times (0.1122, 0.3994)$</td>
<td>$2.9560 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.7218, 0)$</td>
<td>$2.9399 \times 10^{-32}$</td>
<td>9</td>
</tr>
<tr>
<td>2.5</td>
<td>$1.0 \times 10^{-7} \times (0.1113, 0.3994)$</td>
<td>$2.5585 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1165, 0)$</td>
<td>$1.2360 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>3.0</td>
<td>$1.0 \times 10^{-7} \times (0.1106, 0.3899)$</td>
<td>$2.5224 \times 10^{-16}$</td>
<td>9</td>
<td>$(0, 0)$</td>
<td>$2.4652 \times 10^{-32}$</td>
<td>9</td>
</tr>
<tr>
<td>3.5</td>
<td>$1.0 \times 10^{-7} \times (0.1102, 0.387)$</td>
<td>$2.5040 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1162, 0)$</td>
<td>$1.2310 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>4.0</td>
<td>$1.0 \times 10^{-7} \times (0.1099, 0.387)$</td>
<td>$2.4909 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.7209, 0)$</td>
<td>$2.9388 \times 10^{-32}$</td>
<td>9</td>
</tr>
<tr>
<td>4.5</td>
<td>$1.0 \times 10^{-7} \times (0.1097, 0.386)$</td>
<td>$2.4807 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-15} \times (0.0725, 0.1748)$</td>
<td>$7.3648 \times 10^{-32}$</td>
<td>9</td>
</tr>
<tr>
<td>5.0</td>
<td>$1.0 \times 10^{-7} \times (0.1095, 0.383)$</td>
<td>$2.4724 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-15} \times (0.1185, 0)$</td>
<td>$1.2804 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>5.5</td>
<td>$1.0 \times 10^{-7} \times (0.1094, 0.385)$</td>
<td>$2.4656 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.7262, 0.1903)$</td>
<td>$5.5480 \times 10^{-33}$</td>
<td>9</td>
</tr>
<tr>
<td>6.0</td>
<td>$1.0 \times 10^{-7} \times (0.1092, 0.384)$</td>
<td>$2.4596 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1199, 0)$</td>
<td>$1.4804 \times 10^{-31}$</td>
<td>9</td>
</tr>
<tr>
<td>6.5</td>
<td>$1.0 \times 10^{-7} \times (0.1091, 0.2213)$</td>
<td>$2.4549 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1232, 0)$</td>
<td>$1.3827 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>7.0</td>
<td>$1.0 \times 10^{-7} \times (0.1090, 0.384)$</td>
<td>$2.4508 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1259, 0)$</td>
<td>$1.4449 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>7.5</td>
<td>$1.0 \times 10^{-7} \times (0.1090, 0.383)$</td>
<td>$2.4473 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1181, 0)$</td>
<td>$1.2715 \times 10^{-34}$</td>
<td>9</td>
</tr>
<tr>
<td>8.0</td>
<td>$1.0 \times 10^{-7} \times (0.1089, 0.383)$</td>
<td>$2.4444 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.3123)$</td>
<td>$8.1621 \times 10^{-33}$</td>
<td>9</td>
</tr>
<tr>
<td>9.0</td>
<td>$1.0 \times 10^{-7} \times (0.1086, 0.383)$</td>
<td>$2.4399 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-16} \times (0.1255, 0.0082)$</td>
<td>$6.3079 \times 10^{-33}$</td>
<td>9</td>
</tr>
<tr>
<td>10.0</td>
<td>$1.0 \times 10^{-7} \times (0.1087, 0.382)$</td>
<td>$2.4368 \times 10^{-16}$</td>
<td>9</td>
<td>$1.0 \times 10^{-15} \times (0.7305, 0.4090)$</td>
<td>$5.7996 \times 10^{-32}$</td>
<td>9</td>
</tr>
</tbody>
</table>

**Figure 1.** Numerical results for Example 1 by Method 1. The x-axis represents the iteration step; the y-axis represents $\theta(x, y) = \frac{1}{2} \| F(x, y) \|^2$.

**Figure 2.** Numerical results for Example 1 by the STRM method. The x-axis represents the iteration step; the y-axis represents $\theta(x, y) = \frac{1}{2} \| F(x, y) \|^2$. 
From Table 1, we can see that the merit functions associated with \( p \in (1, 2) \), for example \( p = 1.5 \), are more effective than the Fischer–Burmeister merit function, for which exactly \( p = 2 \).

In Table 2, we give the numerical comparison of Method 1 with fmincon, which is a MATLAB tool box for constrained optimization. We use the sequential quadratic programming (SQP) method in the fmincon tool box to solve Example 1 by \( p = 1.1 \) and the same initial points.

<table>
<thead>
<tr>
<th>( p \times 10^{-8} \times (x_0, y_0) )</th>
<th>( \lambda )</th>
<th>( \lambda \times 10^{-16} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.6159, 0.3060 )</td>
<td>0.5</td>
<td>1.7595 \times 10^{-16}</td>
</tr>
<tr>
<td>( 0.0002, 0 )</td>
<td>0.5</td>
<td>1.2188 \times 10^{-14}</td>
</tr>
</tbody>
</table>

From Table 2, we can see that Method 1 is more effective than fmincon.

**Example 2.** Consider \( SLCP(M(\omega), q(\omega)) \) with:

\[
M(\omega) = \begin{pmatrix} 1 - \omega & 0 \\ -\omega & 2 & \omega \\ 0 & \omega & 3 \end{pmatrix}, q(\omega) = \begin{pmatrix} 3 - 2\omega \\ -2 - \omega \\ -3 - \omega \end{pmatrix},
\]

where \( \Omega = \{\omega_1, \omega_2\} = \{0, 1\} \), and \( p_i = P(\omega_i \in \Omega) = 0.5, i = 1, 2 \).

Numerical results are given in Table 3, Figures 2 and 3; \( x_0 \) are chosen randomly in \( \mathbb{R}^3 \); \( y_0 \) are chosen randomly in \( \mathbb{R}^6 \); and \( \lambda = 0.00000001 \).

From Table 3, Figures 3 and 4, we can see that the iterations of Method 1 are less than the STRM method. In Method 1, when \( p = 5 \), the function value falls faster. When \( p \) is larger, a greater number of iterations is needed in the STRM method.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( (x_0, y_0) )</th>
<th>Final Value</th>
<th>Iteration</th>
<th>Final Value</th>
<th>Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.6439 \times 10^{-17} )</td>
<td>17</td>
<td>5.000 \times 10^{-17}</td>
<td>80</td>
</tr>
<tr>
<td>3.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>16</td>
<td>5.000 \times 10^{-17}</td>
<td>80</td>
</tr>
<tr>
<td>4.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0002 \times 10^{-17} )</td>
<td>19</td>
<td>5.000 \times 10^{-17}</td>
<td>72</td>
</tr>
<tr>
<td>5.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>15</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
<tr>
<td>6.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>13</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
<tr>
<td>7.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>14</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
<tr>
<td>8.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>16</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
<tr>
<td>9.0</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>11</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
<tr>
<td>10</td>
<td>( (0, 1, 1) )</td>
<td>( 5.0000 \times 10^{-17} )</td>
<td>11</td>
<td>5.000 \times 10^{-17}</td>
<td>234</td>
</tr>
</tbody>
</table>
Figure 3. Numerical results for Example 2 by Method 1. The $x$-axis represents the iteration step; the $y$-axis represents $\theta(x, y) = \frac{1}{2} \| F(x, y) \|^2$.

Figure 4. Numerical results for Example 2 by the STRM method. The $x$-axis represents the iteration step; the $y$-axis represents $\theta(x, y) = \frac{1}{2} \| F(x, y) \|^2$.

In Table 4, we also give the comparison of Method 1 with fmincon. For the propose of comparison, we fixed $p = 10$ and the same initial points.

Table 4. Numerical results for Example 2 by Method 1 and fmincon.

<table>
<thead>
<tr>
<th>$x^*$</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method 1 (0, 1, 1)</td>
<td>$5.0000 \times 10^{-17}$</td>
</tr>
<tr>
<td>fmincon (0, 1, 1)</td>
<td>$4.9780 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

From Table 4, we can see that Method 1 is also more effective than fmincon.
Example 3. This example is a refinery production problem, which is also considered in [2,13].

The problem is defined as:

\[
M(\omega) = \begin{pmatrix}
0 & 0 & 1 & -2 - \omega_1 & -3 \\
0 & 0 & 1 & -6 & \omega_2 - 3.4 \\
-1 & -1 & 0 & 0 & 0 \\
2 + \omega_1 & 6 & 0 & -\omega_3 & -\omega_3 \\
3 & 3.4 - \omega_2 & 0 & -\omega_4 & \omega_4
\end{pmatrix},
\]

\[
q(\omega) = \begin{pmatrix}
c \\
-\omega_3 \\
-180 - \omega_3 \\
-162 - \omega_4
\end{pmatrix},
\]

where \(\omega_1, \omega_2, \omega_3\) and \(\omega_4\) satisfy the following distribution:

\[
\text{distr}\omega_1 \approx u[-0.8, 0.8],
\]

\[
\text{distr}\omega_2 \approx \exp(\lambda = 2.5),
\]

\[
\text{distr}\omega_3 \approx N(0, 12),
\]

\[
\text{distr}\omega_4 \approx N(0, 9).
\]

• Generate samples \(\omega^k_j, j = 1, 2, 3, 4, k = 1, 2, \ldots, K\), respectively, from their 99% confidence intervals (except uniform distributions):

\[
\omega_1 \in I_1 = [-0.8, 0.8],
\]

\[
\omega_2 \in I_2 = [0.0, 1.84],
\]

\[
\omega_3 \in I_3 = [-30.91, 30.91],
\]

\[
\omega_4 \in I_4 = [-23.18, 23.18],
\]

• For each \(j\), divide the \(I_j\) into \(m_j\) cells \(I_{j,i}, i = 1, 2, \ldots, m_j\).

• For each \((j, i)\), calculate the average \(v_{ji}\) of \(\omega^k_j\); it belongs to \(I_{j,i}\).

• For each \((j, i)\), the estimated probability of \(v_{ji}\) is \(p_{ji} = k_{ji}/K\), where \(k_{ji}\) is the number of \(\omega^k_j \in I_{j,i}\).

• Let \(N = m_1 \times m_2 \times m_3 \times m_4\), and set the joint distribution of \(\{ (\omega^\ell, p^\ell), \ell = 1, 2, \ldots, N \}\),

\[
\omega^\ell = \begin{pmatrix}
v_{1,i_1} \\
v_{2,i_2} \\
v_{3,i_3} \\
v_{4,i_4}
\end{pmatrix},
p^\ell = p_{1,i_1} p_{2,i_2} p_{3,i_3} p_{4,i_4}
\]

for \(i_1 = 1, \ldots, m_1, i_2 = 1, \ldots, m_2, i_3 = 1, \ldots, m_3, i_4 = 1, \ldots, m_4\).

In the following part of this section, we use Method 1 to solve the constrained optimization problem:

\[
\min_{z \geq 0} \theta(z) = \frac{1}{2} \| F(z) \|^2,
\]

where \(z = (x, y)\),

\[
F(x, y) = \begin{pmatrix}
\phi_n(x) \\
M(\omega^\ell)x + q(\omega^\ell) - y, \ell = 1 \cdots N
\end{pmatrix}.
\]
and:
\[
\phi_\omega(x) = \left( \begin{array}{c} \lambda \phi_\omega(x, M_1 x + q_1), i = 1 \ldots 5 \\ \vdots \\ (1 - \lambda) \phi_\omega(x, M_1 x + q_5), i = 1 \ldots 5 \end{array} \right).
\]

Now, we examine the following two conditions:

**Condition 1**: \( \omega_1 = 0, \omega_2 = 0, m_3 = 15, m_4 = 15 \).

**Condition 2**: \( m_1 = 5, m_2 = 9, m_3 = 7, m_4 = 11 \).

The numerical results of Example 3 are given in Tables 5 and 6, where \( \Psi(x) = \frac{1}{2} \| \phi_\omega(x) \|_2^2 \) is the merit function; \( \theta(x,y) = \frac{1}{2} \| F(x, y) \|_2^2 \); \( k = 10, i = 3, 4, 5 \); \( 2x_1 + 3x_2 \) is the initial production cost; and \( \lambda = 0.5 \).

**Table 5.** Numerical results for Example 3 based on Condition 1.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( k )</th>
<th>( x_k )</th>
<th>( \theta(z_k) )</th>
<th>( \Psi(x_k) )</th>
<th>( 2x_1^k + 3x_2^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10(^3)</td>
<td>(42.6730, 15.8000, 0.2848, 0.4688)</td>
<td>18.1932</td>
<td>5.1078</td>
<td>132.7462</td>
</tr>
<tr>
<td>2</td>
<td>10(^4)</td>
<td>(42.6057, 15.8216, 0.2844, 0.4694)</td>
<td>5.2138</td>
<td>5.0089</td>
<td>132.6760</td>
</tr>
<tr>
<td>2</td>
<td>10(^5)</td>
<td>(42.0369, 16.0037, 0.2791, 0.4740)</td>
<td>4.3894</td>
<td>4.2424</td>
<td>132.0850</td>
</tr>
<tr>
<td>4</td>
<td>10(^3)</td>
<td>(42.7120, 15.7720, 0.2872, 0.4829)</td>
<td>1999.1</td>
<td>5.3083</td>
<td>132.7399</td>
</tr>
<tr>
<td>4</td>
<td>10(^4)</td>
<td>(42.6301, 15.7986, 0.2844, 0.4694)</td>
<td>5.2164</td>
<td>5.0103</td>
<td>132.6559</td>
</tr>
<tr>
<td>4</td>
<td>10(^5)</td>
<td>(42.0487, 15.9888, 0.2791, 0.4740)</td>
<td>4.3826</td>
<td>4.2359</td>
<td>132.0628</td>
</tr>
<tr>
<td>6</td>
<td>10(^3)</td>
<td>(42.7277, 15.7628, 0.2853, 0.4687)</td>
<td>5.3534</td>
<td>5.1372</td>
<td>132.7438</td>
</tr>
<tr>
<td>6</td>
<td>10(^4)</td>
<td>(42.6539, 15.7870, 0.2846, 0.4692)</td>
<td>5.2433</td>
<td>5.0360</td>
<td>132.6688</td>
</tr>
<tr>
<td>6</td>
<td>10(^5)</td>
<td>(42.0594, 15.9826, 0.2791, 0.4740)</td>
<td>4.3925</td>
<td>4.2438</td>
<td>132.0667</td>
</tr>
</tbody>
</table>

**Table 6.** Numerical results for Example 3 based on Condition 2.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( k )</th>
<th>( x_k )</th>
<th>( \theta(z_k) )</th>
<th>( \Psi(x_k) )</th>
<th>( 2x_1^k + 3x_2^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10(^3)</td>
<td>(42.6799, 15.7988, 0.2833, 0.4704)</td>
<td>5.3426</td>
<td>5.1369</td>
<td>132.7562</td>
</tr>
<tr>
<td>2</td>
<td>10(^4)</td>
<td>(42.5951, 15.8259, 0.2826, 0.4706)</td>
<td>5.2120</td>
<td>5.0197</td>
<td>132.6679</td>
</tr>
<tr>
<td>2</td>
<td>10(^5)</td>
<td>(41.9961, 16.0177, 0.2773, 0.4752)</td>
<td>4.3428</td>
<td>4.2083</td>
<td>132.0453</td>
</tr>
<tr>
<td>4</td>
<td>10(^3)</td>
<td>(42.7005, 15.7755, 0.2846, 0.4717)</td>
<td>23.7543</td>
<td>5.1012</td>
<td>132.7276</td>
</tr>
<tr>
<td>4</td>
<td>10(^4)</td>
<td>(42.6135, 15.8036, 0.2826, 0.4707)</td>
<td>5.2027</td>
<td>5.0096</td>
<td>132.6377</td>
</tr>
<tr>
<td>4</td>
<td>10(^5)</td>
<td>(41.9980, 16.0049, 0.2772, 0.4753)</td>
<td>4.3207</td>
<td>4.1872</td>
<td>132.0108</td>
</tr>
<tr>
<td>6</td>
<td>10(^3)</td>
<td>(42.7299, 15.7531, 0.2838, 0.4686)</td>
<td>55.2438</td>
<td>5.2217</td>
<td>132.7789</td>
</tr>
<tr>
<td>6</td>
<td>10(^4)</td>
<td>(42.6568, 15.7867, 0.2829, 0.4703)</td>
<td>5.2612</td>
<td>5.0652</td>
<td>132.6738</td>
</tr>
<tr>
<td>6</td>
<td>10(^5)</td>
<td>(42.0153, 15.9977, 0.2773, 0.4753)</td>
<td>4.3411</td>
<td>4.2060</td>
<td>132.0235</td>
</tr>
</tbody>
</table>

In [13], in the case of \( \omega_1 = 0, \omega_2 = 0.4, m_3 = 15, m_4 = 15 \), Kall and Wallace get the optimal solution \( (x_1, x_2) = (38.539, 20.539) \); initial production cost \( 2x_1 + 3x_2 = 138.695 \). Here, by Method 1, we get the optimal solution \( (x_1, x_2) = (41.6939, 16.1036) \), and the production cost is \( 2x_1 + 3x_2 = 131.6985 \).

**Remark 1.** In this paper, we use:
\[
\omega_i = \begin{cases} \omega_j, & i = j, \\ E(\omega_i), & i \neq j. \end{cases}
\]

The computation cost of our method is greatly reduced. In fact, when we think about the general case of \( \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \) varying the random distribution of discrete approximation by a 5-, 9-, 7- and 11-point distribution, respectively. This yields a joint discrete distribution of \( 5 \times 9 \times 7 \times 11 = 3465 \) realizations. Then, \( F(z) \) is a function of \( 17,335 \) (3465 \( \times 5 + 10 \) = 17,335) dimensions. This is a more complex optimization problem.
In the following part of this subsection, we give a large-scale stochastic linear complementarity problem named the stochastic Murty problem. When \( \Omega = \{ \omega | \omega = \frac{1}{2} \} \), the large-scale stochastic linear complementarity problem reduces to the Murty problem, which is intensively studied in [25–29].

**Example 4. (Stochastic Murty problem)** Consider SLCP \( (M(\omega), q(\omega)) \) with:

\[
M(\omega) = \begin{pmatrix} \frac{1}{2} + \omega & 2 & \cdots & 2 \\ 0 & \frac{1}{2} + \omega & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \frac{1}{2} + \omega \end{pmatrix}, \quad q(\omega) = \begin{pmatrix} -\frac{3}{2} + \omega \\ -\frac{3}{2} + \omega \\ \vdots \\ -\frac{3}{2} + \omega \end{pmatrix},
\]

where \( M(\omega) \in \mathbb{R}^{n \times n}, q(\omega) \in \mathbb{R}^n, \Omega = \{ \omega_1, \omega_2 \} = \{ 0, 1 \} \), and \( p_i = P(\omega_i \in \Omega) = 0.5, i = 1, 2 \).

In Table 7, we give the comparison of Method 1 with the SQP method in the fmincon tool box, when the dimensions of Example 4 are 10, 100, 200, 300 and 400; where \( \theta(x, y) = \frac{1}{2} \| F(x, y) \|_2^2 \). \( x_0 \) are chosen randomly in \( \mathbb{R}^n \). \( y_0 \) are chosen randomly in \( \mathbb{R}^{2n} \), \( \lambda = 0.0001 \).

**Table 7.** Numerical results for Example 4.

<table>
<thead>
<tr>
<th>DIM</th>
<th>( p )</th>
<th>Final Value of Method 1</th>
<th>Final Value of Fmincon</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>( 1.6000 \times 10^{-3} )</td>
<td>0.4315</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>( 1.6000 \times 10^{-3} )</td>
<td>0.4315</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>( 1.6000 \times 10^{-3} )</td>
<td>0.4315</td>
</tr>
<tr>
<td>100</td>
<td>2</td>
<td>( 3.8000 \times 10^{-3} )</td>
<td>0.4426</td>
</tr>
<tr>
<td>100</td>
<td>4</td>
<td>( 4.5000 \times 10^{-3} )</td>
<td>0.4461</td>
</tr>
<tr>
<td>100</td>
<td>6</td>
<td>( 4.0000 \times 10^{-3} )</td>
<td>0.4502</td>
</tr>
<tr>
<td>200</td>
<td>2</td>
<td>( 4.6000 \times 10^{-3} )</td>
<td>0.5101</td>
</tr>
<tr>
<td>200</td>
<td>4</td>
<td>( 4.8000 \times 10^{-3} )</td>
<td>0.4123</td>
</tr>
<tr>
<td>200</td>
<td>6</td>
<td>( 4.6000 \times 10^{-3} )</td>
<td>0.5108</td>
</tr>
<tr>
<td>300</td>
<td>2</td>
<td>( 1.3263 \times 10^{-4} )</td>
<td>0.5394</td>
</tr>
<tr>
<td>300</td>
<td>4</td>
<td>( 0.4373 \times 10^{-3} )</td>
<td>0.5665</td>
</tr>
<tr>
<td>300</td>
<td>6</td>
<td>( 6.3331 \times 10^{-4} )</td>
<td>0.5395</td>
</tr>
<tr>
<td>400</td>
<td>2</td>
<td>( 4.6550 \times 10^{-4} )</td>
<td>0.4575</td>
</tr>
<tr>
<td>400</td>
<td>4</td>
<td>( 8.0495 \times 10^{-4} )</td>
<td>0.5365</td>
</tr>
<tr>
<td>400</td>
<td>6</td>
<td>( 3.2255 \times 10^{-4} )</td>
<td>0.5514</td>
</tr>
</tbody>
</table>

**Remark 2.** By the numerical results of Example 4, we can see that Method 1 is very suitable to solve large-scale SLCP. Moreover, Method 1 can be used flexible by adjusting the value of \( p \).

5. Conclusions

In this paper, we introduced a feasible nonsmooth Levenberg–Marquardt-type method to solve the stochastic linear complementarity problems with finitely many elements. This method used a linear least squares reformulation of the stochastic linear complementarity problem and applied a feasible nonsmooth Levenberg–Marquardt-type method to solve the reformulated problem. The finally given numerical results showed that the given method is efficient to solve the large-scale stochastic linear complementarity problem and related refinery production problem. Additionally, the method can choose the initial points in a large scope with less computations and high precision.

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**Author Contributions:** Zhi-min Liu prepared the manuscript. Rui-ying Wang assisted in the work. Shou-qiang Du was in charge of the overall research of the paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**


