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Article

Univariate Cubic L_1 Interpolating Splines: Analytical Results for Linearity, Convexity and Oscillation on 5-Point Windows

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Abstract: We analytically investigate univariate C^1 continuous cubic L_1 interpolating splines calculated by minimizing an L_1 spline functional based on the second derivative on 5-point windows. Specifically, we link geometric properties of the data points in the windows with linearity, convexity and oscillation properties of the resulting L_1 spline. These analytical results provide the basis for a computationally efficient algorithm for calculation of L_1 splines on 5-point windows.

Keywords: convexity; cubic L_1 spline; 5-point window; interpolation; linearity; locally calculated; oscillation; second-derivative-based; univariate

Classification: MSC 65D05, 65D07

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1. Introduction

Shape-preserving techniques for interpolating and approximating multiscale data, that is, data with sudden large changes in magnitude and/or spacing, are important for modeling of natural and urban terrain, geophysical features, biological objects, robotic paths and many other irregular surfaces, processes and functions. Over the past decade, a new class of univariate and bivariate splines, namely, L_1 splines, that have superior shape-preserving properties for interpolating and approximating multiscale data has arisen ([1–21]). The L_1-norm minimization principles on which L_1 splines are based result in non-differentiable convex generalized geometric programs that, so far, have been more complex and more computationally expensive to solve than the programs by which other variants of splines, e.g., conventional and tension splines, T-splines, *etc.*, are solved, but the shape preservation provided by L_1 splines is significantly better than the shape preservation provided by these alternative approaches.

L_1 splines have typically been calculated by minimization of global spline functionals, that is, spline functionals that extend over the whole range of the data to be interpolated. However, there have been three reports in the literature of L_1 splines on local windows. The first such report is in [17], where bivariate L_1 splines were calculated by a non-iterative "domain decomposition" procedure on overlapping 80×80 windows and 40×40 subsets of these windows were pieced together to create global surfaces. With parallel computation, the domain-decomposition procedure results in sharply reduced computing time.

In 2007, a result for univariate L_1 splines on much smaller windows arose. In [2], Auquiert, Gibaru and Nyiri showed that, given five points on a Heaviside function with two to the left of the discontinuity and three to the right, the L_1 spline for these five points is linear over the set of three points ([2], Proposition 9). Even though preservation of linearity is not all of what we desire in shape preservation, it is a large part thereof. This linearity-preservation result suggests that calculation of L_1 splines on small, 5-point windows, is geometrically meaningful. An immediate generalization of Proposition 9 of [2] is that, if, in a set of five points, three consecutive points on one end are collinear, then the L_1 spline through those three points is, except in the case of a V-shaped corner, linear. Such a result does not hold when the five points are embedded in a larger data set and a global L_1 spline functional is minimized. The best result that can be achieved in the case of a global L_1 spline functional is the following.

Theorem 1. (Theorem 2 of [7]) *If four consecutive data points* (x_i, z_i), (x_{i+1}, z_{i+1}), (x_{i+2}, z_{i+2}) *and* (x_{i+3}, z_{i+3}) *lie on a straight line, then a cubic* L_1 *spline* $z(x)$ *preserves linearity over the middle interval* $[x_{i+1}, x_{i+2}]$*. If* $\beta^*_{i-1} \neq \pm\frac{5}{3}$*, then* $z(x)$ *preserves linearity over the first interval* $[x_i, x_{i+1}]$*. If* $\beta^*_{i+2} \neq \pm\frac{5}{3}$*, then* $z(x)$ *preserves linearity over the last interval* $[x_{i+2}, x_{i+3}]$*. (Here* β^*_{i-1} *and* β^*_{i+2} *are components of the optimal dual solution in [7].)*

In this case, one needs four (rather than just three) consecutive collinear points and the L_1 spline is guaranteed to be linear only in the second interval (the interval between the second and the third of the four points). The L_1 spline is linear in the first and third of the three cells only if additional conditions, ones that do not have clear geometric meaning, are fulfilled. Proposition 9 of [2] is thus a significant improvement over Theorem 2 of [7].

Proposition 9 of [2] shows that one can preserve linearity over a larger set of points by calculating the L_1 spline using local 5-point windows rather than globally. This has a potential strategic implication, namely, that one may be able, by replacing a global minimization problem by a set of local minimization problems, to both further improve the shape preservation capabilities of L_1 splines and at the same time reduce the computing time because the local problems are independent of each other and can be solved in parallel. This is the opportunity that this paper wishes to investigate. The authors Auquiert, Gibaru and Nyiri of [2] have followed up on their results of 2007 with an article [22] containing new analytical results about preservation of linearity by windowed, rotation-invariant parametric L_1 splines of degrees 3 and higher. In contrast, this present paper considers linearity, convexity and oscillation for 5-point-window, rotation-dependent nonparametric cubic L_1 splines.

The precise purpose of this present paper is to provide analytical results that link linearity, convexity and oscillatory properties of the data on 5-point windows with linearity, convexity, oscillatory and uniqueness properties of the resulting L_1 spline. In each 5-point window, a local L_1 spline functional is used to determine the first derivative at the middle (or, near boundaries, other) point in this window. After the first derivatives at all of the data points have been determined, a C^1 piecewise cubic interpolant, called the L_1 spline (or "locally calculated cubic L_1 spline"), is set up by Hermite interpolation in each interval. In Section 2, we investigate analytical properties of the spline functional that link local geometric properties of 5-point windows of the data with geometric properties of the local L_1 splines on these windows. Based on the analytical results for 5-point windows, we investigate in Section 3 the properties of the C^1 piecewise cubic interpolant that has derivatives determined by these 5-point-window L_1 splines. In Section 4, we summarize the results presented in the previous sections and describe potential computational implications of these results.

All of the quantities in this paper are real quantities. The nodes x_i, $i = 0, 1, \ldots, I$, are a strictly monotonic but otherwise arbitrary partition of the finite interval $[x_0, x_I]$. Let $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, I-1$. At each node x_i, the function value z_i is given, $i = 0, 1, \ldots, I$. The slope of the line segment connecting (x_i, z_i) and (x_{i+1}, z_{i+1}) is $\triangle z_i := \dfrac{z_{i+1} - z_i}{h_i}$, $i = 0, 1, \ldots, I-1$. The L_1 splines discussed in this paper are cubic polynomials in each interval (x_i, x_{i+1}), $i = 0, 1, \ldots, I-1$, and are C^1 continuous at the nodes. The first derivative of the spline at node x_i, $i = 0, 1, \ldots, I$, is denoted by b_i (to be determined by minimization of the L_1 spline functional). We use δ_i to denote the slope of the chord between neighboring points:

$$\delta_i = \frac{z_{i+1} - z_{i-1}}{x_{i+1} - x_{i-1}}, \quad i = 2, \ldots, I-2 \tag{1}$$

We use ζ to denote the linear spline:

$$\zeta(x) = \frac{(x_{i+1} - x)z_i + (x - x_i)z_{i+1}}{h_i}, \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, \ldots, I-1 \tag{2}$$

2. Analytical Properties of 5-Point-Window L_1 Splines

The splines that we will consider in this paper are calculated locally as described in this paragraph. For the interpolation problem under consideration in the present paper, the function values are given. In the 5-point window with middle point x_i, $2 \le i \le I-2$, the derivative at x_i is calculated by minimizing

$$\int_{x_{i-2}}^{x_{i+2}} \left| \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \right| \mathrm{d}x \tag{3}$$

over the finite-dimensional spline space of C^1 piecewise cubic polynomials z that interpolate the data. The free parameters in the minimization of functional (3) are the derivatives b_i of the spline at the five nodes. The derivative at node x_i that occurs at the minimum of functional (3) is denoted by b_i^*. Whenever the minimum of functional (3) is nonunique, we choose b_i^* to be the scalar in the optimal set (the interval $[b_i^l, b_i^u]$) closest to the slope δ_i of the chord between the neighboring points, that is, median$\{b_i^u, b_i^l, \delta_i\}$. Previously, nonuniqueness was resolved by "regularization" of the spline functional, specifically, by adding to the spline functional (3) a sum consisting of the absolute values of various expressions involving the derivatives at the nodes times a sufficiently small number ε (*cf.* [5,9,16]). The method for resolving nonuniqueness that we use in the present paper differs from the regularization approach used in previous L_1 spline work but leads to both simpler analysis and simpler computational procedures. The derivatives at the points x_0 and x_1 are determined by b_2^* which is calculated by minimizing (3) for $i = 2$. Analogously, the derivatives at the points x_{I-1} and x_I are determined by b_{I-2}^* which is calculated by minimizing (3) for $i = I - 2$. After obtaining all of the b_i^*, a C^1 piecewise cubic interpolant z is set up by Hermite interpolation

$$z(x) = z_i + b_i^*(x - x_i) - \frac{1}{h_i}(2b_i^* + b_{i+1}^* - 3\triangle z_i)(x - x_i)^2 + \frac{1}{h_i^2}(b_i^* + b_{i+1}^* - 2\triangle z_i)(x - x_i)^3 \tag{4}$$

for $x \in (x_i, x_{i+1})$, $i = 0, \ldots, I-1$ (*cf.* [9]). The C^1 piecewise cubic interpolant calculated in this manner is the L_1 spline (locally calculated cubic L_1 spline).

In the remainder of this section, we investigate the relation between the geometry of the 5 points in each window and the derivative at the middle point of the window. For the five points under consideration, we use the notation (x_0, z_0), (x_1, z_1), (x_2, z_2), (x_3, z_3) and (x_4, z_4). For the window with these 5 points, the objective function (3) is

$$E(\mathbf{b}) = \sum_{i=0}^{3} \int_{x_i}^{x_{i+1}} \left| \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \right| \mathrm{d}x = \sum_{i=0}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_{i+1} - b_i) + 6t(b_i + b_{i+1} - 2\triangle z_i)| \; \mathrm{d}t \tag{5}$$

where $\mathbf{b}$ denotes $(b_0, b_1, b_2, b_3, b_4)$. Each term in the summation is a function

$$\theta(p, q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |(q - p) + 6t(p + q)| \; \mathrm{d}t \tag{6}$$

that is continuously differentiable and has the properties stated in the following lemma.

Lemma 2. ([2]) *$\theta(p, q)$ is convex,*

$$\theta(p, q) = \begin{cases} |q - p| & \text{if } |q - p| \geq 3|p + q|, \\ \frac{3}{2}|p + q| + \frac{(q-p)^2}{6|p+q|} & \text{otherwise} \end{cases} \tag{7}$$

and

(1) $\min_{p \in R} \theta(p, q) = \dfrac{2(\sqrt{10} - 1)}{3}|q|$ *with* $p = \dfrac{2 - \sqrt{10}}{\sqrt{10}}q$,

(2) $\min_{q \in R} \theta(p, q) = \dfrac{2(\sqrt{10} - 1)}{3}|p|$ *with* $q = \dfrac{2 - \sqrt{10}}{\sqrt{10}}p$,

(3) $\min_{(p,q) \in R^2} \theta(p, q) = 0$ *with* $p = q = 0$.

On the basis of Lemma 2, we have

$$
\begin{aligned}
\min_{\mathbf{b}\in R^5} E(\mathbf{b}) &= \min_{b_1,b_2,b_3}\left\{\frac{2(\sqrt{10}-1)}{3}|b_1-\triangle z_0| + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_2-b_1)+6t(b_1+b_2-2\triangle z_1)|\ \mathrm{d}t\right. \\
&\quad \left. + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_3-b_2)+6t(b_2+b_3-2\triangle z_2)|\ \mathrm{d}t + \frac{2(\sqrt{10}-1)}{3}|b_3-\triangle z_3|\right\} \\
&= \min_{b_2}\left\{\min_{b_1}\left\{\frac{2(\sqrt{10}-1)}{3}|b_1-\triangle z_0| + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_2-b_1)+6t(b_1+b_2-2\triangle z_1)|\ \mathrm{d}t\right\}\right. \\
&\quad \left. + \min_{b_3}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_3-b_2)+6t(b_2+b_3-2\triangle z_2)|\ \mathrm{d}t + \frac{2(\sqrt{10}-1)}{3}|b_3-\triangle z_3|\right\}\right\}
\end{aligned}
$$

Minimization of $E(\mathbf{b})$ is a two-level minimization problem that can be written in the form

$$\min_{\mathbf{b}} E(\mathbf{b}) = \min_{b_2}\{G_1(b_2)+G_2(b_2)\} \tag{8}$$

where

$$G_1(b_2) = \frac{2(\sqrt{10}-1)}{3}|b_1(b_2)-\triangle z_0| + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_2-b_1(b_2))+6t(b_1(b_2)+b_2-2\triangle z_1)|\ \mathrm{d}t \tag{9}$$

and

$$G_2(b_2) = \frac{2(\sqrt{10}-1)}{3}|b_3(b_2)-\triangle z_3| + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(b_3(b_2)-b_2)+6t(b_2+b_3(b_2)-2\triangle z_2)|\ \mathrm{d}t \tag{10}$$

For later use, we introduce the notation

$$\phi(p,q;c) = \frac{2(\sqrt{10}-1)}{3}|p-c| + \int_{-\frac{1}{2}}^{\frac{1}{2}} |(q-p)+6t(p+q)|\ \mathrm{d}t \tag{11}$$

and

$$G(q;c) = \min_{p}\{\phi(p,q;c)\} = \phi(p(q),q;c)\,, \tag{12}$$

where c is a parameter.

Lemma 3. *The functions $\phi(p,q;c)$ and $G(q;c)$ are both convex. $G(q;c)$ is continuous on $q\in R$ and differentiable except at $q=0$. When $c=0$, we have $p(q)=0$ and*

$$\frac{\mathrm{d}G(q;0)}{\mathrm{d}q} = \begin{cases} \dfrac{5}{3} & \text{if } q>0\,, \\ -\dfrac{5}{3} & \text{if } q<0\,. \end{cases}$$

When $c>0$,

(i) *If $q > \frac{\sqrt{10}+1}{3}c$, then $p(q)=c$ and*

$$\frac{4\sqrt{10}-8}{3} < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = \frac{10q^2+20cq+6c^2}{6(c+q)^2} < \frac{5}{3}\,.$$

(ii) *If* $0 < q \leq \frac{\sqrt{10}+1}{3}c$, *then* $p(q) = \frac{\sqrt{10}-1}{3}q$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = \frac{4\sqrt{10}-8}{3}\,.$$

(iii) *If* $\frac{2-\sqrt{10}}{\sqrt{10}}c \leq q < 0$, *then* $p(q) = \frac{\sqrt{10}}{2-\sqrt{10}}q$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = 0\,.$$

(iv) *If* $-\frac{1}{2}c < q < \frac{2-\sqrt{10}}{\sqrt{10}}c$, *then* $p(q) = c$ *and*

$$-1 < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = \frac{10q^2+20cq+6c^2}{6(c+q)^2} < 0\,.$$

(v) *If* $-2c \leq q \leq -\frac{1}{2}c$, *then* $p(q) = c$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = -1\,.$$

(vi) *If* $q < -2c$, *then* $p(q) = c$ *and*

$$-\frac{5}{3} < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = -\frac{10q^2+20cq+6c^2}{6(c+q)^2} < -1\,.$$

When $c < 0$,

(i) *If* $q > -2c$, *then* $p(q) = c$ *and*

$$1 < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = \frac{10q^2+20cq+6c^2}{6(c+q)^2} < \frac{5}{3}\,.$$

(ii) *If* $-\frac{1}{2}c \leq q \leq -2c$, *then* $p(q) = c$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = 1\,.$$

(iii) *If* $\frac{2-\sqrt{10}}{\sqrt{10}}c < q < -\frac{1}{2}c$, *then* $p(q) = c$ *and*

$$0 < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = -\frac{10q^2+20cq+6c^2}{6(c+q)^2} < 1\,.$$

(iv) *If* $0 < q \leq \frac{2-\sqrt{10}}{\sqrt{10}}c$, *then* $p(q) = \frac{\sqrt{10}}{2-\sqrt{10}}q$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = 0\,.$$

(v) *If* $\frac{\sqrt{10}+1}{3}c \leq q < 0$, *then* $p(q) = \frac{\sqrt{10}-1}{3}q$ *and*

$$\frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = -\frac{4\sqrt{10}-8}{3}\,.$$

(vi) *If* $q < \frac{\sqrt{10}+1}{3}c$, *then* $p(q) = c$ *and*

$$-\frac{5}{3} < \frac{\mathrm{d}G(q;c)}{\mathrm{d}q} = -\frac{10q^2 + 20cq + 6c^2}{6(c+q)^2} < -\frac{4\sqrt{10}-8}{3}.$$

Proof. The function $\phi(p,q;c)$ is the sum of two convex functions, so it is also convex. The convexity of $G(q;c)$ comes from the fact that it is the partial minimization of $\phi(p,q;c)$ (see [23]).

If $q > 0$, we calculate using Lemma 2

$$\frac{\partial\theta(p,q)}{\partial p} = \begin{cases} -\dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2}, & \text{if } p < -2q, \\ -1, & \text{if } -2q \le p \le -\frac{1}{2}q, \\ \dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2}, & \text{if } p > -\frac{1}{2}q, \end{cases}$$

which is a nondecreasing function of p for any fixed q. Moreover, when $p < \frac{\sqrt{10}}{2-\sqrt{10}}q$,

$$\frac{\partial\theta(p,q)}{\partial p} = -\frac{10p^2 + 20pq + 6q^2}{6(p+q)^2} < -\frac{2(\sqrt{10}-1)}{3}.$$

When $p > \frac{\sqrt{10}-1}{3}q$,

$$\frac{\partial\theta(p,q)}{\partial p} = \frac{10p^2 + 20pq + 6q^2}{6(p+q)^2} > \frac{2(\sqrt{10}-1)}{3}.$$

Analogously, if $q < 0$, we calculate from Lemma 2

$$\frac{\partial\theta(p,q)}{\partial p} = \begin{cases} -\dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2}, & \text{if } p < -\frac{1}{2}q, \\ 1, & \text{if } -\frac{1}{2}q \le p \le -2q, \\ \dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2}, & \text{if } p > -2q, \end{cases}$$

which is a nondecreasing function of p for any fixed q. When $p < \frac{\sqrt{10}-1}{3}q$,

$$\frac{\partial\theta(p,q)}{\partial p} = -\frac{10p^2 + 20pq + 6q^2}{6(p+q)^2} < -\frac{2(\sqrt{10}-1)}{3}.$$

When $p > \frac{\sqrt{10}}{2-\sqrt{10}}q$,

$$\frac{\partial\theta(p,q)}{\partial p} = \frac{10p^2 + 20pq + 6q^2}{6(p+q)^2} > \frac{2(\sqrt{10}-1)}{3}.$$

Note that

$$\phi(p,q;c) = \frac{2(\sqrt{10}-1)}{3}|p-c| + \theta(p,q).$$

When $c = 0$ and $q > 0$,

$$\frac{\partial\phi(p,q;c)}{\partial p} = \begin{cases} \dfrac{2(\sqrt{10}-1)}{3} + \dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2} > 0, & \text{if } p > 0, \\ -\dfrac{2(\sqrt{10}-1)}{3} + \dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2} < 0, & \text{if } -\frac{1}{2}q < p < 0, \\ -\dfrac{2(\sqrt{10}-1)}{3} - 1 < 0, & \text{if } -2q \le p \le -\frac{1}{2}q, \\ -\dfrac{2(\sqrt{10}-1)}{3} - \dfrac{10p^2 + 20pq + 6q^2}{6(p+q)^2} < 0, & \text{if } p < -2q. \end{cases}$$

Therefore, $p(q) = 0$ and

$$G(q;0) = \frac{5}{3}q\,,$$

which implies that

$$\frac{\mathrm{d}G(q;0)}{\mathrm{d}q} = \frac{5}{3}\,.$$

When $c = 0$ and $q < 0$,

$$\frac{\partial\phi(p,q;c)}{\partial p} = \begin{cases} \dfrac{2(\sqrt{10}-1)}{3} + \dfrac{10p^2+20pq+6q^2}{6(p+q)^2} > 0, & \text{if } p > -2q, \\ \dfrac{2(\sqrt{10}-1)}{3} + 1 > 0, & \text{if } -\frac{1}{2}q \le p \le -2q, \\ \dfrac{2(\sqrt{10}-1)}{3} - \dfrac{10p^2+20pq+6q^2}{6(p+q)^2} > 0, & \text{if } 0 < p < -\frac{1}{2}q, \\ -\dfrac{2(\sqrt{10}-1)}{3} - \dfrac{10p^2+20pq+6q^2}{6(p+q)^2} < 0, & \text{if } p < 0. \end{cases}$$

Therefore, $p(q) = 0$ and

$$G(q;0) = -\frac{5}{3}q\,,$$

which implies that

$$\frac{\mathrm{d}G(q;0)}{\mathrm{d}q} = -\frac{5}{3}\,.$$

The proofs for $c > 0$ and $c < 0$ are similar to the proof for $c = 0$ and are omitted. $\square$

Now, we let

$$\begin{aligned} p_1 &= b_1 - \triangle z_1\,, & p_2 &= b_3 - \triangle z_2\,, \\ q_1 &= b_2 - \triangle z_1\,, & q_2 &= b_2 - \triangle z_2\,, \\ c_1 &= \triangle z_0 - \triangle z_1\,, & c_2 &= \triangle z_3 - \triangle z_2 \end{aligned} \tag{13}$$

With this notation, we have

$$G_1(b_2) + G_2(b_2) = G(q_1;c_1) + G(q_2;c_2) = G(b_2 - \triangle z_1; c_1) + G(b_2 - \triangle z_2; c_2) \tag{14}$$

Remark. Later in this paper, we will use $\triangle z_1 - \triangle z_0$, $\triangle z_2 - \triangle z_1$ and $\triangle z_3 - \triangle z_2$ to classify cases of linearity, convexity and oscillation. However, for clarity of the analysis in much of the remainder of this section, we use c_1 to denote $\triangle z_0 - \triangle z_1$ instead of $\triangle z_1 - \triangle z_0$ because G_1 and G_2 are defined in a symmetric manner in (9) and (10) and b_0 and b_4 are determined by b_1 and b_3, which are in turn determined by b_2 (progression outward from the middle point).

From Lemma 3, $G_1(b_2) + G_2(b_2)$ is convex and continuous for $b_2 \in R$ and is differentiable except at $b_2 = \triangle z_1$ and $b_2 = \triangle z_2$. If $b_2 < \min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}|c_1|, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}|c_2|\}$, then

$$\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0\,.$$

If $b_2 > \max\{\triangle z_1 - \frac{2-\sqrt{10}}{\sqrt{10}}|c_1|, \triangle z_2 - \frac{2-\sqrt{10}}{\sqrt{10}}|c_2|\}$, then

$$\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0\,.$$

Therefore, the scalars

$$\begin{aligned}&\min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}|c_1|, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}|c_2|\}\\ &= \min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}|\triangle z_0 - \triangle z_1|, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}|\triangle z_3 - \triangle z_2|\}\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\max\{\triangle z_1 - \frac{2-\sqrt{10}}{\sqrt{10}}|c_1|, \triangle z_2 - \frac{2-\sqrt{10}}{\sqrt{10}}|c_2|\}\\ &= \max\{\triangle z_1 - \frac{2-\sqrt{10}}{\sqrt{10}}|\triangle z_0 - \triangle z_1|, \triangle z_2 - \frac{2-\sqrt{10}}{\sqrt{10}}|\triangle z_3 - \triangle z_2|\}\end{aligned} \tag{16}$$

form a lower and upper bound, respectively, for b_2^*, the optimal b_2. Since $G_1(b_2) + G_2(b_2)$ is convex, we could (if the lower bound is less than the upper bound) use any line search method to find b_2^*. However, simply using line search methods at this point does not reveal geometric properties of the spline and does not lead to efficient calculation of b_2^*.

The geometric properties of the set of 5 data points can be classified by looking at $\triangle z_1 - \triangle z_0$, $\triangle z_2 - \triangle z_1$ and $\triangle z_3 - \triangle z_2$. For example, $\triangle z_1 - \triangle z_0 = 0$ means that the the first three points lie on a straight line; $\triangle z_1 - \triangle z_0 > 0$ means that the first three points are convex. When $\triangle z_1 - \triangle z_0 > 0$, $\triangle z_2 - \triangle z_1 > 0$ and $\triangle z_3 - \triangle z_2 > 0$, all five points are convex. When $\triangle z_1 - \triangle z_0 > 0$, $\triangle z_2 - \triangle z_1 < 0$ and $\triangle z_3 - \triangle z_2 > 0$, the five points "oscillate." As shown in Table 1, there are 27 cases to consider, of which, due to symmetry, only 10 cases need be analyzed. We will analyze the location of b_2^* in these 10 cases. Recall that b_2^* is the unique optimal solution after applying the choice procedure to resolve nonuniqueness, if it occurs.

Remark. The portions of the following results related to linearity (Cases 1, 2, 4, 5, 6, 11 and 12 and cases that are equivalent to these cases) overlap with analogous linearity results in [22]. In the present paper, however, these linearity results are presented in a wider context where not only linearity but also convexity and oscillation, measured by increases and decreases in the Δz_i, are considered.

Recall that from equation (1), we have

$$\delta_2 = \frac{z_3 - z_1}{x_3 - x_1}.$$

Case 1. In this case, $\triangle z_0 = \triangle z_1 = \triangle z_2 = \triangle z_3$ and $c_1 = c_2 = 0$. From Lemma 3,

$$\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} = \begin{cases} \dfrac{10}{3} & \text{if } b_2 - \triangle z_1 > 0,\\ -\dfrac{10}{3} & \text{if } b_2 - \triangle z_1 < 0.\end{cases}$$

The unique optimal solution is therefore $b_2^* = \triangle z_1$.

Case 2. In this case, $\triangle z_0 = \triangle z_1 = \triangle z_2 < \triangle z_3$, $c_1 = 0$ and $c_2 > 0$. From Lemma 3,

$$\begin{aligned}\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> \frac{5}{3} > 0 && \text{if } b_2 - \triangle z_1 > 0,\\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &\le -\frac{5}{3} < 0 && \text{if } b_2 - \triangle z_1 < 0.\end{aligned}$$

Table 1. 27 cases in the 5-point window method.

Case	Sign of $\triangle z_1 - \triangle z_0$	Sign of $\triangle z_2 - \triangle z_1$	Sign of $\triangle z_3 - \triangle z_2$	Same as Case	Linearity	Convexity / Concavity	Oscillation
1	0	0	0		Yes	Yes	No
2	0	0	+		Yes	Yes	No
3	0	0	–	2	Yes	Yes	No
4	0	+	0		Yes	Yes	No
5	0	+	+		Yes	Yes	No
6	0	+	–		Yes	No	No
7	0	–	0	4	Yes	Yes	No
8	0	–	+	6	Yes	No	No
9	0	–	–	5	Yes	Yes	No
10	+	0	0	2	Yes	Yes	No
11	+	0	+		Yes	Yes	No
12	+	0	–		Yes	No	No
13	+	+	0	5	Yes	Yes	No
14	+	+	+		No	Yes	No
15	+	+	–		No	No	No
16	+	–	0	6	Yes	No	No
17	+	–	+		No	No	Yes
18	+	–	–	15	No	No	No
19	–	0	0	2	Yes	Yes	No
20	–	0	+	12	Yes	No	No
21	–	0	–	11	Yes	Yes	No
22	–	+	0	6	Yes	No	No
23	–	+	+	15	No	No	No
24	–	+	–	17	No	No	Yes
25	–	–	0	5	Yes	Yes	No
26	–	–	+	15	No	No	No
27	–	–	–	14	No	Yes	No

The unique optimal solution is $b_2^* = \triangle z_1$.

Case 4. In this case, $\triangle z_0 = \triangle z_1 < \triangle z_2 = \triangle z_3$, $c_1 = 0$ and $c_2 = 0$. From Lemma 3,

$$\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} = \begin{cases} \dfrac{10}{3} & \text{if } b_2 - \triangle z_2 > 0, \\ 0 & \text{if } \triangle z_1 \leq b_2 \leq \triangle z_2, \\ -\dfrac{10}{3} & \text{if } b_2 - \triangle z_1 < 0. \end{cases}$$

Any solution in $[\triangle z_1, \triangle z_2]$ is optimal. Since $\triangle z_1 < \delta_2 < \triangle z_2$, the unique solution (the b_2 in the optimal interval closest to δ_2) is $b_2^* = \delta_2$.

Case 5 and 6. In Case 5, $\triangle z_0 = \triangle z_1 < \triangle z_2 < \triangle z_3$, $c_1 = 0$ and $c_2 > 0$. In Case 6, $\triangle z_0 = \triangle z_1 < \triangle z_2$, $\triangle z_2 > \triangle z_3$, $c_1 = 0$ and $c_2 < 0$. In both cases, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> \frac{5}{3} - \frac{5}{3} = 0 && \text{if } b_2 - \triangle z_1 > 0, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &\le -\frac{5}{3} < 0 && \text{if } b_2 - \triangle z_1 < 0. \end{aligned}$$

The unique optimal solution in both cases is $b_2^* = \triangle z_1$.

Case 11. In this case, $\triangle z_0 < \triangle z_1 = \triangle z_2 < \triangle z_3$, $c_1 < 0$ and $c_2 > 0$. From Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 && \text{if } b_2 - \triangle z_1 > 0, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 && \text{if } b_2 - \triangle z_1 < 0. \end{aligned}$$

The unique optimal solution is $b_2^* = \triangle z_1$.

Case 12. In this case, $\triangle z_0 < \triangle z_1 = \triangle z_2$, $\triangle z_2 > \triangle z_3$, $c_1 < 0$ and $c_2 < 0$. From Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 && \text{if } b_2 - \triangle z_1 > \min\{\tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &= 0 && \text{if } 0 < b_2 - \triangle z_1 \le \min\{\tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 && \text{if } b_2 - \triangle z_1 < 0. \end{aligned}$$

Any solution that lies in $[\triangle z_1, \triangle z_1 + \min\{\frac{2-\sqrt{10}}{\sqrt{10}}c_1, \frac{2-\sqrt{10}}{\sqrt{10}}c_2\}]$ is optimal. Since $\triangle z_1 = \delta_2 = \triangle z_2$, the unique solution is $b_2^* = \delta_2 = \triangle z_1$.

Case 14. In this case, $\triangle z_0 < \triangle z_1 < \triangle z_2 < \triangle z_3$, $c_1 < 0$ and $c_2 > 0$. From Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 && \text{if } b_2 - \triangle z_2 > 0, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 && \text{if } b_2 - \triangle z_1 < 0. \end{aligned}$$

Therefore, the optimal b_2 lies in $[\triangle z_1, \triangle z_2]$. This case is divided into 4 subcases as follows.

Subcase 14-1. If $\triangle z_2 - \triangle z_1 \le \frac{\sqrt{10}-2}{\sqrt{10}}(|c_1| + |c_2|)$, then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 && \text{if } b_2 > \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &= 0 && \text{if } \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2\} \le b_2 \le \max\{\triangle z_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 && \text{if } b_2 < \max\{\triangle z_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}. \end{aligned}$$

From the condition $\frac{\sqrt{10}-2}{\sqrt{10}}(|c_1|+|c_2|) \ge \triangle z_2 - \triangle z_1$, the interval $\big[\max\{\triangle z_1, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2\}\big]$ is not empty, and any b_2 in this interval is optimal. The unique solution is

$$b_2^* = \text{median}\{\max\{\triangle z_1, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2\}, \delta_2\}\,.$$

Subcase 14-2. If $\frac{\sqrt{10}-2}{\sqrt{10}}(|c_1|+|c_2|) < \triangle z_2 - \triangle z_1 < \frac{1}{2}(|c_1|+|c_2|)$, then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0 \qquad & \text{if } b_2 > \min\{\triangle z_1 - \tfrac{1}{2}c_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0 \qquad & \text{if } b_2 < \max\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 - \tfrac{1}{2}c_2\}. \end{aligned}$$

From the condition $\frac{\sqrt{10}-2}{\sqrt{10}}(|c_1|+|c_2|) < \triangle z_2 - \triangle z_1 \leq \frac{1}{2}(|c_1|+|c_2|)$, the interval

$$\left[\max\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 - \frac{1}{2}c_2\}, \min\{\triangle z_1 - \frac{1}{2}c_1, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}c_2\}\right]$$

is not empty. Since $\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2}$ is strictly increasing on this interval, there exists exactly one b_2^* in this interval such that $\frac{\mathrm{d}G_1(b_2^*)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2^*)}{\mathrm{d}b_2} = 0$.

Subcase 14-3. If $\frac{1}{2}(|c_1|+|c_2|) \leq \triangle z_2 - \triangle z_1 \leq 2(|c_1|+|c_2|)$, then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0 \qquad & \text{if } b_2 > \min\{\triangle z_1 - 2c_1, \triangle z_2 - \tfrac{1}{2}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} = 0 \qquad & \text{if } \max\{\triangle z_1 - \tfrac{1}{2}c_1, \triangle z_2 - 2c_2\} \leq b_2 \leq \min\{\triangle z_1 - 2c_1, \triangle z_2 - \tfrac{1}{2}c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0 \qquad & \text{if } b_2 < \max\{\triangle z_1 - \tfrac{1}{2}c_1, \triangle z_2 - 2c_2\}. \end{aligned}$$

From the condition $\frac{1}{2}(|c_1|+|c_2|) \leq \triangle z_2 - \triangle z_1 \leq 2(|c_1|+|c_2|)$, the interval

$$\left[\max\{\triangle z_1 - \frac{1}{2}c_1, \triangle z_2 - 2c_2\}, \min\{\triangle z_1 - 2c_1, \triangle z_2 - \frac{1}{2}c_2\}\right]$$

is not empty and any b_2 in this interval is optimal. The unique solution is

$$b_2^* = \text{median}\{\max\{\triangle z_1 - \frac{1}{2}c_1, \triangle z_2 - 2c_2\}, \min\{\triangle z_1 - 2c_1, \triangle z_2 - \frac{1}{2}c_2\}, \delta_2\}\,.$$

Subcase 14-4. If $2(|c_1|+|c_2|) < \triangle z_2 - \triangle z_1$, then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0 \qquad & \text{if } b_2 > \triangle z_2 - 2c_2, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0 \qquad & \text{if } b_2 < \triangle z_1 - 2c_1. \end{aligned}$$

From the condition $\triangle z_2 - \triangle z_1 > 2(|c_1|+|c_2|)$, the interval

$$[\triangle z_1 - 2c_1, \triangle z_2 - 2c_2]$$

is not empty. Since $\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2}$ is strictly increasing on this interval, there exists exactly one b_2^* in this interval such that $\frac{\mathrm{d}G_1(b_2^*)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2^*)}{\mathrm{d}b_2} = 0$.

Case 15. In this case, $\triangle z_0 < \triangle z_1 < \triangle z_2$, $\triangle z_2 > \triangle z_3$, $c_1 < 0$ and $c_2 < 0$. Then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0 \qquad & \text{if } b_2 > \max\{\triangle z_2, \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0 \qquad & \text{if } b_2 < \min\{\triangle z_1 - \tfrac{7+\sqrt{10}}{3}c_1, \triangle z_2\}. \end{aligned}$$

Therefore, b_2^* lies in

$$\left[\min\{\triangle z_1 - \frac{7+\sqrt{10}}{3}c_1, \triangle z_2\}, \max\{\triangle z_2, \min\{\triangle z_1 + \frac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 + \frac{2-\sqrt{10}}{\sqrt{10}}c_2\}\}\right].$$

This case is divided into 3 subcases as follows.

Subcase 15-1. If $\triangle z_2 - \triangle z_1 < \frac{2-\sqrt{10}}{\sqrt{10}}c_1$, then, from Lemma 3,

$$\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} = 0,$$

when b_2 lies in the interval

$$\begin{aligned}&\left[\min\{\triangle z_1 - \tfrac{7+\sqrt{10}}{3}c_1, \triangle z_2\}, \max\{\triangle z_2, \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}\}\right]\\ &= \left[\triangle z_2, \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}\right].\end{aligned}$$

Therefore, any b_2 in this interval is optimal. The unique solution is $b_2^* = \triangle z_2$.

Subcase 15-2. If $\frac{2-\sqrt{10}}{\sqrt{10}}c_1 < \triangle z_2 - \triangle z_1 < -\frac{7+\sqrt{10}}{3}c_1$, then, from Lemma 3,

$$\begin{aligned}\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 \qquad \text{if } b_2 - \triangle z_2 > 0,\\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 \qquad \text{if } b_2 - \triangle z_2 < 0.\end{aligned}$$

The unique optimal solution is $b_2^* = \triangle z_2$.

Subcase 15-3. If $-\frac{7+\sqrt{10}}{3}c_1 < \triangle z_2 - \triangle z_1$, then b_2^* is in the interval

$$\begin{aligned}&\left[\min\{\triangle z_1 - \tfrac{7+\sqrt{10}}{3}c_1, \triangle z_2\}, \max\{\triangle z_2, \min\{\triangle z_1 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_1, \triangle z_2 + \tfrac{2-\sqrt{10}}{\sqrt{10}}c_2\}\}\right]\\ &= \left[\triangle z_1 - \tfrac{7+\sqrt{10}}{3}c_1, \triangle z_2\right].\end{aligned}$$

Since $\dfrac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \dfrac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2}$ is strictly increasing on this interval, there exists exactly one b_2^* in this interval such that $\dfrac{\mathrm{d}G_1(b_2^*)}{\mathrm{d}b_2} + \dfrac{\mathrm{d}G_2(b_2^*)}{\mathrm{d}b_2} = 0$.

Case 17. In this case, $\triangle z_0 < \triangle z_1$, $\triangle z_1 > \triangle z_2$, $\triangle z_2 < \triangle z_3$, $c_1 < 0$ and $c_2 > 0$. Then, from Lemma 3,

$$\begin{aligned}\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 \qquad \text{if } b_2 - \triangle z_1 > 0,\\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 \qquad \text{if } b_2 - \triangle z_2 < 0.\end{aligned}$$

Therefore, b_2^* lies in $[\triangle z_2, \triangle z_1]$. This case is divided into 2 subcases as follows.

Subcase 17-1. If $\triangle z_1 - \triangle z_2 > \frac{\sqrt{10}+1}{3}(|c_1| + |c_2|)$, then, from Lemma 3,

$$\begin{aligned}\frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &> 0 \qquad \text{if } b_2 > \triangle z_1 + \tfrac{\sqrt{10}+1}{3}c_1,\\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} &< 0 \qquad \text{if } b_2 < \triangle z_2 + \tfrac{\sqrt{10}+1}{3}c_2.\end{aligned}$$

From the condition $\triangle z_1 - \triangle z_2 > \frac{\sqrt{10}+1}{3}(|c_1| + |c_2|)$, the interval

$$[\triangle z_2 + \frac{\sqrt{10}+1}{3}c_2, \triangle z_1 + \frac{\sqrt{10}+1}{3}c_1]$$

is not empty. Since $\dfrac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \dfrac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2}$ is strictly increasing on this interval, there exists exactly one b_2^* in this interval such that $\dfrac{\mathrm{d}G_1(b_2^*)}{\mathrm{d}b_2} + \dfrac{\mathrm{d}G_2(b_2^*)}{\mathrm{d}b_2} = 0$.

Subcase 17-2. If $\triangle z_1 - \triangle z_2 \leq \frac{\sqrt{10}+1}{3}(|c_1| + |c_2|)$, then, from Lemma 3,

$$\begin{aligned} \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} > 0 \qquad & \text{if } b_2 > \min\{\triangle z_1, \triangle z_2 + \tfrac{\sqrt{10}+1}{3} c_2\}, \\ \frac{\mathrm{d}G_1(b_2)}{\mathrm{d}b_2} + \frac{\mathrm{d}G_2(b_2)}{\mathrm{d}b_2} < 0 \qquad & \text{if } b_2 < \max\{\triangle z_2, \triangle z_1 + \tfrac{\sqrt{10}+1}{3} c_1\}. \end{aligned}$$

From the condition $\triangle z_1 - \triangle z_2 \leq \frac{\sqrt{10}+1}{3}(|c_1| + |c_2|)$, the interval

$$\left[\max\{\triangle z_2, \triangle z_1 + \frac{\sqrt{10}+1}{3} c_1\}, \min\{\triangle z_1, \triangle z_2 + \frac{\sqrt{10}+1}{3} c_2\}\right]$$

is not empty and any b_2 in this interval is optimal. The unique solution is

$$b_2^* = \mathrm{median}\{\max\{\triangle z_2, \triangle z_1 + \frac{\sqrt{10}+1}{3} c_1\}, \min\{\triangle z_1, \triangle z_2 + \frac{\sqrt{10}+1}{3} c_2\}, \delta_2\}\,.$$

3. Linkage of Geometric Properties of Data Points and L_1 Spline

In this section, based on the analytic results for the solution at the middle point in each 5-point window, we present two theorems that link the local linearity, convexity and oscillatory properties of the original data set with the local linearity, convexity and oscillatory properties of the locally calculated L_1 spline. In particular, we show that the locally calculated L_1 spline does not "over-oscillate".

The capability of the 5-point local window method to preserve linearity is described in the following theorem.

Theorem 4. *(Proposition 3 of [22]) If any three consecutive points in a five-point window are collinear with slope $\triangle z$, then $b_i^* = \triangle z$ except in Cases 4 and 7.*

Proof. See Cases 1, 2, 4, 5, 6, 11 and 12 in Section 2. □

Theorem 4 indicates that local linearity of the data is preserved in the 5-point-window L_1 spline with the "reasonable" exception of when two lines intersect at the point (x_i, z_i). C^1 continuity of the spline prevents linearity from being preserved in both intervals bordering on a corner (x_i, z_i).

Convexity is not as simple as linearity. To study the convexity of the L_1 spline, we need to consider not just a node x_i but the whole interval $[x_i, x_{i+1}]$. In this interval, the L_1 spline is determined by b_i^* and b_{i+1}^*, which are calculated using the six data points (x_k, z_k), $k = i-2, i-1, i, i+1, i+2, i+3$ in the two overlapping 5-point windows for b_i^* and b_{i+1}^*. In the remainder of this section, we focus on the L_1 spline in the interval $[x_i, x_{i+1}]$ and assume that these six data points (and, therefore, also their linear spline interpolant) are convex on $[x_{i-2}, x_{i+3}]$. The analysis in the rest of this section will reveal that the spline in $[x_i, x_{i+1}]$ is not always convex, but, when not, the oscillation is not large.

Lemma 5. *The following statements are equivalent:*

(i) *The cubic spline function is convex on the interval* $[x_i, x_{i+1}]$*;*

(ii) $(b^*_{i+1} - b^*_i) \geq 3|(b^*_{i+1} - \triangle z_i) + (b^*_i - \triangle z_i)|$*;*

(iii) $0 \leq -\frac{1}{2}(b^*_i - \triangle z_i) \leq (b^*_{i+1} - \triangle z_i) \leq -2(b^*_i - \triangle z_i)$.

Remark. Condition (iii) in Lemma 5 is equivalent to Proposition 3.1 in [24].

Proof. Recall from the definition in Section 1 that $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, I-1$.

The second derivative of the cubic spline function on $[x_i, x_{i+1}]$ is

$$\begin{aligned} &-\frac{2}{h_i}(2b^*_i + b^*_{i+1} - 3\triangle z_i) + \frac{6}{h_i^2}(b^*_i + b^*_{i+1} - 2\triangle z_i)(x - x_i) \\ &= \frac{1}{h_i}(b^*_{i+1} - b^*_i) + \frac{6}{h_i^2}(b^*_i + b^*_{i+1} - 2\triangle z_i)(x - x_i - \frac{h_i}{2}). \end{aligned}$$

Let $x - x_i = \lambda h_i$, $0 \leq \lambda \leq 1$, then

$$\begin{aligned} &\frac{1}{h_i}(b^*_{i+1} - b^*_i) + \frac{6}{h_i^2}(b^*_i + b^*_{i+1} - 2\triangle z_i)(x - x_i - \frac{h_i}{2}) \\ &= \frac{1}{h_i}(b^*_{i+1} - b^*_i) + \frac{6}{h_i}(b^*_i + b^*_{i+1} - 2\triangle z_i)(\lambda - \frac{1}{2}). \end{aligned}$$

Hence the cubic spline function is convex on the interval $[x_i, x_{i+1}]$ if and only if

$$\begin{aligned} &\frac{1}{h_i}(b^*_{i+1} - b^*_i) + \frac{6}{h_i}(b^*_i + b^*_{i+1} - 2\triangle z_i)(\lambda - \frac{1}{2}) \geq 0, \quad \forall \lambda \in [0, 1], \\ \Leftrightarrow \quad &(b^*_{i+1} - b^*_i) \geq 3|b^*_i + b^*_{i+1} - 2\triangle z_i| \\ \Leftrightarrow \quad &0 \leq -\tfrac{1}{2}(b^*_i - \triangle z_i) \leq (b^*_{i+1} - \triangle z_i) \leq -2(b^*_i - \triangle z_i). \end{aligned}$$

□

Every contiguous set of six points comes from two 5-point windows. Let Case $\alpha \leftrightarrow \beta$ denote Case/Subcase α for the left window (27 cases and 9 subcases) and Case/Subcase β for the right window (also 27 cases and 9 subcases). After applying Lemma 5 to all convex situations and eliminating equivalent cases, we can identify that the L_1 spline is convex on $[x_i, x_{i+1}]$ in Cases 1↔1, 1↔2, 2↔5, 10↔2, 11↔5, 5↔14-3, is not convex in Cases 2↔4, 4↔11, 5↔14-1, 5↔14-2, 5↔14-4, 14-1↔14-4, 14-2↔14-4, and is not determined in Cases 5↔13, 14-1↔14-1, 14-1↔14-2, 14-1↔14-3, 14-2↔14-3. However, the L_1 spline does not have extraneous oscillation on $[x_i, x_{i+1}]$ as is shown in the remainder of this section.

Lemma 6. $b^*_i \in [b^l_i, b^u_i]$, *where* $b^l_i = \min\{\triangle z_{i-1}, \triangle z_i\}$ *and* $b^u_i = \max\{\triangle z_{i-1}, \triangle z_i\}$.

Proof. The proof comes directly from the analysis of the 27 cases. □

Remark. Lemma 6 does not hold for global L_1 splines. Consider, for example, the 11 data points $(-5, 4)$, $(-4, 3)$, $(-3, 2)$, $(-2, 1)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, -1)$, $(3, -2)$, $(4, -3)$ and $(5, -4)$. By the 5-point-window method, b^*_5 (the derivative at $x = 0$) is 0. In contrast, the b^*_5 of the global L_1 spline is 0.37304.

Lemma 7. *If $b_i^* \leq \triangle z_i \leq b_{i+1}^*$, then the cubic L_1 spline is bounded above by the linear spline $\zeta(x)$ on the interval $[x_i, x_{i+1}]$.*

Proof. Given $b_i^* \leq \triangle z_i \leq b_{i+1}^*$, the cubic L_1 spline on $[x_i, x_{i+1}]$ can be written as

$$z(x) = z_i + b_i^*(x - x_i) - \frac{1}{h_i}(2b_i^* + b_{i+1}^* - 3\triangle z_i)(x - x_i)^2 + \frac{1}{h_i^2}(b_i^* + b_{i+1}^* - 2\triangle z_i)(x - x_i)^3 \,.$$

Let $x - x_i = \lambda h_i$, $0 \leq \lambda \leq 1$, then

$$\begin{aligned}
\zeta(x) - z(x) \;&= (\triangle z_i - b_i^*)(x - x_i) + \tfrac{1}{h_i}(2b_i^* + b_{i+1}^* - 3\triangle z_i)(x - x_i)^2 \\
&\quad - \tfrac{1}{h_i^2}(b_i^* + b_{i+1}^* - 2\triangle z_i)(x - x_i)^3 \\
&= \lambda h_i(\triangle z_i - b_i^*) + \lambda^2 h_i(2b_i^* + b_{i+1}^* - 3\triangle z_i) - \lambda^3 h_i(b_i^* + b_{i+1}^* - 2\triangle z_i) \\
&= \lambda h_i\big[(1 - 2\lambda + \lambda^2)(\triangle z_i - b_i^*) + (\lambda - \lambda^2)(b_{i+1}^* - \triangle z_i)\big] \\
&\geq 0, \quad \forall \ 0 \leq \lambda \leq 1 \,.
\end{aligned}$$

□

Theorem 8. *If the linear spline is convex on the interval $[x_{i-1}, x_{i+2}]$, that is, $\triangle z_{i-1} \leq \triangle z_i \leq \triangle z_{i+1}$, then the cubic L_1 spline is bounded above by the linear spline on the interval $[x_i, x_{i+1}]$.*

Proof. The proof comes from Lemmas 6 and 7. □

The results in this section indicate that the L_1 splines produced by the 5-point-window method with the proposed choice procedure for resolving nonuniqueness preserve linearity and convexity in many cases and do not oscillate excessively. From Lemma 6, we see that the b_i^* calculated by this method is always bounded by $\triangle z_{i-1}$ and $\triangle z_i$. This property is a prime factor in restricting oscillation of L_1 splines and may in the future lead to additional theoretic results about the properties of L_1 splines for non-over-oscillating interpolation of oscillatory data.

4. Conclusions

In summary, the results presented in this paper indicate that a new class of univariate L_1 interpolating splines calculated using 5-point windows as suggested by [2] has superior geometric shape preservation properties—better than those of L_1 splines calculated using global functionals. Lemma 6 ensures that the optimal solution b_i^* (the first derivative at node x_i) of 5-point-window L_1 splines is bounded by $\triangle z_{i-1}$ and $\triangle z_i$. This property does not hold for globally calculated L_1 splines and is not known to hold for locally calculated L_1 splines with uniqueness being enforced by adding a regularization term to the spline functional as was done in [7,9]. Theorems analogous to Theorem 8 that will assist in understanding how local convexity and oscillation in the data set translate into local convexity and oscillation of the L_1 spline are an excellent topic for future research. The results presented here for univariate interpolation are a basis for development of locally calculated univariate L_1 approximating splines and locally calculated bivariate L_1 interpolating and approximating splines.

The algorithmic implications of the analytical results of the present paper are large. In the past, there have been a few published reports and more unpublished reports about deficiencies of the primal affine, primal-dual and active-set algorithms that have been used to minimize L_1 splines. The convergence of these algorithms for medium to large data sets is often unsatisfactory. In addition, the discretization required by the primal affine and primal-dual algorithms is not desirable. The results of the present paper are a basis on which an efficient algorithm that minimizes the original continuum spline functional (not a discretization thereof) can be constructed. In a companion [25] article, we present such an algorithm and provide computational results for it.

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