Recognizing the Repeatable Configurations of Time-Reversible Generalized Langton’s Ant is PSPACE-Hard

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Abstract: Chris Langton proposed a model of an artificial life that he named “ant”: an agent called ant that is over a square of a grid moves by turning to the left (or right) accordingly to black (or white) color of the square where it is heading, and the square then reverses its color. Bunimovich and Troubetzkoy proved that an ant’s trajectory is always unbounded, or equivalently, there exists no repeatable configuration of the ant’s system. On the other hand, by introducing a new type of color where the ant goes straight ahead and the color never changes, repeatable configurations are known to exist. In this paper, we prove that determining whether a given finite configuration of generalized Langton’s ant is repeatable or not is PSPACE-hard. We also prove the PSPACE-hardness of the ant’s problem on a hexagonal grid.

Keywords: cellular automata; computational complexity; Langton’s ant; Lorentz lattice gas; PSPACE-hard

1. Introduction

1.1. Generalized Langton’s Ant

The virtual ant defined by Chris Langton [1–3] is the following cellular automaton. The “ant”, represented by a pair \((z, D)\) of a lattice point \(z \in \mathbb{Z}^2 := \mathbb{Z} + i\mathbb{Z}\) and a direction \(D \in \{e^0, e^{2\pi i}, e^{\pi i}, e^{3\pi i}\}\),
moves around on a two-dimensional square lattice $\mathbb{Z}^2$, where each lattice point, referred to as a “cell”, is colored by white or black (later we will also introduce gray cells). Initially, the ant is sitting on a given cell with a given direction, say $z = 0$ and $D = 1$. It proceeds to travel from cell to cell according to the following rule (see Figure 1): the ant $(z, D)$ moves to $(z + D, e^{i\pi/8}D)$ when $z + D$ is a white cell, and to $(z + D, e^{-i\pi/8}D)$ when $z + D$ is a black cell; the cell $z + D$ then reverses its color. In other words, the ant moves in the direction it is heading; when it lands on a white (or black) cell it rotates its direction $90^\circ$ to the right (left); after that, the color of the cell changes to black (or white).

Langton’s ant has been investigated independently as one model of Lorentz Lattice Gas Cellular Automata (LLGCA). Langton’s ant corresponds to the Flipping Rotator (FR) model on $[4]$. In more general terms, the FR model was investigated on a triangular lattice $T^2 = \mathbb{Z} + e^{i\pi/3}\mathbb{Z}$ [5] and on a hexagonal lattice $H^2 = T^2 - \sqrt{3}e^{i\pi/6}T^2$ [6], too. On $H^2$ the ant $(z, D)$, where $z \in H^2$ and $D \in \{e^{2\pi i k/3} : k = 0, 1, 2\}$, moves to $(z + D, e^{2\pi i/3}D)$ and $(z + D, e^{-2\pi i/3}D)$ when $z + D$ is white and black, respectively (see Figure 1), and the cell $z + D$ then reverses its color. Another generalization of Langton’s ant on $\mathbb{Z}^2$ is to introduce a third type of cell, called “gray” cell [7]. The ant $(z, D)$ moves to $(z + D, D)$ if $z + D$ is a grey cell, and the cell $z + D$ never changes its color but remains gray forever (see Figure 1). In LLGCA models, gray cells were introduced naturally as empty lattice points. We remark that the $H^2$ topology does not allow having such a gray cell.

We denote by R, L and S the ant’s valid moves corresponding to the Right-turn, Left-turn and Straight-ahead respectively under these transaction rules; e.g., by R, L and S, the ant $(z, D)$ on $\mathbb{Z}^2$ moves to $(z + D, e^{i\pi/8}D)$, $(z + D, e^{-i\pi/8}D)$ and $(z + D, D)$, respectively.

**Figure 1.** Transaction rules on each topology and each color of the cell that the ant is heading to.

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<tr>
<th>$\mathbb{Z}^2$</th>
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1.2. Recognizing the Repeatable Configurations of GLA

These transaction rules assure that Generalized Langton’s Ant (GLA) is a time-reversible cellular automaton: the current configuration of GLA, consisting of a coloring of the cells and an ant’s starting cell and direction, determines the past configurations as well as the future ones. As a consequence, the
configurations of GLA are divided into the following two kinds: an ant’s trajectory starting from a configuration of one kind is unbounded, never repeating the same configuration again; an ant’s trajectory starting from a configuration of the other kind is bounded, repeating a finite series of configurations an infinitive number of times.

A “finite” configuration of GLA is defined by a finite coloring of the cells and an ant's starting cell and direction. Here, a coloring is finite if it has only a finite number of non-background-color cells. In this paper, we use the all white, all black, and half-and-half coloring as the background, where the half-and-half coloring gives white (black) color to the cells on the upper-half (lower-half) plane (see Figure 2). We define size of a configuration by the minimum size of a closed square which contains the ant’s starting cell and the all non-background-color cells. In this paper, we study the computational complexity of determining whether a given finite configuration of GLA is repeatable or not.

Figure 2. The half-and-half coloring.

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<tr>
<td><img src="image1.png" alt="Half-and-Half Coloring" /></td>
<td><img src="image2.png" alt="Half-and-Half Coloring" /></td>
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1.3. Previous Results

Bunimovich and Troubetskoy [4] proved the following: when there is no gray cell, an ant's trajectory on $\mathbb{Z}^2$ is always unbounded, or equivalently, there exists no repeatable configuration of the ant’s system. As a matter of fact, the set of repeatable configurations of GLA on $\mathbb{Z}^2$ with no gray cell is empty, hence its recognition problem is trivial. On the other hand, repeatable configurations exist of GLA on $\mathbb{Z}^2$ with some gray cells (see Figure 3, see also [8]). The $\mathbb{H}^2$ model is also known to have repeatable configurations (see Figure 3, see also [6]).

The long-run behavior of Langton’s ant on $\mathbb{Z}^2$ has been studied using both theories and experiments for more than two decades, yet it is still highly unpredictable. As a result, indicating hardness of the prediction, Gajardo, Moreira and Goles [9] proved that the following problem on $\mathbb{Z}^2$ with no gray cell is PTIME-hard: “Does the ant ever visit this given cell?”.

Figure 3. Repeatable configurations of GLA.

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<th>$\mathbb{Z}^2$</th>
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<td><img src="image3.png" alt="Repeatable Configurations" /></td>
<td><img src="image4.png" alt="Repeatable Configurations" /></td>
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1.4. New Results

In this paper, we prove the following theorems:

**Theorem 1.** Recognizing the repeatable configurations of GLA on $\mathbb{Z}^2$ with gray cells is PSPACE-hard.

**Theorem 2.** Recognizing the repeatable configurations of GLA on $\mathbb{H}^2$ is PSPACE-hard.

To prove these theorems, we should have unbounded trajectories of the ant on each of the topologies. For the half-and-half background, Figure 2 shows such unbounded trajectories where the ant, starting from the arrow, walks from left to right by repeating the LLRR (LLLRRRR) moves forever on $\mathbb{Z}^2$ ($\mathbb{H}^2$). On the other hand, for the monochromatic background, we do have the famous diagonal highway on $\mathbb{Z}^2$ (see Figure 4), but no provable unbounded trajectory is known on $\mathbb{H}^2$ [6]. For this reason, we can prove Theorem 1 for both of the monochromatic and half-and-half backgrounds, but Theorem 2 for only the half-and-half background.

**Figure 4.** The ant starting from the arrow proceeds as RLRLRLRLRL, and then starts repeating 104 steps forever, forming the famous diagonal highway going in a southeast direction.

We will prove Theorem 1 for “everywhere sparse” gray cells. For a given function $f : \mathbb{N} \to \mathbb{N}$, we say that a configuration is colored by $f$-sparse gray cells if the configuration size is $n$ and its coloring has no more than one gray cell within any closed square of size no more than $\frac{n}{\sqrt{f(n)}}$. For example, a configuration colored by $k$-sparse gray cells, for any constant $k \in \mathbb{N}$, contains no more than $k$ gray cells, and a configuration colored by $n^2$-sparse gray cells could have an arbitrary number of gray cells. We prove the following theorem:

**Theorem 3.** For any $\epsilon > 0$ recognizing the repeatable configurations of GLA on $\mathbb{Z}^2$ colored by $n^\epsilon$-sparse gray cells is PSPACE-complete.

1.5. Reduction

To prove Theorems 1–3, it is enough to reduce a known PSPACE-hard problem to the ant’s problems on square and hexagonal grids. In this paper, we will reduce QBF (Quantified Boolean Formula) evaluation problem to the ant’s problems. An instance of QBF is given by a closed CNF (Conjunctive Normal Form) formula, which is written as $Q_1x_1Q_2x_2\cdots Q_nx_n\phi(x_1,\ldots,x_n)$ by an open CNF formula $\phi(x_1,\ldots,x_n)$ and arbitrary Boolean quantifiers $Q_j \in \{\exists, \forall\}$. Then, QBF evaluation problem asks
the truth value of a given closed CNF formula. QBF evaluation problem is a well known PSPACE-complete problem [10].

1.6. Preparation

A (ant’s walking) course is a sequence of ant’s consecutive valid moves; it also represents a sequence of the induced coloring. When a coloring $S$ of the ant’s system turns to a coloring $T$ by an ant’s walking course $w$, we write as $S \xrightarrow{w} T$. When the ant, at a place $I$, moves to another place $O$ by $w$, we write it as $I \xrightarrow{w} O$. The reverse of the ant $(z, D)$ is $(z + D, -D)$. By time reversibility, we can define the reverse $w^{-1}$ of $w$ by the following walking course: reverse the order of the coloring $w$, and reverse the ants therein. We write $S \xleftarrow{w^{-1}} T$, meaning that $S$ turns to $T$ by $w$, and $T$ turns to $S$ by $w^{-1}$.

2. Gadgets

A gadget is a collection of GLA’s coloring written on the all white background with an associated transition diagram and several input and output marked arrows. A polynomial number of gadgets are seamlessly connected to form an entire coloring of GLA, where some gadgets may be used after rotation or reflection. Note that the colors of reflected gadgets should be switched.

2.1. Path

We connect the rotations and reflections of PATH gadgets (see Figure 5) to form a long path along which the ant is guided. Its coloring is initially $INIT$, which turns to $USED$ by a course $IN \xrightarrow{w} OUT$.

Figure 5. PATH.

2.2. Switch & Pass

A Switch & Pass (S&P) gadget (see Figure 6) can memorize 1 bit information by its coloring state. The coloring is initially $OFF$, which turns to $ON$ by a walking course $I_{S&P} \xrightarrow{w_1} O_{S&P}$; in other words, the ant is “Switching” the coloring state and “Passing” through it. When the coloring is $OFF$, the ant entering at $IN$ walks along $w_2$ and exits at $O_{OFF}$, changing the coloring to $UOFF$. When the coloring is $ON$, the ant entering at $IN$ walks along $w_3$ and exits at $O_{ON}$, changing the coloring to $UON$. 
2.3. Switch & Turn

A Switch & Turn (S&T) gadget (see Figure 7) can memorize 1 bit information by its coloring state, too. The coloring is initially OFF, which turns to ON by an ant’s walking course \( I_{\text{S&T}} \rightarrow O_{\text{S&T}} \); in other words, since \( O_{\text{S&T}} \) is the reverse of \( I_{\text{S&T}} \), the ant is “Switching” the coloring state and “Turning” around. When the coloring is OFF, the ant entering at IN walks along \( w_2 \) and exits at \( OUT_{OFF} \). When the coloring is ON, the ant entering at IN walks along \( w_3 \) and exits at \( OUT_{ON} \).

2.4. Conjunction

The CONJunction (CONJ) gadget (see Figure 8, see also [9]) has two entrances and one exit. The coloring is initially \( INIT \). The ant entering at \( IN_j \) walks along \( w_j \) and exits at \( OUT \), for each \( j = 1, 2 \).
2.5. Pseudo-Crossing

A Pseudo-Crossing (PC) gadget (see Figure 9, see also [9]) has two entrances \(IN_1\) and \(IN_2\) and two exits \(OUT_1\) and \(OUT_2\) such that the ant entering at \(IN_j\) walks along \(w_j\) and exits at \(OUT_j\) for each
Since $IN_1$, $IN_2$, $OUT_1$, $OUT_2$ are placed clockwise in this order in 2D plane, these two walking courses $w_1$ and $w_2$ should be mutually crossing. Beginning from the initial coloring $INIT$, the ant can take mutually intersecting courses, first $IN_1 \rightarrow w_1 \rightarrow OUT_1$ and secondly $IN_1 \rightarrow w_2 \rightarrow OUT_{1,2}$ in order, changing the coloring to $INIT \rightarrow USED_1 \rightarrow USED_{1,2}$.

**Figure 9.** PC.

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<td>$H^2$</td>
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2.6. Crossing

A CROSSing (CROSS) gadget (see Figure 10) is placed at each crossing point of two intersecting paths on 2D plane. It is built by one S&P gadget and three PC gadgets, named PC$_1$, PC$_2$ and PC$_3$. The coloring of a CROSS gadget can be represented by the coloring of all gadgets composing it, that are S&P, PC$_1$, PC$_2$, PC$_3$, two CONJ gadgets and many PATH gadgets connecting them. We indicate the coloring of a CROSS gadget only by those of (S&P, PC$_1$, PC$_2$, PC$_3$). The coloring of CONJ (PATH) gadgets and PATH gadgets are initially $INIT$, that turn to $USED_3$ ($USED$) when the ant has passed through them. The ant can take mutually intersecting courses by first $IN_1 \rightarrow w_1 \rightarrow OUT_1$ and secondly $IN_2 \rightarrow w_3 \rightarrow OUT_2$, as well as by first $IN_2 \rightarrow w_2 \rightarrow OUT_2$ and secondly $IN_1 \rightarrow w_4 \rightarrow OUT_1$. In Figure 10, each of these paths are depicted by thick lines, where those gadgets that passed through are shown by the thick lines that have turned to used coloring, while the other gadgets remain in their initial coloring.
3. CNF Formulae Evaluation

In this section, we construct an \( \text{EVAL}_\phi \) gadget for evaluating the truth value of a CNF formula \( \phi \) by inputting a given truth-value assignment \( a \). It has one entrance \( \text{IN} \) and two exits \( \text{OUT}_{\text{FALSE}} \) and \( \text{OUT}_{\text{TRUE}} \). The ant entering at \( \text{IN} \) is routed to exit at \( \text{OUT}_{\text{FALSE}} \) (\( \text{OUT}_{\text{TRUE}} \)) if \( \phi(a) \) is FALSE (TRUE).

Let \( \text{POS} = \{x_1, \ldots, x_m\} \) and \( \text{NEG} = \{-x_1, \ldots, -x_n\} \). Let \( y_{i,j} \in \text{POS} \cup \text{NEG} \) be the \( j \)th literal in the \( i \)th term of \( \phi(x_1, \ldots, x_n) \), where \( 1 \leq i \leq m \) and \( 1 \leq j \leq k_i \). These literals \( y_{i,j} \) are implemented in \( \text{EVAL}_\phi \) by S&P gadgets, named \( \text{S&P}_{i,j} \). Then, draw \textsc{Paths} between \( \text{S&P}_{i,j} \) in the following manner to define an evaluator \( \text{EVAL}_\phi \) of \( \phi: \text{IN} \rightarrow \text{IN} \) of \( \text{S&P}_{1,1} \) (see Figure 11-1). When \( y_{i,j} \in \text{POS} \) (NEG), \( \text{OUT}_{\text{OFF}} \) (\( \text{OUT}_{\text{ON}} \)) of \( \text{S&P}_{i,j} \rightarrow \text{IN} \) of \( \text{S&P}_{i,j+1} \) (see Figure 11-2). When \( y_{i,k_i} \in \text{POS} \) (NEG), \( \text{OUT}_{\text{OFF}} \) (\( \text{OUT}_{\text{ON}} \)) of \( \text{S&P}_{i,k_i} \rightarrow \text{OUT}_{\text{FALSE}} \) (see Figure 11-3). When \( y_{m,j} \in \text{POS} \) (NEG), \( \text{OUT}_{\text{ON}} \) (\( \text{OUT}_{\text{OFF}} \)) of \( \text{S&P}_{m,j} \rightarrow \text{OUT}_{\text{TRUE}} \) (see Figure 11-4).

For each \( x_k \in \text{POS} \), \( \text{EVAL}_\phi \) has an entrance \( I_{x_k} \) and an exit \( O_{x_k} \) such that the ant entering at \( I_{x_k} \) switches the coloring states of all S&P gadgets corresponding to the literals \( y_{i,j} \in \{x_k, -x_k\} \) from \( \text{OFF} \) to \( \text{ON} \), and exits at \( O_{x_k} \). Figure 11-5 is drawing such \textsc{Paths}, where the ports \( I_{x_1}, O_{x_1}, I_{x_2}, O_{x_2}, \ldots, I_{x_n}, O_{x_n} \) should be placed in this order for further development in the next section.

Suppose that the coloring of the composing gadgets of \( \text{EVAL}_\phi \) are all initialized. For every \( a \in \{\text{FALSE}, \text{TRUE}\}^n \), we denote by \( \text{INIT}_a \) the coloring of \( \text{EVAL}_\phi \) after taking the walking courses.
\[ I_{x_k} \rightarrow O_{x_k} \text{ for all } x_k \text{ such that } a_{x_k} = \text{TRUE} \text{ in the descending order of } k. \] Consequently, \( INIT_a \) sets the coloring of every S&P\( _{i,j} \) with \( y_{i,j} \in \{ x_k, -x_k \} \ OFF(ON) \) if \( a_k = \text{FALSE} \ (\text{TRUE}). \)

**Figure 11. EVAL-\( \phi \).**
Lemma 1. If the coloring of \( \text{EVAL}_\phi \) is \( \text{INIT}_a \) and \( \phi(a) = \text{FALSE} \) (TRUE), then the ant entering at \( \text{IN} \) exits at \( \text{OUT}_\text{FALSE} \) (\( \text{OUT}_\text{TRUE} \)).

Proof. First, suppose that \( \phi(a) = \text{FALSE} \). We can assume that the \( \ell \)th term of \( \phi \) is \( \text{FALSE} \) and the \( i \)th term of \( \phi \) is \( \text{TRUE} \) for every \( 1 \leq i < \ell \). So, we can assume that for every \( 1 \leq j \leq k_\ell \), if \( y_{\ell,j} \in \text{POS} \) (\( \text{NEG} \)) then \( \text{S&P}_{\ell,j} \) is \( \text{OFF} \) (\( \text{ON} \)), and for every \( 1 \leq i < \ell \) there exists \( 1 \leq j_i \leq k_i \) such that if \( y_{i,j_i} \in \text{POS} \) (\( \text{NEG} \)) then \( \text{S&P}_{i,j_i} \) is \( \text{ON} \) (\( \text{OFF} \)); in addition, for every \( 1 \leq j < j_i \), if \( y_{i,j} \in \text{POS} \) (\( \text{NEG} \)) then \( \text{S&P}_{i,j} \) is \( \text{OFF} \) (\( \text{ON} \)). Given these colorings of the \( \text{S&P} \)s, the ant walks from \( \text{IN} \) to \( \text{OUT}_\text{FALSE} \) in the following way: \( \text{IN} \rightarrow \text{IN} \) of \( \text{S&P}_{1,1} \); \( \text{IN} \) of \( \text{S&P}_{i,j} \rightarrow \text{OUT}_\text{OFF} \) (\( \text{OUT}_\text{ON} \)) of \( \text{S&P}_{i,j} \rightarrow \text{IN} \) of \( \text{S&P}_{i,j+1} \); \( \text{IN} \) of \( \text{S&P}_{i,j} \rightarrow \text{OUT}_\text{ON} \) (\( \text{OUT}_\text{OFF} \)) of \( \text{S&P}_{i,j} \rightarrow \text{IN} \) of \( \text{S&P}_{i+1,1} \); \( \text{IN} \) of \( \text{S&P}_{\ell,j} \rightarrow \text{OUT}_\text{OFF} \) (\( \text{OUT}_\text{ON} \)) of \( \text{S&P}_{\ell,j} \rightarrow \text{IN} \) of \( \text{S&P}_{\ell,k_\ell} \); \( \text{IN} \) of \( \text{S&P}_{\ell,1} \rightarrow \text{OUT}_\text{OFF} \) \( \text{OUT}_\text{ON} \) of \( \text{S&P}_{\ell,1} \rightarrow \text{OUT}_\text{OFF} \). Figure 11-6 illustrates this walking course by a thick line when \( \phi = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \) and \( (a_1, a_2) = (\text{TRUE}, \text{FALSE}) \), where the gadgets have passed through, shown by the thick line, have turned to used coloring states, while the other gadgets still remain in their initial coloring.

Secondly, suppose that \( \phi(a) = \text{TRUE} \). Then, all terms of \( \phi \) are \( \text{TRUE} \). So, we can assume that for every \( 1 \leq i < m \) there exists \( 1 \leq j_i \leq k_i \) such that if \( y_{i,j_i} \in \text{POS} \) (\( \text{NEG} \)) then \( \text{S&P}_{i,j_i} \) is \( \text{ON} \) (\( \text{OFF} \)); in addition, for every \( 1 \leq j < j_i \), if \( y_{i,j} \in \text{POS} \) (\( \text{NEG} \)) then \( \text{S&P}_{i,j} \) is \( \text{OFF} \) (\( \text{ON} \)). Given these colorings of the \( \text{S&P} \)s, the ant walks from \( \text{IN} \) to \( \text{OUT}_\text{TRUE} \) in the following way: \( \text{IN} \rightarrow \text{IN} \) of \( \text{S&P}_{1,1} \); \( \text{IN} \) of \( \text{S&P}_{i,j} \rightarrow \text{OUT}_\text{OFF} \) (\( \text{OUT}_\text{ON} \)) of \( \text{S&P}_{i,j} \rightarrow \text{IN} \) of \( \text{S&P}_{i,j+1} \); \( \text{IN} \) of \( \text{S&P}_{i,j} \rightarrow \text{OUT}_\text{ON} \) (\( \text{OUT}_\text{OFF} \)) of \( \text{S&P}_{i,j} \rightarrow \text{IN} \) of \( \text{S&P}_{i+1,1} \); \( \text{IN} \) of \( \text{S&P}_{m,j} \rightarrow \text{OUT}_\text{OFF} \) \( \text{OUT}_\text{ON} \) of \( \text{S&P}_{m,j} \rightarrow \text{IN} \) of \( \text{S&P}_{m,j+1} \); \( \text{IN} \) of \( \text{S&P}_{m,j} \rightarrow \text{OUT}_\text{ON} \) \( \text{OUT}_\text{OFF} \) of \( \text{S&P}_{m,j} \rightarrow \text{OUT}_\text{ON} \). Figure 11-7 illustrate this walking course when \( \phi = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \) and \( (a_1, a_2) = (\text{TRUE}, \text{TRUE}) \).

Let \( \text{USED}_a \) be the coloring of an \( \text{EVAL}_\phi \) gadget derived from \( \text{INIT}_a \) by the ant’s walking course from the entrance to an exit of \( \text{EVAL}_\phi \).

4. Boolean Quantifiers Evaluation

Let \( \phi(x_1, \ldots, x_n) \) be an arbitrary CNF formula of the variables \( x_1, \ldots, x_n \). For arbitrary taken quantifiers \( Q_i \in \{ \exists, \forall \} \), let \( \phi_i(x_1, \ldots, x_i) := Q_{i+1}x_{i+1}Q_{i+2}x_{i+2}\ldots Q_nx_n\phi(x_1, \ldots, x_n) \). In particular, \( \phi_n(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n) \). Let \( \text{EVAL}_\phi := \text{EVAL}_\phi \) built in the previous section. We have already defined \( \text{INIT}_{n,a} := \text{INIT}_a \) as a coloring of \( \text{EVAL}_n \) for every \( a \in \{ \text{FALSE}, \text{TRUE} \}^n \). In the following, for each \( i = n, n-1, \ldots, 0 \), we define inductively an \( \text{EVAL}_i \) gadget and its coloring \( \text{INIT}_{i,a} \) for every \( a \in \{ \text{FALSE}, \text{TRUE} \}^i \). Figure 12 depicts our construction of an \( \text{EVAL}_i \) gadget when \( Q_i = \forall \), containing an already constructed \( \text{EVAL}_{i+1} \) gadget. We describe the coloring of \( \text{EVAL}_i \) by those of the \( \{ \text{S&T}, \text{S&P}, \text{CONJ}, \text{EVAL}_{i+1} \} \) gadgets therein. For \( a \in \{ \text{FALSE}, \text{TRUE} \}^i \) and \( b \in \{ \text{FALSE}, \text{TRUE} \}^i \), let \( (a,b) = (a_1, \ldots, a_i, b) \in \{ \text{FALSE}, \text{TRUE} \}^{i+1} \). Let \( \text{INIT}_{i,a} \) be the coloring of \( \text{EVAL}_i \) that sets the coloring of the \( \{ \text{S&T}, \text{S&P}, \text{CONJ}, \text{EVAL}_{i+1} \} \) gadgets as \( \{ \text{OFF}, \text{OFF}, \text{INIT}, \text{INIT}_{i+1, (a, \text{FALSE})} \} \); the coloring of the other gadgets are set to be initialized. Then, we denote by \( \text{USED}_{i,a} \) the coloring derived from \( \text{INIT}_{i,a} \) by the ant’s walking course from the entrance to an exit of \( \text{EVAL}_i \). We have already defined \( \text{USED}_{n,a} := \text{USED}_a \). In the following lemma, \( \text{USED}_{i,a} \) are inductively defined, too.

Lemma 2. For every \( a \in \{ \text{FALSE}, \text{TRUE} \}^i \), if the coloring of \( \text{EVAL}_i \) is \( \text{INIT}_{i,a} \) and \( \phi_i(a_1, \ldots, a_i) \) = \( \text{FALSE} \) (\( \text{TRUE} \)) then the ant entering to the \( \text{EVAL}_i \) gadget at \( \text{IN}_i \) exits at \( \text{OUT}_i, \text{FALSE} \) (\( \text{OUT}_i, \text{TRUE} \)).
Proof. Lemma 1 proves the $i = n$ case of Lemma 2. Let $a \in \{\text{FALSE, TRUE}\}^i$. By the backward induction hypothesis, Lemma 2 is assumed to hold for both $\phi_{i+1}(a, \text{FALSE})$ and $\phi_{i+1}(a, \text{TRUE})$. In the following proof, we prove Lemma 2 for $\phi_i(a)$ when $Q_i = Q_{\text{Q}}$.

Case 1. Suppose that $\phi_{i+1}(a, \text{FALSE}) = \text{FALSE}$. The ant walks as follows (Figure 12-1): $IN_i \rightarrow S&T \rightarrow \text{CONJ} \rightarrow IN_{i+1}$, changing the coloring of $(S&T, \text{CONJ})$ from $(U\text{OFF, INIT})$ to $(\text{OFF, USED}_i)$; next, since the coloring of $\text{EVAL}_{i+1}$ is $\text{INIT}_{i+1}(a, \text{FALSE})$, by Lemma 2 for $\phi_{i+1}(a, \text{FALSE})$, the ant starting from $IN_{i+1}$ reaches to $OUT_{i+1, \text{FALSE}}$, which changes the coloring of $\text{EVAL}_{i+1}$ from $\text{INIT}_{i+1}(a, \text{FALSE})$ to $\text{USED}_{i+1}(a, \text{FALSE})$; finally, the ant walks from $OUT_{i+1, \text{FALSE}}$ to $OUT_{i, \text{FALSE}}$.

Case 2. Suppose $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$ and $\phi_{i+1}(a, \text{TRUE}) = \text{FALSE}$. First, the ant walks as follows (see Figure 12-2): $IN_i \rightarrow S&T \rightarrow \text{CONJ} \rightarrow IN_{i+1}$; next, since $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$, the ant entering $\text{EVAL}_{i+1}$ at $IN_{i+1}$ reaches to $OUT_{i+1, \text{TRUE}}$, changing the coloring of $\text{EVAL}_{i+1}$ from $\text{INIT}_{i+1}(a, \text{FALSE})$ to $\text{USED}_{i+1}(a, \text{FALSE})$; after that, the ant walks as $OUT_{i+1, \text{TRUE}} \rightarrow IN$ of $S&P \rightarrow OUT_{\text{OFF}}$ of $S&P \rightarrow I_{S&T}$, which change the coloring of the $S&P$ gadget from $\text{OFF}$ to $\text{ON}$, and the coloring of the $\text{PATHs}$ taken from $\text{INIT}$ to $\text{USED}$; then, getting into the $S&T$ gadget, the ant turns around and exits at $O_{S&T}$, switching the coloring of the $S&T$ gadget from $\text{OFF}$ to $\text{ON}$.

Secondly, the ant takes the following reversed walking course (see Figure 12-3): $O_{S&T} \rightarrow S&P \rightarrow \text{EVAL}_{i+1} \rightarrow \text{CONJ} \rightarrow IN$ of $S&T$, changing the colorings of $(S&P, \text{EVAL}_{i+1}, \text{CONJ})$ from $(U\text{OFF, USED}_{i+1}(a, \text{FALSE}), \text{USED}_i)$ to $(\text{ON, INIT}(a, \text{FALSE}), \text{INIT})$; the coloring of the taken $\text{PATH}$ and $\text{CROSS}$ gadgets are initialized, too.

We remark that the gadgets that passed through, shown by the dotted line in Figure 12-3, are those gadgets that changed to the used color by the ant’s forward walking, but then initialized by the ant’s backward walking. These gadgets can be used again without violating the rules of the transition diagram given in Section 2; in other words, the already completely used gadgets, e.g., $\text{USED}$ $\text{PATH}$ gadgets or $\text{USED}_{1,2}$ $\text{CROSS}$ gadgets, will never be used without being initialized in this manner.

Finally, the ant walks as follows (see Figure 12-4): $IN$ of $S&T \rightarrow OUT_{\text{ON}}$ of $S&T \rightarrow I_{S&T} \rightarrow O_{S&P} \rightarrow I_{x_i}$ of $\text{EVAL}_{\phi} \rightarrow O_{x_j}$ of $\text{EVAL}_{\phi} \rightarrow \text{CONJ} \rightarrow IN$ of $S&T$, changing the coloring of $(S&T, S&P, \text{EVAL}_{i+1}, \text{CONJ})$ from $(\text{ON, OFF, INIT}(a, \text{FALSE}), \text{INIT})$ to $(\text{UON, ON, INIT}(a, \text{TRUE}), \text{USED}_2)$; we remark that, as shown in Figure 11, the walking course $I_{x_j} \rightarrow O_{x_j}$ going through the $\text{EVAL}_{\phi}$ gadget have polynomially many crossing points with other such courses $I_{x_j} \rightarrow O_{x_j}$ inside the $\text{EVAL}_{\phi}$ gadget; next, since $\phi_{i+1}(a, \text{TRUE}) = \text{FALSE}$, the ant entering to $\text{EVAL}_{i+1}$ at $IN_{i+1}$ reaches $OUT_{i+1, \text{FALSE}}$; finally, the ant walks as $OUT_{i+1, \text{FALSE}} \rightarrow OUT_{i, \text{FALSE}}$.

Case 3. Suppose $\phi_{i+1}(a, \text{FALSE}) = \text{TRUE}$ and $\phi_{i+1}(a, \text{TRUE}) = \text{TRUE}$. First, the ant walks as in Case 2: $IN_i \rightarrow S&T \rightarrow \text{CONJ} \rightarrow IN_{i+1} \rightarrow OUT_{i+1, \text{TRUE}} \rightarrow IN$ of $S&P \rightarrow OUT_{\text{OFF}}$ of $S&P \rightarrow I_{S&T} \rightarrow O_{S&T} \rightarrow S&P \rightarrow \text{EVAL}_{i+1} \rightarrow \text{CONJ} \rightarrow IN$ of $S&T \rightarrow OUT_{\text{ON}}$ of $S&T \rightarrow I_{S&P} \rightarrow O_{S&P} \rightarrow I_{x_i}$ of $\text{EVAL}_{\phi} \rightarrow O_{x_j}$ of $\text{EVAL}_{\phi} \rightarrow \text{CONJ} \rightarrow IN_{i+1}$.

Secondly, the ant walks as follows (see Figure 12-5): since the coloring of $\text{EVAL}_{i+1}$ has become $\text{INIT}(a, \text{TRUE})$ and $\phi_{i+1}(a, \text{TRUE}) = \text{TRUE}$, the ant reaches $OUT_{i+1, \text{TRUE}}$; after that, since the coloring of the $S&P$ gadget has become $\text{ON}$, the ant walks as $OUT_{i+1, \text{TRUE}} \rightarrow IN$ of $S&P \rightarrow OUT_{\text{ON}}$ of $S&P \rightarrow OUT_{i, \text{TRUE}}$.
By these three Cases, Lemma 2 has shown to hold for $\phi_i(a)$ when $Q_i = \forall$. Since $\exists$ is a logical dual of $\forall$, a gadget of the $Q_i = \exists$ case is obtained from Figure 12 by switching the labels $OUT_{i',\text{true}}$ and $OUT_{i',\text{false}}$ for $i' = i$ and $i+1$. Accordingly, rewriting the above proof attains that of the $Q_i = \exists$ case.

**Figure 12.** $\text{EVAL}_i$ when $Q_i = \forall$. 
5. Polynomial-Time Reduction

For a given closed QBF formula $\phi_0$, Lemma 2 and its proof gives a polynomial-time construction of an EVAL$_0$ gadget such that if $\phi_0 = \text{FALSE}$ (TRUE) then the ant placed at INIT$_0$ of INIT$_0$,FALSE$^*$ coloring of EVAL$_0$ finally reaches OUT$_0$,FALSE (OUT$_0$,TRUE). So, as illustrated in Figure 13, plugging reflectors (see Figure 3) in both INIT$_0$ and INIT$_0$,FALSE$^*$, and a diagonal highway (see Figure 4) in OUT$_0$,TRUE gives an initial configuration of GLA such that $\phi_0 = \text{FALSE}$ (TRUE) if, and only if, the ant stays in a bounded area (goes out of any bounded area). This establishes an efficient reduction from the QBF evaluation problem to the recognition problem of the repeatable coloring of GLA on $\mathbb{Z}^2$ with gray cells, proving Theorem 1. For the $H^2$ model, plugging a highway along the horizon of the half-and-half background (see Figure 1) to OUT$_0$,TRUE gives an efficient reduction, too, proving Theorem 2.

Among our gadgets on $\mathbb{Z}^2$ given in Section 2, only Switch & Turn gadget uses gray cell. In addition, the Switch & Turn gadget contains only one gray cell. So, putting these Switch & Turn gadgets mutually away from each other makes a size-$n$ configuration of GLA colored by $n^c$-sparse gray cells for the reduction, proving Theorem 3.

**Figure 13.** Reduction from QBF evaluation problem to the recognition problem of the repeatable coloring of GLA.

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<tr>
<th>$\mathbb{Z}^2$</th>
<th>$H^2$</th>
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6. Open Questions

We can construct all gadgets shown in Section 2 on the $T^2$ (triangular lattice) model with gray cells, excepting the Switch & Turn gadget. Although we are lacking the Switch & Turn gadget, we believe that the recognition problem of the repeatable configurations of GLA on $T^2$ with gray cells is PSPACE-hard. The experimental results by Wang and Cohen [6] showed that randomly generated configurations of GLA on $H^2$ for the monochromatic background, fall into the repeatable configurations with high probability. As far as we know, it is challenging to find even one provably unrepeatable configuration of GLA on $H^2$ for the monochromatic background. Perhaps it is more challenging to prove the following: “an ant’s trajectory starting from a repeatable size- $n$ configuration
of GLA is always at most a polynomial of \( n^r \). If this were true, then the recognition problem of the repeatable configurations of GLA would belong to PSPACE.

References


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