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# Biogeography-Based Optimization of the Portfolio Optimization Problem with Second Order Stochastic Dominance Constraints

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**Abstract:** The portfolio optimization problem is the central problem of modern economics and decision theory; there is the Mean-Variance Model and Stochastic Dominance Model for solving this problem. In this paper, based on the second order stochastic dominance constraints, we propose the improved biogeography-based optimization algorithm to optimize the portfolio, which we called $\varepsilon$BBO. In order to test the computing power of $\varepsilon$BBO, we carry out two numerical experiments in several kinds of constraints. In experiment 1, comparing the Stochastic Approximation (SA) method with the Level Function (LF) algorithm and Genetic Algorithm (GA), we get a similar optimal solution by $\varepsilon$BBO in $[0, 0.6]$ and $[0, 1]$ constraints with the return of 1.174% and 1.178%. In $[-1, 2]$ constraint, we get the optimal return of 1.3043% by $\varepsilon$BBO, while the return of SA and LF is 1.23% and 1.26%. In experiment 2, we get the optimal return of 0.1325% and 0.3197% by $\varepsilon$BBO in $[0, 0.1]$ and $[-0.05, 0.15]$ constraints. As a comparison, the return of FTSE100 Index portfolio is 0.0937%. The results prove that $\varepsilon$BBO algorithm has great potential in the field of financial decision-making, it also shows that $\varepsilon$BBO algorithm has a better performance in optimization problem.

**Keywords:** biogeography-based optimization; second order stochastic dominance; portfolio optimization

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## 1. Introduction

In solving the problem of uncertainty, the Expected Utility Theory describes the rational people how to determine the optimal decisions and then take action when faced with the uncertainty of risk and return. The Expect Utility Theory assumes that the individual is risk aversion, that is, its utility function is concave. In modern economics, how to choose the optimal portfolio is one of its central problems. Based on the relationship between the risk of assets and the return on assets, Markowitz [1] solve the portfolio optimization problem through mathematic statistic method and propose the Mean-Variance model. The Mean-Variance model measures the expected revenue at the required rate of return, and measures the risk size with the variance of required rate of return. However, the model gets poor applications because of its harsh assumptions.

To supply a gap, Fishburn [2] propose the Stochastic Dominance theory and apply it into portfolio optimization problem. The Stochastic Dominance theory takes the risk appetite into consideration, which has been widely used in the financial field. Thereinto, second order stochastic dominance (SSD) is a branch of Stochastic Dominance theory.

Darinka Dentcheva and Andrzej Ruszczyński [3,4] introduce the second order stochastic dominance into portfolio optimization model. They also take the portfolio selection theory as a constraint and apply it to the risk repugnance asset optimization problem. While the second order stochastic dominance constraint requires that the investable assets should be compared with each other. When the number of investable assets is large, the calculation requirements becomes very harsh. Rudolf and Ruszczy [5] propose a new duality theory for this problem and optimize the model by cutting plane methods. Homem-De-Mello and Mehrotra [6] and Fábián, Mitra and Roman [7] also use the cutting plane methods to solve the portfolio optimization problem. The cutting plane method has a great efficiency in portfolio optimization problem, while this method is too restrictive. Besides, the probability space is discrete, which greatly limits the application of this method. Dentcheva and Ruszczyński [8] introduce a subsidiary variable into SSD model and transform it into a linear programming model. However, the number of variables in the model depends on the number of investable assets, which increases the pressure of calculation. Meskarian, Xu and Fliege [9] penalize SSD constraints to the objective under Slater's constraint qualification and apply the SA method and LF method to solve the penalized problem. Hu [10] apply the Sample Average Approximation method into SSD model, which introduces new variables into the model. Besides, Roman, Mitra and Zverovich [11] present an empirical study which analyses the effectiveness in the context of enhanced indexation.

Inspired by Biogeography [12], Dan [13] propose the Biogeography-based optimization (BBO) which is a sort of bionic optimization algorithm consists of three operators: migration operator, mutation operator and elimination operator. As a swarm intelligence optimization algorithm, the Biogeography-based optimization has advantages of good ability of robustness, easy to implement and the process of algorithm is relatively simple. Besides, the mutation operator helps to increase the diversity of solutions. Because the BBO algorithm does not depend on a particular problem, it is widely used in various fields, such as pathology test [14,15], distributed generation [16], intelligent identification [17–19] and other fields recently. While BBO algorithm also has some disadvantages, including easy to fall into the local optimization, poor exploration ability, strong randomness and lacking theoretical basis. In this paper, to increase the exploration ability of algorithm, we propose the $\varepsilon$BBO algorithm based on the DE algorithm. We improve the mutation operator to prevent the algorithm falling into the local optimization. Besides, we introduce it into portfolio optimization problem.

The rest of the paper is organized as follows. In Section 2, we discuss the SSD constrains model. In Section 3, we discuss the $\varepsilon$BBO algorithm. In Section 4, we apply the proposed methods to portfolio optimization problems and report some numerical test results. In Section 5, we summarize the performance of $\varepsilon$BBO on SSD constrains model.

## 2. The Model of Second Order Stochastic Dominance

In this section, we discuss the SSD constraints model for portfolio optimization problem. We start by discussing preliminaries needed.

### 2.1. The Expected Utility Theory and Mean-Variance Model

The expected utility is proposed by Bernoulli in 1738. Based on the explanation to St. Petersburg Paradox, Neumann and Morgenstern [20] discuss the strict axiomatic assumptions and propose the Expected Utility Theory. Let $U$ denote the expected utility of a certain decision:

$$U = u(\sum_{i=1}^{n} p_i w_i) = \sum_{i=1}^{n} p_i u(w_i) \tag{1}$$

where $u$ is utility, $p_i$ and $w_i$ is the probability and return of various circumstances. However, the Expected Utility Theory can not explain some investors' behavior, such as in buying lottery tickets and insurance at the same time. The emergence of Allais Paradox and Ellsberg Paradox proves that the Expected Utility Theory has a big loophole in derivation of theoretical model. Laciana and Weber [21] introduce the Regret and Disappointment Theory into expected utility function, which explain the

Allais Paradox and Ellsberg Paradox. Generally speaking, the main problem of Expected Utility Theory is lacking practical application.

It is generally believed that modern investment theory begins with the portfolio selection theory proposed by Markowitz. Markowitz mainly solves the two problems: why we need to select portfolio and how to choose optimal portfolio. Therefore, Markowitz propose the following portfolio optimization model:

$$\begin{aligned} min \quad & \delta^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i,j=1, i \neq j}^{n} x_i x_j \rho_{ij} \sigma_i \sigma_j \\ s.t. \quad & \sum_{i=1}^{n} x_i R_i = r \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n \end{aligned} \tag{2}$$

The mean-variance model is based on several assumptions, such as effective securities market, risk-aversion investors and so on, which means the mean-variance model has poor applications. Therefore, in this paper, we mainly discuss the portfolio optimization problem based on the stochastic dominance theory.

### 2.2. The Stochastic Dominance Theory

The stochastic dominance theory originates from discrete stochastic variables optimization theory, which is later developed into generalized stochastic variables optimization theory and widely used in economics and finance. In the stochastic dominance theory, the comparison among the random variables is carried out the point-wise comparison of their k-order distribution function $F^{(k)}$. Assuming that there are only two portfolios $X$ and $Y$, and the utility function of all investors is monotonically increasing. If all investors prefer portfolio $X$ or believe that only part of the portfolio $X$ and $Y$ is no difference, then we can say the portfolio $X$ stochastically dominates portfolio $Y$ in the first order.

From the mathematical point of view, let $x$ and $y$ be the decision vectors and $\xi$ be the random variable. It is said that $g(x,\xi)$ stochastically dominates $g(y,\xi)$ in the first order, denoted by $g(x,\xi) \succeq_1 g(y,\xi)$, if

$$F(g(x,\xi);\eta) \leq F(g(y,\xi);\eta), \forall \eta \in \mathbb{R} \tag{3}$$

where $g(x,\xi)$ is the concave continuous function both in $x$ and $\xi$, $F(g(x,\xi);\eta)$ and $F(g(y,\xi);\eta)$ are the cumulative distribution function of $g(x,\xi)$ and $g(y,\xi)$. For a random variable $X \in \mathbb{R}$, the first order distribution function of $X$ is its right-continuous cumulative distribution function:

$$F_x^{(1)}(\eta) = F_x(\eta) = \int_{-\infty}^{\eta} P_x(d\xi) = P\{x \leq \eta\}, \forall \eta \in \mathbb{R} \tag{4}$$

The first order of stochastic dominance is applicable to investors with arbitrary risk appetite, while the second order stochastic dominance is only for the risk averse decision makers whose marginal utility of expected utility function decreases. Based on the definition of first order stochastic dominance, similarly, $g(x,\xi)$ stochastically dominates $g(y,\xi)$ in the second order, denoted by $g(x,\xi) \succeq_2 g(y,\xi)$ , if

$$\int_{-\infty}^{\eta} F(g(x,\xi);\alpha)d\alpha \leq F(g(y,\xi);\alpha)d\alpha, \forall \eta \in \mathbb{R} \tag{5}$$

The corresponding strict dominance relation $\succ_{(k)}$ is defined in the usual way:

$$X \succ_{(k)} Y \Leftrightarrow X \succeq_{(k)} Y \text{ and } Y \not\succeq X \tag{6}$$

In this paper, we mainly discuss the second order stochastic dominance. Assuming that the rate of return on $g(x,\xi)$ and $g(y,\xi)$ in portfolio $x$ and $y$ having a finite expectation. In the portfolio optimization problem, it is said that portfolio $x$ stochastically dominates portfolio $y$ in the second order, denoted by $g(x,\xi) \succeq_2 g(y,\xi)$, its requirement is the same as model (5).

### *2.3. The Second Order Stochastic Dominance Model*

Dentcheva and Ruszczyński [4] propose the following fundamental model of second order stochastic dominance:

$$\begin{aligned} max \quad & f(x) \\ s.t. \quad & g(x,\xi) \succeq_2 g(y,\xi) \\ & x \in X \end{aligned} \tag{7}$$

Here $f : X \to \mathbb{R}$ is a concave continuous function. In particular, we may use

$$f(x) = E[f(x,\xi)] \tag{8}$$

where $g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ are concave continuous functions both in $x$ and $\xi$, $x \in X$ is a decision vector with $X$ being a nonempty convex subset of $\mathbb{R}^n$, and $\xi$ is a random vector defined on probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with support $\Xi$, $y \in X$ is a predefined vector, $E(\cdot)$ denotes the expected value w.r.t. the probability distribution of $\xi$ [9].

The constraints of model (7) are based on the definition of second order stochastic dominance, Fábián [7] and Ogryczak [22] prove that the constraint $g(x,\xi) \succeq_2 g(y,\xi)$ is equivalent to the following two inequalities:

$$E[(\eta - X)_+] \leq E[(\eta - Y)_+], \forall \eta \in \mathbb{R}, \text{where } [t - R_x]_+ = max\{t - R_x, 0\} \tag{9}$$

$$E(U(R_x)) \geq E(U(R_y)), \text{for any increasing and concave utility function U} \tag{10}$$

Combined with the constraint (9), the model (7) can be formulated as a stochastic semi-infinite programming problem [3,9] :

$$\begin{aligned} min \quad & -E[f(x,\xi)] \\ s.t. \quad & G(x,\eta) := E[(\eta - g(x,\xi))_+] - E[(\eta - g(y,\xi))_+] \leq 0, \forall \eta \in \mathbb{R} \\ & x \in X \end{aligned} \tag{11}$$

For the random variable $\eta \in \mathbb{R}$, it is clear that there is no $x \in X$ seriously satisfies the constraint condition when $\eta$ takes a sufficiently small value. So the inequality does not hold and the model (11) does not satisfy the Slater condition. In the actual research, in order to overcome the serious technical difficulties and seek the optimal solution of algorithm, we get the following relaxed model:

$$\begin{aligned} min \quad & -E[f(x,\xi)] \\ s.t. \quad & G(x,\eta) \leq 0, \forall \eta \in [a,b] \\ & x \in X \end{aligned} \tag{12}$$

where $[a,b]$ is a closed interval in $\mathbb{R}$. Darinka Dentcheva et al. [3] propose that model (12) is equivalent to model (11) for some appropriate interval $[a,b]$ if $\xi$ has uniformly bounded distribution. As a relation of model (11), model (12) has a larger set of feasible solutions and subsequently its optimal value gives a lower bound for model. Besides, the relaxed model is more likely to satisfy the Slater condition which is closely related to numerical stability [9].

Although the constraints are relaxed, the model (12) is a stochastic semi-infinite and non-smooth programming problem in nature. So the difficulty of solving is still large. Dentcheva and Ruszczynski [4] change the model into a liner programming problem which solves the problem of non-smooth of SSD constraints. Besides, this method also applies to non-linear cases. However, facing the multi-dimensional problems, due to the increase in variables, the efficiency of the method will be greatly affected.

In this paper, we mainly discuss the discrete condition, assuming the random variable $\xi$ satisfies the discrete distribution, that is $P(\xi = \xi^i) = p_i, i = 1, \ldots, m$. Fábián [7] proposes the following model:

$$\begin{aligned} min \quad & \{-\sum_{i=1}^{m} p_i f(x, \xi^i)\} \\ s.t. \quad & \sum_{i=1}^{m} p_i(\eta_j - g(x, \xi^i))_+ \leq \sum_{i=1}^{m} p_i(\eta_j - g(y, \xi^i))_+, \forall j = 1, \ldots, m \\ & x \in X \end{aligned} \tag{13}$$

where $\eta_j = g(y, \xi^j), j = 1, \ldots, m$ . In particular, $p_i = \frac{1}{m}$. In this case, model (13) is the sample average approximation of model (12). However, the model (13) is still non-smooth and does not satisfy the Slater condition.

## 2.4. Portfolio Optimization Model

In the portfolio optimization problem, our aim is to invest our capital in some assets in order to obtain some desirable characteristics of the total return on investment.

Let $n$ denote the number of assets available for investment at the beginning of a fixes time period and assume that we have a fixed capital to be invested in $n$ assets. To simplify the discussion, we use $x = (x_1, \ldots, x_n)^T$ to denote the fractions of initial capital invested in different assets. $_i = \frac{w_i}{w}(i = 1, \ldots, n)$ where $w_i$ is the capital invested in asset $i$ and $w$ is the total amount of capital to be invested. Let $X$ denote a set of feasible portfolios and it is clear that $X \in \mathbb{R}^n$ is a bound convex polyhedron. Let $R_i(\xi), i = 1, \ldots, n$ denote the return of asset $i$ in the case of discrete distribution, we assume that $E[| R_j |] < \infty$ for all $j = 1, \ldots, n$. Therefore, if we have a fixed capital, the return of portfolio can then be formulated as:

$$g(x, \xi) := R_1(\xi)x_1 + R_2(\xi)x_2 + \cdots + R_n(\xi)x_n \tag{14}$$

In economics and related disciplines, the transaction cost is a cost in making any economic trades when participating in the market. If we take the transaction cost into account, let $c(x)$ denote the V-type function of transaction cost and the return of portfolio can be formulated as:

$$g(x, \xi) = x^T\xi - c(x) \tag{15}$$

where $x$ is the investment proportion vector and $\xi$ is a vector represents the return of assets. Clearly, the set of possible asset allocations can be defined as follows:

$$X = \{(x_1, \ldots, x_n) \mid \sum_{j=1}^{n} x_j = 1, x_j \geq 0, \forall j \in 1, \ldots, n\} \tag{16}$$

In order to ensure the diversity of portfolio, the bounds are always set to 0.6 and 0:

$$X = \{(x_1, \ldots, x_n) \mid \sum_{j=1}^{n} x_j = 1, 0 \leq x_j \leq 0.6, \forall j \in 1, \ldots, n\} \tag{17}$$

The two constraints proposed above based on the short-selling is prohibited. Short-selling is the sale of a security that is not owned by the seller, or the seller has borrowed. If the short-selling is allowed and investors do not want to invest very small amounts in an asset, the upper and lower bounds on the fraction of capital invested in each assets are set to 2 and $-1$ [9]:

$$X = \{(x_1, \ldots, x_n) \mid \sum_{j=1}^{n} x_j = 1, -1 \le x_j \le 2, \forall j \in 1, \ldots, n\} \tag{18}$$

Assumed that a reference random return $Y$ having a finite expected value which may be an index such as FTSE 100 index, or a current portfolio. Let $y$ denote a benchmark investment, normally $y_i = \frac{1}{m}$, for $i = 1, \ldots, n$. Based on the model (13) and the discussion above, we propose the following portfolio optimization model with SSD constraints:

$$\begin{aligned} min \quad & f(x, \xi) \\ s.t. \quad & \sum_{i=1}^{m} \frac{1}{m}(g(y, \xi^j) - g(x, \xi^i))_+ \le \sum_{i=1}^{m} \frac{1}{m}(g(y, \xi^j) - g(y, \xi^i))_+ \\ & x \in X \end{aligned} \tag{19}$$

Firstly we consider the simplest case:

$$f(x, \xi) = -g(x, \xi) \tag{20}$$

Then we consider a sightly more complicated condition. In view of the impact of transaction cost and the small amounts investment in portfolio, we consider the following performance function [9]:

$$f(x, \xi) = -g(x, \xi) - \sum_{i=1}^{n} x_i^2 \tag{21}$$

## 3. The εBBO Algorithm for SSD Model

Aiming at the portfolio optimization problem based on SSD constraints, drawn on the experience of mutation operator of the Differential Evolution (DE) algorithm, we propose the εBBO algorithm. In this section, we discuss the fitness function, migration operator and mutation operator for SSD model. Besides, we introduce the main procedure of εBBO algorithm for SSD model.

### *3.1. The Fitness Function for SSD Model*

BBO is a population-based algorithm in which a population of candidate individuals is used for solving the global optimization problem. In the BBO algorithm, each habitat is considered to be an individual and has its habitat suitability index (HSI) to show the degree of its goodness. Habitat with a high level of HSI represents a good solution to the problem, while habitat with a low level of HSI represents a poor solution to the problem. The HSI is determined by several vectors, such as rainfall and the diversity of vegetation, which is named as Suitability Index Variable (SIV).

In this paper, we mainly discuss the portfolio optimization problem based on the SSD constraints. Therefore, the SIV of the portfolio optimization problem is the return of assets and transaction cost. If we do not consider the influence of transaction cost, the HSI can be express as:

$$g(x) = R_1 x_1 + R_2 x_2 + \cdots + R_n x_n \tag{22}$$

In the actual investment market, in order to avoid petty investments, we take the transaction cost into account and the HSI can be express as:

$$g(x) = R_1 x_1 + R_2 x_2 + \cdots + R_n x_n + \lambda \sum_{i=1}^{n} x_i^2 \tag{23}$$

where $\lambda$ is a constant which is determined by physical truth.

*3.2. Migration Operator for SSD Model*

Modified migration operator is a generalization of standard BBO operator. The idea inspired by blended migration operator [23], the coefficient of solution $\mathbf{H}_i$ is a constant. In our algorithm, along the number of iterations increases, solution $\mathbf{H}_i$ is much fitter than solution $\mathbf{H}_j$, and solution $\mathbf{H}_i$ is more affected by itself. The migration operator based on the number of iterations is designed to accelerate the speed of convergence to global optimal solution. In $\varepsilon$BBO, the Modified migration is defined as:

$$\mathbf{H}_i^{'}(j) = \frac{t}{t_{max}} \mathbf{H}_i(j) + (1 - \frac{t}{t_{max}}) \mathbf{H}_e(j) \tag{24}$$

where $\mathbf{H}_i$ is a immigrated island, $\mathbf{H}_e$ is a emigrated island, $\mathbf{H}_i(j)$ is the $j$th dimension of the $i$th solution, and $t$ is the number of iterations, $t_{max}$ is the maximum number of iterations. Equation (24) means that the features of solution are changed by solution $\mathbf{H}_i$ and solution $\mathbf{H}_e$. At first, the new solution is more affected by another better solution which accelerates the convergence characteristics of the algorithm. Along with the growth of generation, the new solution is more affected by itself.

After the migration operation, the portfolio in the immigrated island (a bad solution) accepts the sharing information from the emigrated island (a better solution). However, another portfolio in immigrated island may have the same value as immigrated portfolio. To keep the new solution feasible, therefore, we need to adjust the portfolio which has identical vector as the sharing information. That is to say, if the updated portfolio has the same value as another portfolio in the immigrated island, the portfolio swaps places with the immigrated portfolio.

*3.3. Mutation Operator for SSD Model*

In BBO, if a solution is selected for mutation, it will be replaced by another randomly generated new solution set. The hybridization between the BBO and Differential Evolution Algorithm (DE) has achieved many great results [24–26]. However, they mostly incorporate DE into the migration procedure. Our algorithm incorporates the DE into the mutation procedure. The algorithm adjusts near the optimal solutions, so that it can find the global optimal solution. In $\varepsilon$BBO, A mutated individual ($\mathbf{H}_i(j)$) is generated according to the following equation:

$$\mathbf{H}_i^{'}(j) = \mathbf{H}_i(j) - \mid c_1 * (\mathbf{H}_{best}(j) - \mathbf{H}_i(j)) + c_1 * (\mathbf{H}_{r1}(j) - \mathbf{H}_{r2}(j)) \mid \tag{25}$$

where '$| \cdot |$' means absolute value, $\mathbf{H}_i(j)$ is selected for mutation, $c_1$ is the mutation scaling factor and its value is usually set as 0.5. $\mathbf{H}_{r1}(j)$ and $\mathbf{H}_{r2}(j)$ are the two solutions randomly selected, $\mathbf{H}_{best}(j)$ is the best solution in this generation. In $\varepsilon$BBO, this mutation scheme tends to increase the diversity among the population. It acts as a fine tuning unit and helps to achieve global optimal solution.

*3.4. Adaptive BBO for SSD Model*

There are modification probability and mutation probability factors in the BBO algorithm. Modification probability is denoted as $P_{modif}$ and mutation probability is denoted as $P_{muta}$. The two factors which range from 0 to 1 are defined by users. When generated random number is less than modification probability, the program executes migration operator; when generated random number is less than mutation probability, the program executes mutation operator. In original BBO algorithm, the modification probability and mutation probability are user-defined parameters. The setting of the parameters are very relevant to the experience of the user, and they are unfavorable for the selection of migration individuals. In order to choose better migration individuals, these parameters are related to the fitness function. When the value of fitness function is better than the average, we will reduce the probability of modification and mutation operations, or else we will increase the probability of modification and mutation operations. In other terms, in migration operations, when the value of fitness function is larger than the average, we set constant factor $k_1$ to smaller value, and the probability of modification operations is decided by Formula (26). When the value of fitness function is less than the average, we set constant factor $k_2$ to larger value, so as to search extensively in the solution space. Similarly, the selection of mutation probability is similar to the modification probability. When the value of the fitness function is better than the average, we set constant factor $k_3$ to smaller value. Otherwise, we set constant factor $k_4$ to larger value. As the mutation operation is adjusted near the optimal solution, we set the maximum mutation probability $k_4$ to 0.25. So modification probability and mutation probability are adjusted adaptively by the fitness function.

$$P_{modif} = \begin{cases} k_1 * \dfrac{MaxCost\text{-}FitnessCost}{MaxCost\text{-}AvgCost} & FitnessCost \geq AvgCost \\ k_2 & FitnessCost < AvgCost \end{cases} \tag{26}$$

$$P_{muta} = \begin{cases} k_3 * \dfrac{MaxCost\text{-}FitnessCost}{MaxCost\text{-}AvgCost} & FitnessCost \geq AvgCost \\ k_4 & FitnessCost < AvgCost \end{cases} \tag{27}$$

In the adaptive Biogeography-based optimization algorithm, modification probability and mutation probability are modified according to the relation between the cost of fitness function of randomly selected habitat and average cost of fitness function of all habitats in last generation. In other words, if the cost of fitness function is equal or greater than average cost, modification probability and mutation probability are modified by Equations (26) and (27) respectively. Otherwise, modification probability and mutation probability are taken as constant $k_2$ and $k_4$ respectively. In Equations (26) and (27), constant factors $k_1$, $k_2$, $k_3$ and $k_4$ which range from 0 to 1 are defined by users. In another literature [27], we used the values of adaptive factors and achieved good results. Therefore, in this paper, we adapt the same adaptive factors. We set $k_1 = 0.4$, $k_2 = 0.95$, $k_3 = 0.1$, $k_4 = 0.25$.

### *3.5. Main Procedure of εBBO for SSD Model*

By incorporating the above-mentioned migration operator and mutation operator into Biogeography-based optimization, the inverse operation is performed after the mutation operator; in addition, modification probability and mutation probability are modified in term of Equations (26) and (27) respectively. Meanwhile, immigration rate and emigration rate based on cosine curve are modified. The εBBO algorithm is able to explore the new search space with the mutation operator of DE algorithm and to exploit the population information with the migration operator of BBO algorithm. This feature overcomes the lack of exploration of the original BBO algorithm.

The procedure chat of the εBBO algorithm is shown in Figure 1.

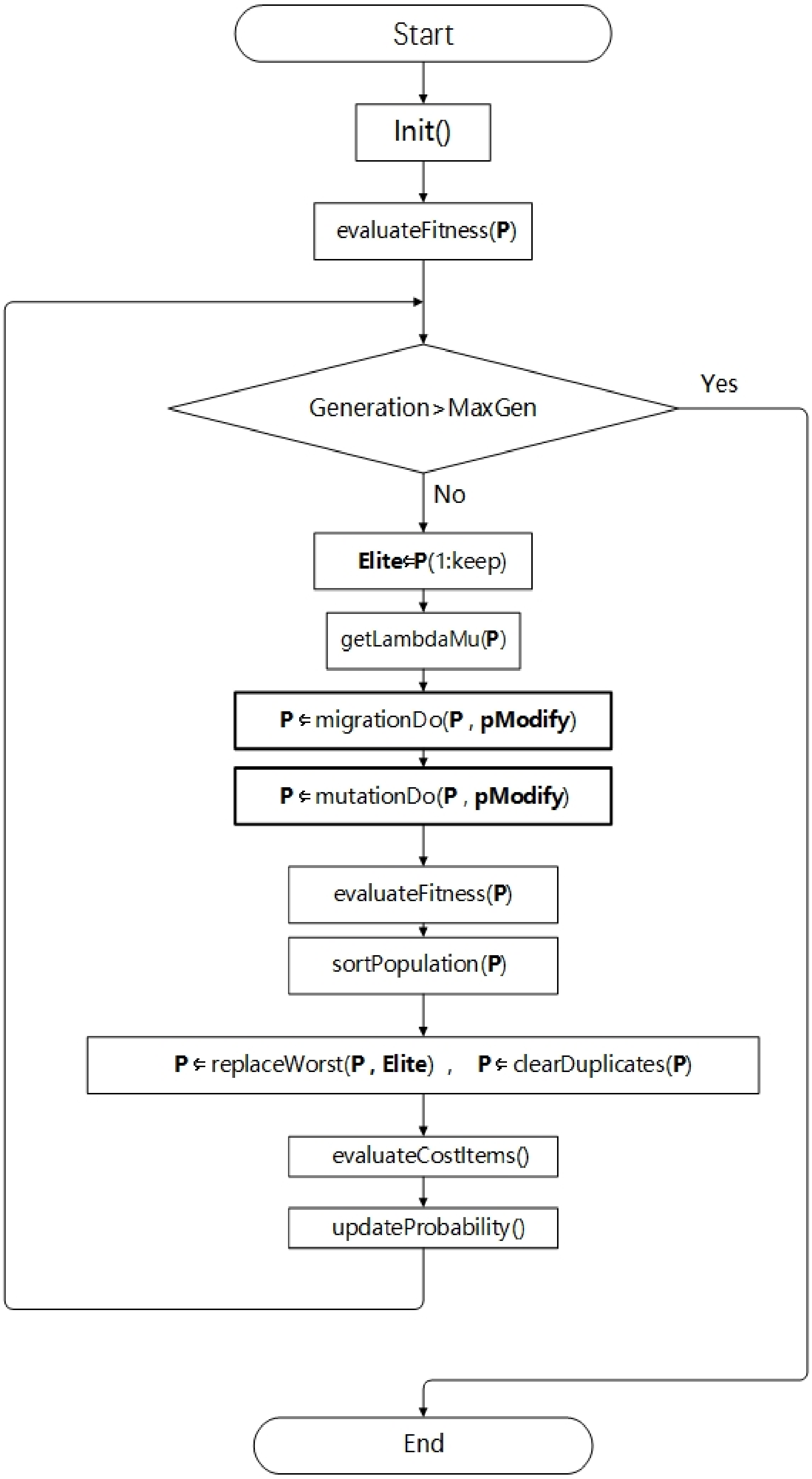



**Figure 1.** The procedure chat of the εBBO algorithm.

The εBBO approach is described in Algorithm 1 and SortPopulation (Population $P$) is described in Algorithm 2.

**Algorithm 1** the main procedure of $\varepsilon$BBO for the SSD.

**Input:** modification probability, **pModify**; mutation probability, **pMutate**; elitism parameter, **keep**; number of iterations
**Output:** the best portfolio and the return of the portfolio

**P** $\leftarrow$ generate Initial Random Population()
evaluateFitness(**P**)
$Generation_{current} \leftarrow 0$
**while** halting criterion not being satisfied **do**
    $Generation_{current} \leftarrow Generation_{current} + 1$
    **Elite** $\leftarrow$ **P**(1:keep)
    $[\lambda_i, \mu_i] \leftarrow$ getLambdaMu(**P**)
    **P** $\leftarrow$ migrationDo(**P** , **pModify**)
    **P** $\leftarrow$ mutationDo(**P** , **pMutate**)
    evaluateFitness(**P**)
    sortPopulation(**P**)
    **P** $\leftarrow$ replaceWorst(**P** , **Elite**)
    **P** $\leftarrow$ clearDuplicates(**P**)
    [maximum cost, minimum cost , average cost] $\leftarrow$ evaluateCostItems()
    **[pModify , pMutate]** $\leftarrow$ updateProbability()
**end while**

**Algorithm 2** the main algorithm of sortPopulation (Population $P$).

$Pun \leftarrow$ Punish($P$)
**sortPun** $(Pun, P)$
$\varepsilon_0 \leftarrow Pun_{(1/5)NP}$
$\varepsilon \leftarrow \varepsilon_0 \times (\frac{Generation_{current}}{Generation_{max}})^{cp}$
**for** $i$ = 1 to **NP do**
    **for** $j$ = 1 to **NP-1 do**
        **if** $(Pun_j, Pun_{j+1} \leq \varepsilon)$ or $(Pun_j = Pun_{j+1})$ and $(f_{j+1} < f_j)$ **then**
            **swap** $(P_j, P_{j+1})$
        **else if** $(Pun_j > Pun_{j+1})$ **then**
            **swap** $(P_j, P_{j+1})$
        **end if**
    **end for**
    **if** no swap **then**
        **break**
    **end if**
**end for**

## 4. Numerical Experiments

We have carried out a number of numerical tests by using $\varepsilon$BBO algorithms in MATLAB2014b install of a Lenovo PC with Windows 8.1 operating system and 8 GB of RAM. In this section, we report the test results.

### 4.1. Example 1

At first, we consider the portfolio optimization problem in low dimensional. We consider a history of percentage returns, for $m = 10$ time periods and a group of $n = 5$ assets in Table 1 [9].

**Table 1.** Rates of return on five assets over ten period.

| | Returns % for Period | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| Asset 1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.1 | 1.2 | 1.1 | 1.0 | 1.0 | 1.1 |
| Asset 2 | 1.3 | 1.0 | 0.8 | 0.9 | 1.4 | 1.3 | 1.2 | 1.1 | 1.2 | 1.1 |
| Asset 3 | 0.9 | 1.1 | 1.0 | 1.1 | 1.1 | 1.3 | 1.2 | 1.1 | 1.0 | 1.1 |
| Asset 4 | 1.1 | 1.1 | 1.2 | 1.3 | 1.2 | 1.2 | 1.1 | 1.0 | 1.1 | 1.2 |
| Asset 5 | 0.80 | 0.75 | 0.65 | 0.75 | 0.80 | 0.90 | 1.00 | 1.10 | 1.10 | 1.20 |

Our aim is to find an optimal investment strategy for a fixed capital in n assets which maximizes the expected return. If we do not consider the impacts of transaction cost, based on Table 1 and model (19), we propose the following SSD constraints model:

$$\begin{aligned} min \quad & -g(x,\xi) \\ s.t. \quad & \sum_{i=1}^{10}\frac{1}{10}(g(y,\xi^j)-g(x,\xi^i))_+ \le \sum_{i=1}^{10}\frac{1}{10}(g(y,\xi^j)-g(y,\xi^i))_+, j=1,\ldots,10 \\ & x\in X \end{aligned} \tag{28}$$

where $X$ has three conditions which we have discussed above. We apply the $\varepsilon$BBO algorithm to the model (28), and we let $y=(0.2,0.2,0.2,0.2,0.2)$ as the reference group. Comparing the $\varepsilon$BBO algorithm with Genetic Algorithm (GA), stochastic approximation (SA) method and level function (LF) methods, we get the test results shown in Table 2 [9,28].

**Table 2.** The model (28) using date in Table 1. Time is in minute and the expected return of the benchmark portfolio $E[g(y,\xi)]=1.073$.

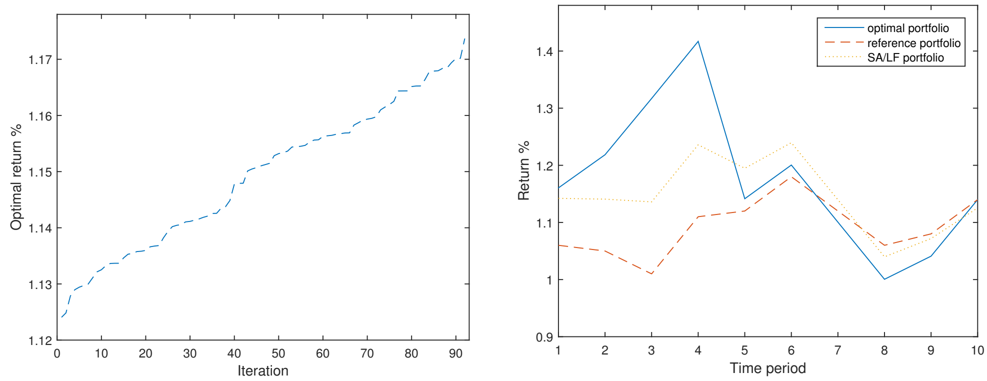
| X Constraint | Algorithm | Iteration | Time | x | E[g(x,ξ)] |
|---|---|---|---|---|---|
| | εBBO | 92 | 0.2833 | (0.595,0.005,0,0.4,0) | 1.1736 |
| $[0,0.6]$ | SA | 115 | 5 | (0.325,0.231,0.177,0.266,0) | 1.147 |
| | LF | 5 | 0.6355 | (0.325,0.231,0.177,0.266,0) | 1.148 |
| | εBBO | 67 | 0.217 | (0.799,0.201,0,0,0) | 1.1779 |
| $[0,1]$ | GA(100) | 4 | 0.1322 | (0.576,0.203,0.027,0.188,0.006) | 1.165 |
| | GA(1000) | 3 | 0.6536 | (0.75,0.175,0,0.075,0) | 1.177 |
| $[-1,2]$ | εBBO | 59 | 0.25 | (0.097,0.468,0.49,0.701,-0.756) | 1.300 |

The Figures 2–4 show the performance of the $\varepsilon$BBO and the optimal portfolio in three kinds of $X$ constraints of model (28).

In model (28) we do not consider the transaction cost, if we take it into account, in order to weaken the contribution of small transactions we propose the following model:

$$\begin{aligned} min \quad & -g(x,\xi)-\sum_{i=1}^{10}x_i^2 \\ s.t. \quad & \sum_{i=1}^{10}\frac{1}{10}(g(y,\xi^j)-g(x,\xi^i))_+ \le \sum_{i=1}^{10}\frac{1}{10}(g(y,\xi^j)-g(y,\xi^i))_+, j=1,\ldots,10 \\ & x\in X \end{aligned} \tag{29}$$

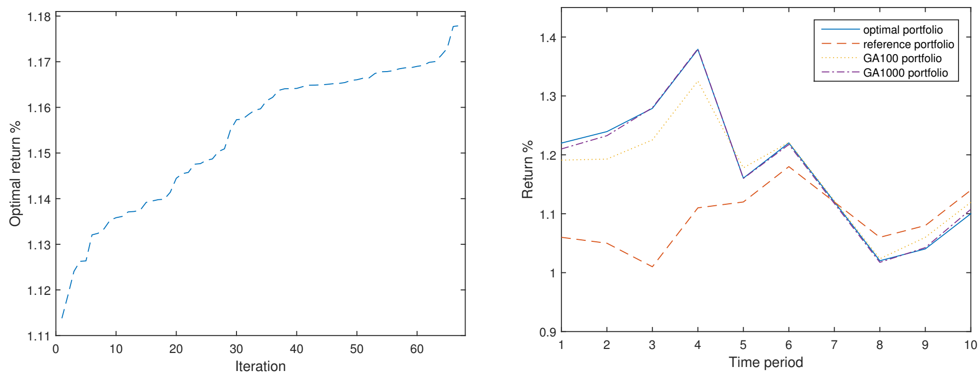
The same as the model (28), we get the test results of model (29) shown in Table 3 [9].




(a) The change of the the return of optimal portfolio between different iterative

(b) The return of optimal portfolio by εBBO in each period with benchmark portfolio and the optimal portfolio by SA and LF algorithm

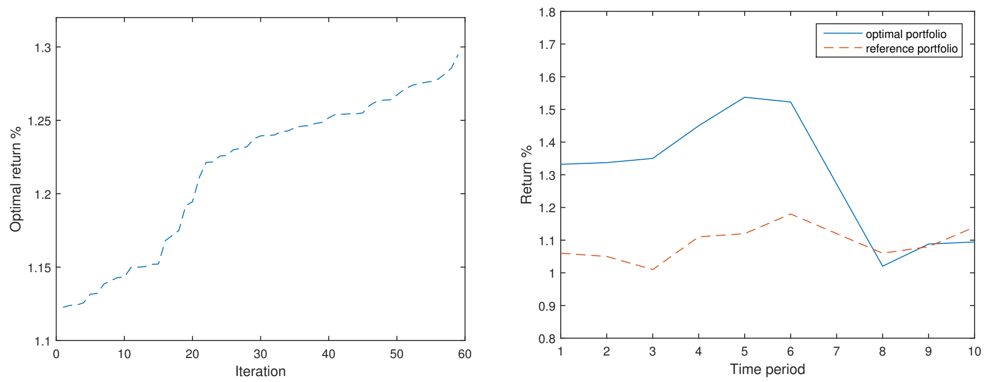
**Figure 2.** The performance of the εBBO and the optimal portfolio in $X \in [0, 0.6]$ of model (28).




(a) The change of the the return of optimal portfolio between different iterative

(b) The return of optimal portfolio by εBBO in each period with benchmark portfolio and the optimal portfolio by GA algorithm

**Figure 3.** The performance of the εBBO and the optimal portfolio in $X \in [0, 1]$ of model (28).




(a) The change of the the return of optimal portfolio between different iterative

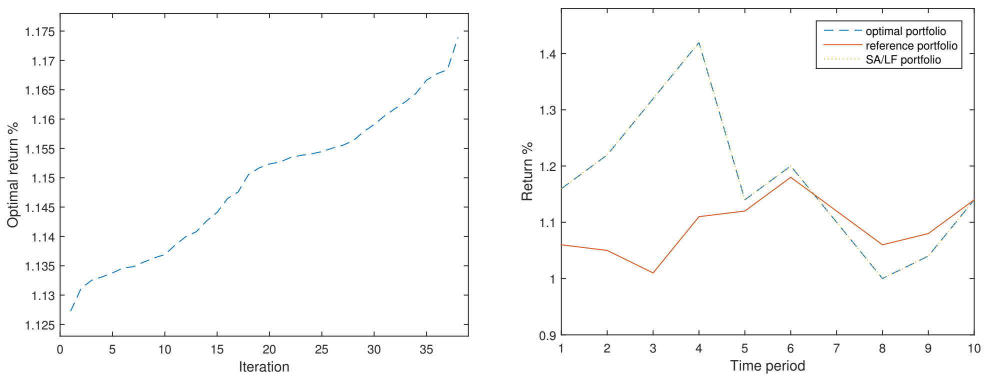
(b) The return of optimal portfolio by εBBO in each period with benchmark portfolio

**Figure 4.** The performance of the εBBO and the optimal portfolio in $X \in [-1, 2]$ of model (28).

**Table 3.** The model (29) using date in Table 1. Time is in minute and the expected return of the benchmark portfolio $E[g(y,\xi)] = 1.073$.

| X constraint | Algorithm | Iteration | Time | x | E[g(x,ξ)] |
|---|---|---|---|---|---|
| | εBBO | 38 | 0.367 | (0.599,0,0.001,0.4,0) | 1.1739 |
| $[0, 0.6]$ | SA | 226 | 9 | (0.6,0,0,0.4,0) | 1.174 |
| | LF | 4 | 0.577 | (0.6,0,0,0.4,0) | 1.174 |
| $[0, 1]$ | εBBO | 48 | 0.65 | (0.799,0.2,0,0.001,0) | 1.1779 |
| | εBBO | 47 | 0.5 | (0.8,0.359,-0.529,0.769,-0.399) | 1.3043 |
| $[-1, 2]$ | SA | 684 | 16 | (0.127,0.495,0.550,0.380,-0.553) | 1.23 |
| | LF | 4 | 0.649 | (0.39,0.527,0.287,0.3,-0.5) | 1.26 |

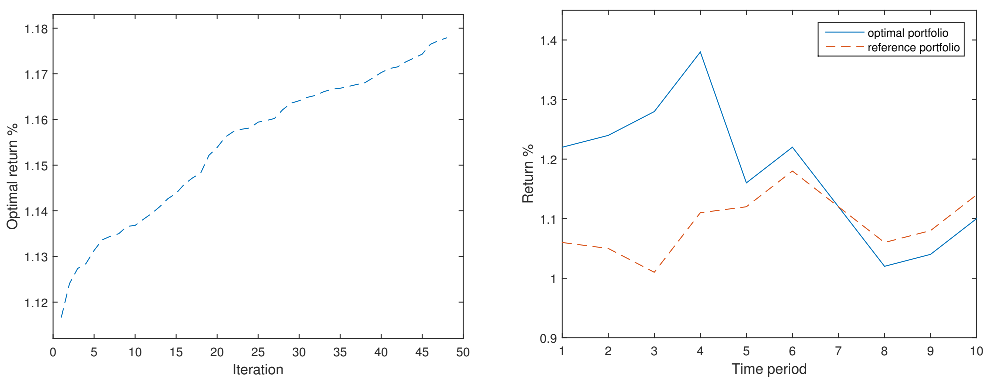
The Figures 5–7 show the performance of the εBBO and the optimal portfolio in the three kinds of $X$ constraints of model (29).




(a) The change of the the return of optimal portfolio between different iterative

(b) The return of optimal portfolio by εBBO in each period with benchmark portfolio and the optimal portfolio by SA and LF algorithm

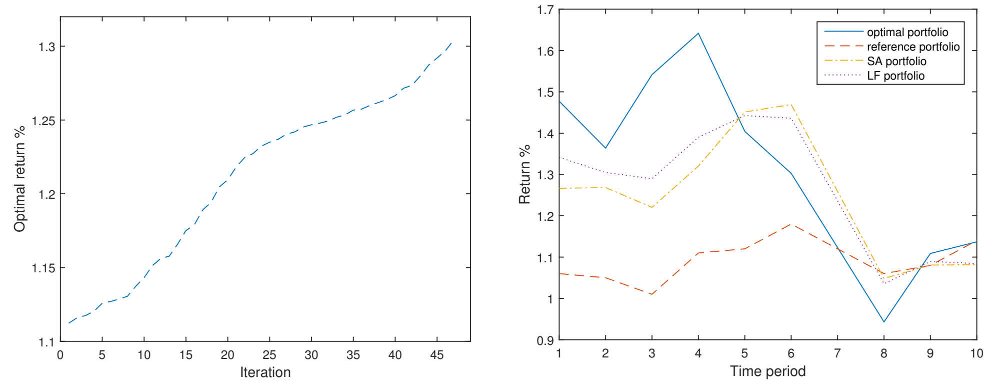
**Figure 5.** The performance of the εBBO and the optimal portfolio in $X \in [0, 0.6]$ of model (29).




(a) The change of the the return of optimal portfolio between different iterative

(b) The return of optimal portfolio by εBBO in each period with benchmark portfolio

**Figure 6.** The performance of the εBBO and the optimal portfolio in $X \in [0, 1]$ of model (29).




(a) The change of the the return of optimal portfolio between different iterative

(b) The return of optimal portfolio by $\varepsilon$BBO in each period with benchmark portfolio and the optimal portfolio by SA and LF algorithm

**Figure 7.** The performance of the $\varepsilon$BBO and the optimal portfolio in $X \in [-1, 2]$ of model (29).

## *4.2. Example 2*

In the practical application of portfolio optimization, often faced with a large number of assets for investors to choose, which requires the algorithm maintain a good performance to large-scale operations. In order to test the computational power of $\varepsilon$BBO algorithm on large-scale SSD constraints, we collect 249 daily historical returns of 101 FTSE100 assets prior to December 2016 to construct the portfolio strategy. Therefore, we get a historical of percentage returns for $m = 249$ time periods and a group of $n = 101$ assets. Besides, we use $y = (\frac{1}{101}, \ldots, \frac{1}{101})$ as the benchmark and we discuss the following SSD constraints model:

$$\begin{aligned} min \quad & 2 - g(x,\xi) \\ s.t. \quad & \sum_{i=1}^{249} \frac{1}{249}(g(y,\xi^j) - g(x,\xi^i))_+ \leq \sum_{i=1}^{249} \frac{1}{249}(g(y,\xi^j) - g(y,\xi^i))_+, j = 1,\ldots,249 \\ & x \in X \end{aligned} \tag{30}$$

Due to the fact that the stock certificate can not always guarantee positive returns, we discuss the suitability function as $2 - g(x, \xi)$ to ensure its value is positive. In order to ensure the diversity of the portfolio, we discuss the X constraint as $[0, 0.1]$ and $[-0.05, 0.15]$ which stand for two conditions: short-selling is allowed and short-selling is prohibited. Comparing with the FTSE100 Index we get the test results shown in Table 4. The specific index weight data of FTSE100 Index is shown in Appendix A.

**Table 4.** The model (30) using date of 249 daily historical returns of 101 FTSE100 assets prior to December 2016. Time is in minute and the expected return of the benchmark portfolio $E[g(y, \xi)] = 0.0595$.

| X constraint | Portfolio | Iteration | Time | NO.Assets | E[g(x,$\xi$)] |
|---|---|---|---|---|---|
| $[0, 0.1]$ | $\varepsilon$BBO | 92 | 1.6 | 101 | 0.1325 |
| $[-0.05, 0.15]$ | $\varepsilon$BBO | 150 | 2.64 | 101 | 0.3197 |
| / | FTSE100 | / | / | 101 | 0.0937 |

The Figures 8 and 9 show the specific asset structure of optimal portfolio in two kinds of X constraints of model (30). The Figures 10–13 show the performance of the $\varepsilon$BBO and the optimal portfolio in the two kinds of $X$ constraints of model (30).

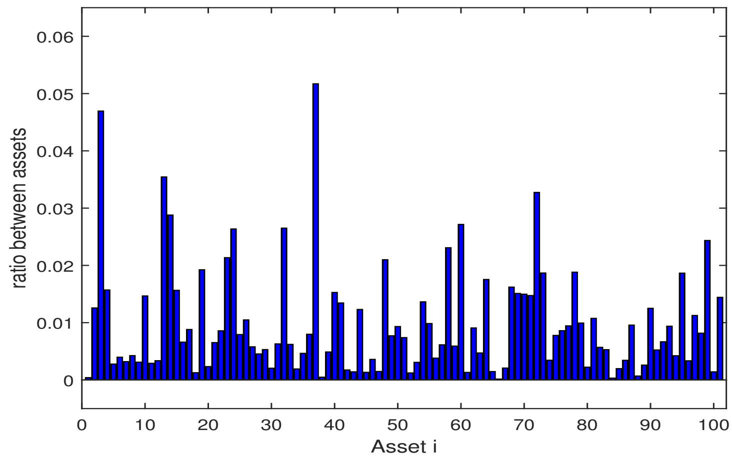



**Figure 8.** The specific asset structure of optimal portfolio in $X \in [0, 0.1]$ of model (30) for 101 assets.

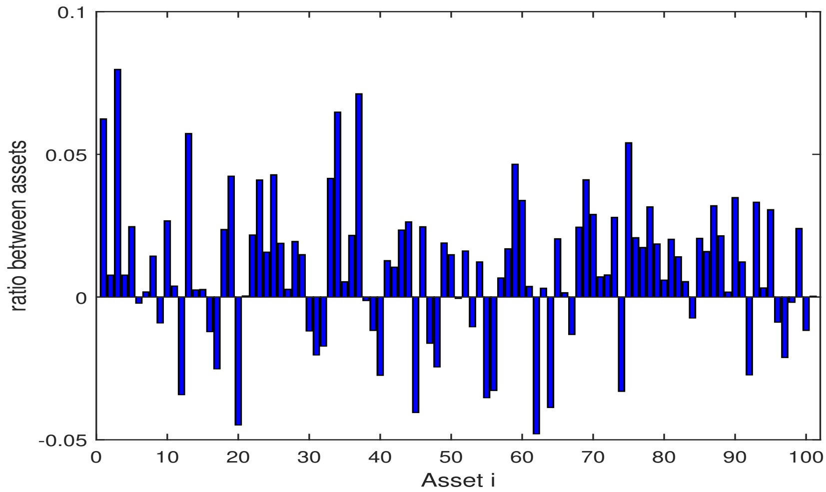



**Figure 9.** The specific asset structure of optimal portfolio in $X \in [-0.05, 0.15]$ of model (30) for 101 assets.

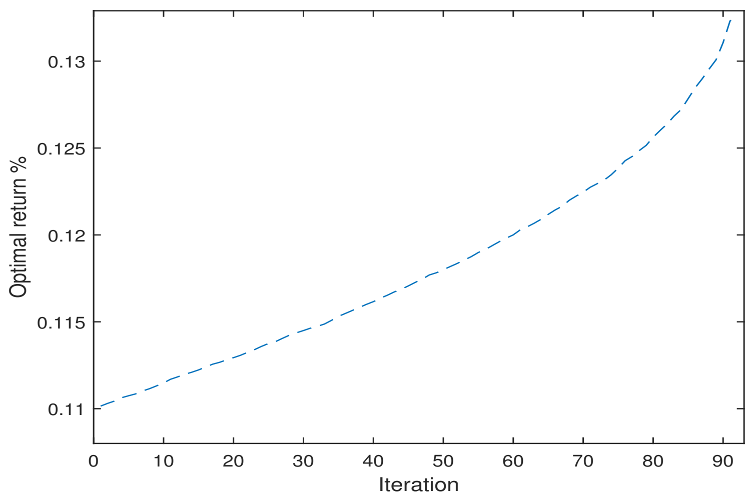



**Figure 10.** The return of optimal portfolio in different iterations in $X \in [0, 0.1]$ of model (30).

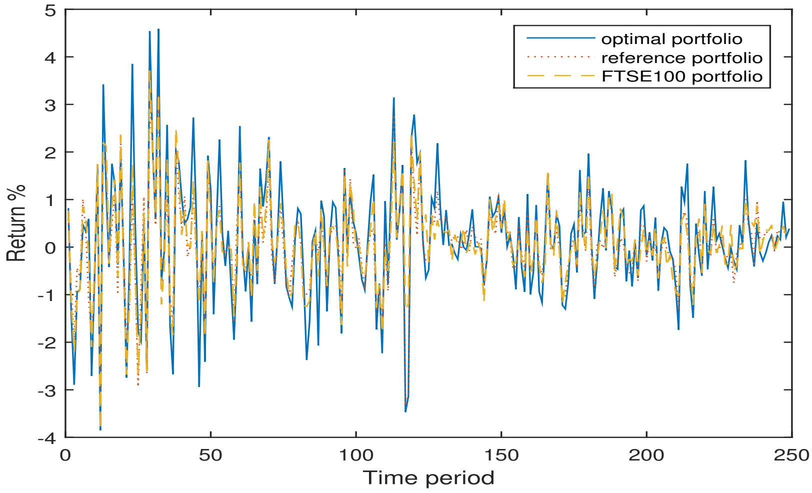



**Figure 11.** The return of optimal portfolio by $\varepsilon$BBO in each period with benchmark portfolio and the FTSE100 Index portfolio in $X \in [0, 0.1]$ of model (30).

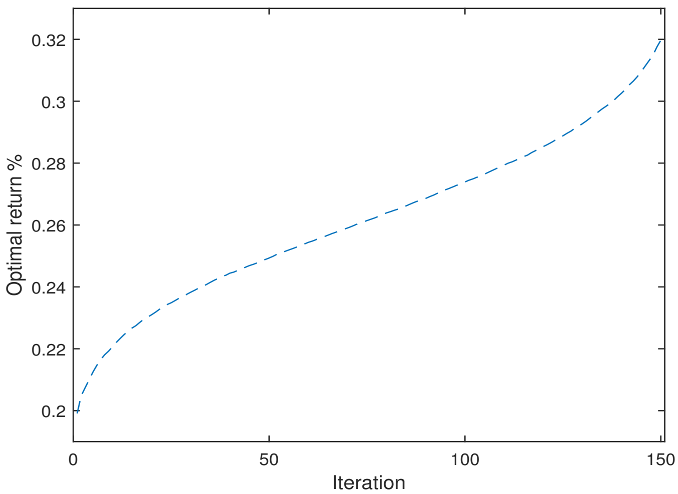



**Figure 12.** The return of optimal portfolio in different iterations in $X \in [-0.05, 0.15]$ of model (30).

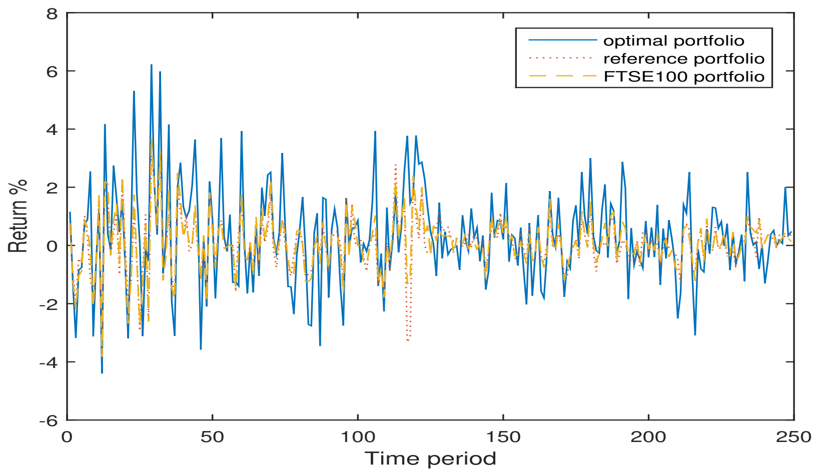



**Figure 13.** The return of optimal portfolio by $\varepsilon$BBO in each period with benchmark portfolio and the FTSE100 Index portfolio in $X \in [-0.05, 0.15]$ of model (30).

### *4.3. Numerical Analysis*

In the numerical experiment, we try the number of initial populations in different orders of magnitude. Taking the efficiency of algorithm into account, we finally set the number of initial population to 100 in the experiment. Besides, we hold the best 10 populations in each iteration while the others carry on migrating and mutating. The model (28) and model (29) discusses the level of optimization of $\varepsilon$BBO algorithm under the SSD in low dimension. Form the result of model (28), under the constraint of $[0, 0.6]$, the return of optimal portfolio by $\varepsilon$BBO is 1.1736%, while the returns of portfolio by SA and LF are 1.147% and 1.148%. Besides, the $\varepsilon$BBO is faster in running, but the number of iteration is much more than LF. Under the constraint of $[0, 1]$, the return of optimal portfolio by $\varepsilon$BBO is 1.1779%, while the returns of portfolio by GA are 1.165% and 1.177%. The GA(100) has the same initial population number as $\varepsilon$BBO, while the efficiency of $\varepsilon$BBO is better than the GA. Under the constraint of $[-1, 2]$, because the short-selling is allowed, the return of optimal portfolio gets greatly improved. Model (29) is an improvement to model (28) essentially, whose enhanced suitability function can weak the small transactions. Under the constraint of $[0, 0.6]$, three kinds of algorithm get the similar portfolio, while the $\varepsilon$BBO is faster in running than SA and LF. Under the constraint of $[0, 1]$, the return of optimal portfolio by $\varepsilon$BBO is 1.1779%, which is better than the model (28). Under the constraint of $[-1, 2]$, the return of optimal portfolio by $\varepsilon$BBO is 1.3043%, while the returns of portfolio by SA and LF are 1.23% and 1.26%. The $\varepsilon$BBO is also fast in running. In the comparison of model (28) and model (29), model (29) can approximate the optimal portfolio in less number of iteration than model (28). Because the model (29) is more complex, the running speed of model (29) is slower. However, the portfolio performance of model (29) is much better. Compared with the SA, the $\varepsilon$BBO has great advantages in all respects. Compared with the LF, the $\varepsilon$BBO requires large number of iteration to approximate the optimal portfolio due to its randomness. Therefore, the LF has a advantage of iterative procedure. Besides, the performance of GA is similar to LF. If having the same initial population, the efficiency of $\varepsilon$BBO is higher than GA. the $\varepsilon$BBO also has a strong ability to approximate the optimal portfolio.

The model (30) discusses the effectiveness of $\varepsilon$BBO algorithm in high dimension. Under the constraint of $[0, 0.1]$, the return of optimal portfolio by $\varepsilon$BBO is 0.1325%, the number of asset whose investment proportion more than 2% is 12, the number of asset whose investment proportion more than 1% is 34, the number of asset whose investment proportion less than 0.5% is 39, and the maximum investment proportion of optimal portfolio is 5.17%. Under the constraint of $[-0.05, 0.15]$, the return of optimal portfolio by $\varepsilon$BBO is 0.3197%, the number of asset whose investment proportion more than 4% is 12, the number of asset whose investment proportion more than 2% is 35, the number of short-selling asset is 29, the maximum investment proportion of optimal portfolio is 7.97% and the minimum investment proportion of optimal portfolio is $-4.79$%. As a comparison, the return of FTSE100 Index portfolio is 0.0937%, the number of asset whose investment proportion more than 2% is 16, and the maximum investment proportion of portfolio is 7.3%. Compared with the FTSE100 Index, the optimal portfolio of $\varepsilon$BBO is inclined to the concentrated investment and avoid the small amount of investment. Besides, once the short-selling is allowed, the return of the optimal portfolio has been greatly improved, and the capital is tending to focus on some assets with excellent performance. It can be seen from the experimental results that the $\varepsilon$BBO algorithm is less efficient than its performance in the low dimension.

Generally speaking, the $\varepsilon$BBO algorithm gets an excellent performance. However, the $\varepsilon$BBO algorithm has a large chance of searching the optimal solution. The multi-directionality of the mutation and migration leads to the inability to guarantee the positive optimization of the solution. Besides, we can increase the number of initial populations to reduce the number of iterations but it will increase the performance period of the algorithm. From the performance of the optimal portfolio, the optimal portfolio increase the risk of portfolio which can not guarantee the stability of the benefits in each period.

## 5. Conclusions and Future Research

The second order stochastic dominance poses a great challenge to the numerical processing ability of the algorithm because of its unique judgment method. Based on the BBO algorithm, in this paper we discuss the portfolio optimization problem with the second order stochastic dominance constraints by using the improved $\varepsilon$BBO algorithm, and give a detailed numerical experiment. The experimental results show that the $\varepsilon$BBO algorithm is effective on portfolio optimization problem. In this paper, we prove the feasibility in the field of economic decision-making and explore the practice scope of BBO algorithm.

In this paper, we mainly discuss the simple target portfolio optimization problem. In the future, we will try the multiple target portfolio optimization problem. The target is not only the return of portfolio, but also can be the investment cycle, conditional value at risk (CVaR) and so on. Besides, in terms of transaction costs, we will discuss different types of transaction cost functions and introduce them into the second order stochastic dominance model. Furthermore, we also try to combine the $\varepsilon$BBO algorithm with some prediction algorithms, such as Neural network algorithm, to predict the performance of the portfolio in the future.

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**Author Contributions:** Siling Feng contributed the new processing method, conceived and designed the experiments; Tao Ye and Ziqiang Yang designed and performed the experiments; Siling Feng and Tao Ye analyzed the data; and Tao Ye wrote the paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| SSD | Second Order Stochastic Dominance |
| BBO | Biogeography-based Optimization |
| DE | Differential Evolution |
| HSI | Habitat Suitability Index |
| SIV | Suitability Index Variable |
| GA | Genetic Algorithm |
| SA | Stochastic Approximation |
| LF | Level Function |
| CVaR | conditional value at risk |

## Appendix A

**Table A1.** The specific index weight data of FTSE100 Index.

| Constitution | Index Weight (%) | Constitution | Index Weight (%) |
|---|---|---|---|
| 3i Group | 0.38 | Admiral Group | 0.2 |
| Anglo American | 0.84 | Antofagasta | 0.13 |
| Ashtead Group | 0.44 | Associated British Foods | 0.53 |
| AstraZeneca | 3.1 | Aviva | 1.09 |
| Babcock International Group | 0.27 | BAE Systems | 1.04 |
| Barclays | 2.09 | Barratt Developments | 0.26 |
| BHP Billition | 1.53 | BP | 5.3 |
| British American Tobacco | 4.77 | British Land Co | 0.36 |
| BT Group | 1.7 | Bunzl | 0.39 |
| Burberry Group | 0.37 | Capita | 0.19 |
| Carnival | 0.42 | Centrica | 0.7 |
| Coca-Cola HBC AG | 0.19 | Compass Group | 1.37 |
| ConvaTec Group | 0.08 | CRH | 1.3 |
| Croda International | 0.23 | DCC | 0.3 |
| Diageo | 2.94 | Direct Line Insurance Group | 0.28 |
| Dixons Carphone | 0.2 | Easyjet | 0.14 |
| Experian | 0.84 | Fresnillo | 0.11 |
| GKN | 0.31 | GlaxoSmithKline | 4.21 |
| Glencore | 1.79 | Hammerson | 0.25 |
| Hargreaves Lansdown | 0.16 | Hikma Pharmaceuticals | 0.15 |
| HSBC HIdgs | 7.3 | Imperial Brands | 1.89 |
| Informa | 0.31 | InterContinental Hotels Group | 0.4 |
| International Consolidated Airlines Group | 0.41 | Intertek Group | 0.31 |
| Intu Properties | 0.14 | ITV | 0.43 |
| Johnson Matthey | 0.34 | Kingfisher | 0.44 |
| Land Securities Group | 0.46 | Legal & General Group | 0.81 |
| LIoyds Banking Group | 2.22 | London Stock Exchange Group | 0.51 |
| Marks & Spencer Group | 0.31 | Mediclinic International pIc | 0.17 |
| Merlin Entertainments | 0.18 | Micro Focus International | 0.27 |
| Mondi | 0.34 | Morrison (Wm) Supermarkets | 0.28 |
| National Grid | 1.99 | Next | 0.39 |
| Old Mutual | 0.56 | Paddy Power Betfair | 0.4 |
| Pearson | 0.37 | Persimmon | 0.3 |
| Provident Financial | 0.23 | Prudential | 2.33 |
| Randgold Resources | 0.33 | Reckitt Benckiser Group | 2.4 |
| RELX | 0.88 | Rio Tinto | 2.12 |
| Rolls-Royce Holdings | 0.61 | Royal Bank Of Scotland Group | 0.41 |
| Royal Dutch Shell A | 5.43 | Royal Dutch Shell B | 4.89 |
| Royal Mail | 0.23 | RSA Insurance Group | 0.33 |
| Sage Group | 0.39 | Sainsbury(J) | 0.23 |
| Schroders | 0.19 | Severn Trent | 0.29 |
| Shire | 2.34 | Sky | 0.58 |
| Smith & Nephew | 0.6 | Smiths Group | 0.31 |
| Smurfit Kappa Group | 0.24 | SSE | 0.87 |
| St. James's Place | 0.29 | Standard Chartered | 0.99 |
| Standard Life | 0.41 | Taylor Wimpey | 0.28 |
| Tesco | 0.93 | TUI AG | 0.3 |
| Unilever | 2.2 | United Utilities Group | 0.34 |
| Vodafone Group | 2.94 | Whitbread | 0.38 |
| Wolseley | 0.69 | Worldpay Group | 0.25 |
| WPP | 1.29 | | |

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