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Extending the Applicability of the MMN-HSS Method for Solving Systems of Nonlinear Equations under Generalized Conditions

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Abstract: We present the semilocal convergence of a multi-step modified Newton-Hermitian and Skew-Hermitian Splitting method (MMN-HSS method) to approximate a solution of a nonlinear equation. Earlier studies show convergence under only Lipschitz conditions limiting the applicability of this method. The convergence in this study is shown under generalized Lipschitz-type conditions and restricted convergence domains. Hence, the applicability of the method is extended. Moreover, numerical examples are also provided to show that our results can be applied to solve equations in cases where earlier study cannot be applied. Furthermore, in the cases where both old and new results are applicable, the latter provides a larger domain of convergence and tighter error bounds on the distances involved.

Keywords: MMN-HSS method; semilocal convergence; system of nonlinear equations; generalized Lipschitz conditions; Hermitian method

MSC: 65F10; 65W05

1. Introduction

Let $F : D \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be Gateaux-differentiable and D be an open set. Let also $x_0 \in D$ be a point at which $F'(x)$ is continuous and positive definite. Suppose that $F'(x) = H(x) + S(x)$, where $H(x) = \frac{1}{2}(F'(x) + F'(x)^*)$ and $S(x) = \frac{1}{2}(F'(x) - F'(x)^*)$ are the Hermitian and Skew-Hermitian parts of the Jacobian matrix $F'(x)$, respectively. Many problems can be formulated like the equation

$$F(x) = 0, \tag{1}$$

using mathematical modelling [1–22]. The solution x_* of Equation (1) can rarely be found in explicit form. This is why most solution methods of Equation (1) are usually iterative. In particular, Hermitian and Skew-Hermitian Splitting (HSS) methods have been shown to be very efficient in solving large sparse non-Hermitian positive definite systems of linear equations [11,12,17,19,22].

We study the semilocal convergence of the multi-step modified Newton-HSS (MMN-HSS) method defined by

$$\begin{aligned} x_k^{(0)} &= x_k, \\ x_k^{(i)} &= x_k^{(i-1)} - \left(I - T(\alpha; x)^{l_k^{(i)}}\right) F'(x_k)^{-1} F(x_k^{(i-1)}), \ \ 1 \le i \le m, \\ x_{k+1} &= x_k^{(m)}, \ i = 1, 2, \dots m, \ \ k = 0, 1, \dots, \end{aligned} \tag{2}$$

where $x_0 \in D$ is an initial point, $T(\alpha; x) = (\alpha I + S(x))^{-1}(\alpha I - H(x))(\alpha I + H(x))^{-1}(\alpha I - H(x))$, $l_k^{(i)}$ is a sequence of positive integers, and α and tol are positive constants

$$\|F(x_k)\| \leq tol\|F(x_0)\|$$

and

$$\|F(x_k) + F'(x_k)d_{k,l_k}\| \leq \eta_k\|F(x_k)\|, \ \ \eta_k \in [0,1), \ \ \eta_k \leq \eta \leq 1.$$

The local and semilocal convergence analysis of method (2) was given in [19] using Lipschitz continuity conditions on F. Later, we extended the local convergence of method (2) using generalized Lipschitz continuity conditions [8].

In the present study, we show that the results in [19] can be extended as the ones for MN-HSS in [8]. Using generalized Lipschitz-type conditions, we present a new semilocal convergence analysis with advantages (A):

(a) Larger radius of convergence,
(b) More precise error estimates on $\|x_k - x_*\|$,
(c) The new results can be used in cases where the old ones in [19] cannot be used to solve Equation (1).

The advantages (A) are obtained under the same computational cost as in [19]. Hence, the applicability of the MMN-HSS method is extended.

The rest of the paper is structured as follows: Section 2 contains the semilocal convergence analysis of the MMN-HSS method. Section 3 contains the numerical examples.

2. Semilocal Convergence

The following hypotheses shall be used in the semilocal convergence analysis (H):

(H1) Let $x_0 \in \mathbb{C}^n$. There exist $\beta_1 > 0$, $\beta_2 > 0$, $\gamma > 0$ and $\mu > 0$ such that

$$\|H(x_0)\| \leq \beta_1, \ \|S(x_0)\| \leq \beta_2, \ \|F'(x_0)^{-1}\| \leq \gamma, \ \|F(x_0)\| \leq \mu.$$

(H2) There exist $v_1 : [0, +\infty) \to \mathbb{R}$, $v_2 : [0, +\infty) \to \mathbb{R}$, continuous and nondecreasing functions with $v_1(0) = v_2(0) = 0$ such that, for each $x, y \in D$,

$$\|H(x) - H(x_0)\| \leq v_1(\|x - x_0\|),$$

$$\|S(x) - S(x_0)\| \leq v_2(\|x - x_0\|).$$

Define functions w and v by $w(t) = w_1(t) + w_2(t)$ and $v(t) = v_1(t) + v_2(t)$.

$$\text{Let } r_0 = \sup\{t \geq 0 : \gamma v(t) < 1\}$$

and set

$$D_0 = D \cap U(x_0, r_0).$$

(H3) There exist $w_1 : [0, +\infty) \to \mathbb{R}$, $w_2 : [0, +\infty) \to \mathbb{R}$, continuous and nondecreasing functions with $w_1(0) = w_2(0) = 0$ such that, for each $x, y \in D_0$,

$$\|H(x) - H(y)\| \leq w_1(\|x - y\|),$$

$$\|S(x) - S(y)\| \leq w_2(\|x - y\|).$$

We need the following auxiliary results for the semilocal convergence analysis that follows.

Lemma 1. *Under the (H) hypotheses, the following items hold for each $x, y \in D_0$:*

$$\|F'(x) - F'(y)\| \leq w(\|x-y\|), \tag{3}$$

$$\|F'(x) - F'(x_0)\| \leq v(\|x-y\|), \tag{4}$$

$$\|F'(x)\| \leq v(\|x-y\|) + \beta_1 + \beta_2, \tag{5}$$

$$\|F'(x) - F(y) - F'(y)(x-y)\| \leq \int_0^1 w(\|x-y\|\xi)d\xi\|x-y\|, \tag{6}$$

and

$$\|F'(x)^{-1}\| \leq \frac{\gamma}{1-\gamma v(\|x-x_0\|)}. \tag{7}$$

Proof. By hypothesis (H_3) and $F'(x) = H(x) + S(x)$, we have that

$$\begin{aligned} \|F'(x) - F'(y)\| &= \|\Big(H(x) - H(y)\Big) + \Big(S(x) - S(y)\Big)\| \\ &\leq \|H(x) - H(y)\| + \|S(x) - S(y)\| \\ &\leq w_1(\|x-y\|) + w_2(\|x-y\| = w(\|x-y\|) \end{aligned}$$

and by (H_2)

$$\begin{aligned} \|F'(x) - F'(x_0)\| &\leq \|H(x) - H(x_0)\| + \|S(x) - S(x_0)\| \\ &\leq v_1(\|x-x_0\|) + v_2(\|x-x_0\|) \\ &= v(\|x-x_0\|), \end{aligned}$$

which show inequalities (3) and (4), respectively.

Then, we get, by (H_1) and (H_3),

$$\begin{aligned} \|F'(x)\| &= \|\Big(F'(x) - F(x_0)\Big) + F'(x_0)\| \\ &\leq \|F'(x) - F'(x_0)\| + \|H(x_0)\| + \|S(x_0)\| \\ &\leq v(\|x-x_0\|) + \beta_1 + \beta_2, \end{aligned}$$

which shows inequality (5). Using (H_3), we obtain that

$$\begin{aligned} \|F(x) - F(y) - F'(y)(x-y)\| &= \|\int_0^1 F'\Big(y + \xi(x-y) - F'(y)\Big)d\xi(x-y)\| \\ &\leq \int_0^1 w(\|x-y\|\xi)d\xi\|x-y\|, \end{aligned}$$

which shows inequality (6). By (H_1), (H_2) and inequality (4), we get, in turn, that for $x \in D_0$:

$$\|F'(x_0)^{-1}\|\|F'(x) - F'(x_0)\| \leq \gamma(\|x-x_0\|) \leq \gamma v(r_0) < 1. \tag{8}$$

It follows from inequality (8) and the Banach lemma on invertible operators [4] that $F'(x)^{-1}$ exists so that inequality (7) is satisfied. □

We shall define some scalar functions and parameters to be used in the semilocal convergence analysis. Let $t_0 = 0$ and $s_0^{(1)} = (1+\eta)\gamma\mu$. Define scalar sequences $\{t_k\}, \{s_k^{(1)}\}, \ldots, \{s_k^{(m-1)}\}$ by the following schemes:

$$t_0 = 0,\ s_k^{(0)} = t_k,\ t_{k+1} = s_k^m,$$

$$s_k^{(i)} = s_k^{(i-1)} + \frac{[(\int_0^1 w((s_k^{(i-1)} - s_k^{(i-2)})\xi)d\xi + w(s_k^{(i-2)} - t_k))(1+\eta)\gamma + \eta(1-\gamma v(t_k))](s_k^{(i-1)} - s_k^{(i-2)})}{1-\gamma v(t_k)}$$

$$t_{k+1} = s_k^{(i)} + (1-\gamma v(t_k))(s_k^{(i)} - s_k^{(i-1)}),\ i = 0,1,2,\ldots,m-1,\ k = 0,1,2,\ldots. \tag{9}$$

Moreover, define functions q and h_q on the interval $[0, r_0)$ by

$$q(t) = \frac{(1+\eta)\gamma\int_0^1 w((1+\eta)\gamma\mu\xi)d\xi + (1+\eta)\gamma w(t) + \eta(1-\gamma v(t))}{1-\gamma v(t)}$$

and

$$h_q(t) = q(t) - 1.$$

We have that $h_q(0) = \eta - 1 < 0$ and $h_q \to \infty$ as $t \to r_0^-$. It follows from the intermediate value theorem that function h_q has zeros in interval $(0, r_0)$. Denote by r_q the smallest such zero. Then, we have that for each $t \in [0, r_0)$

$$0 \le q(t) \le 1. \tag{10}$$

Lemma 2. *Suppose that equation*

$$t(1-q(t)) - \left((1+\eta)\gamma\mu + (1+\eta)\gamma\int_0^1 w((1+\eta)\gamma\mu\xi)d\xi + \eta\right) = 0 \tag{11}$$

has zeros in interval $(0, r_q)$. Denote by r the smallest such zero. Then, sequence $\{t_k\}$, generated by Equation (9) is nondecreasing, bounded from above by r_q and converges to its unique least upper bound r^*, which satisfies

$$0 < r^* \le r < r_q. \tag{12}$$

Proof. Equation (11) can be written as

$$\frac{t_1 - t_0}{1-q(r)} = r, \tag{13}$$

since, by Equation (9),

$$t_1 = (1+\eta)\gamma\mu + (1+\eta)\gamma\int_0^1 w((1+\eta)\gamma\mu\tau)d\tau + \eta + w((1+\eta)\gamma\mu)$$

and r solves Equation (11). It follows from the definition of sequence $\{t_k\}$, functions w_1, w_2, v_1, v_2 and inequality (10) that

$$0 \le t_0 \le s_0 \le t_1 \le s_1 \le \cdots \le t_k \le s_k \le t_{k+1} < r,$$
$$t_{k+2} - t_{k+1} = q(r)(t_{k+1} - t_k) \le q(r)^{k+1}(t_1 - t_0),$$

and

$$\begin{aligned} t_{k+2} \le\, & t_{k+1} + q(r)^{k+1}(t_1 - t_0) \le t_k + q(r)^k(t_1 - t_0) + q(r)^{k+1}(t_1 - t_0) \\ \le\, & \cdots \le t_1 + q(r)(t_1 - t_0) + \cdots + q(r)^{k+1}(t_1 - t_0) \\ \le\, & \frac{t_1 - t_0}{1-q(r)}(1 - q(r)^{k+2}) < \frac{t_1 - t_0}{1-q(r)} = r. \end{aligned}$$

Therefore, sequences $\{t_k\}$ converges to r^*, which satisfies inequality (12). □

Next, we present the semilocal convergence analysis of the MMN-HSS method.

Theorem 1. *Suppose that the hypotheses (H) and hypotheses of Lemma 2 hold. Define* $\bar{r} = min\{r_1^+, r^*\}$, *where* r_1^+ *is defined in ([7], Theorem 2.1) and* r^* *is given in Lemma 2. Let* $u = min\{m_*, l_*\}$, $m_* = \liminf_{k\to\infty} m_k$, $l_* = \liminf_{k\to\infty} l_k$. *Moreover, suppose*

$$u > \left\lfloor \frac{ln\eta}{ln((\tau+1)\theta)} \right\rfloor, \tag{14}$$

where the symbol $\lfloor . \rfloor$ *denotes the smallest integer no less than the corresponding real number,* $\tau \in (0, \frac{1-\theta}{\theta})$ *and*

$$\theta := \theta(\alpha; x_0) = \|T(\alpha; x_0\| < 1. \tag{15}$$

Then, the sequence $\{x_k\}$ *generated by the MMN-HSS method is well defined, remains in* $U(x_0, \bar{r})$ *for each* $k = 0, 1, 2, \ldots$ *and converges to a solution* x_* *of Equation* $F(x) = 0$.

Proof. Notice that we showed in ([8], Theorem 2.1) that for each $x \in U(x_0, \bar{r})$

$$\|T(\alpha; x)\| \le (\tau + 1)\theta < 1. \tag{16}$$

The following statements shall be shown using mathematical induction:

$$\begin{cases} \|x_k - x_0\| \le t_k - t_0, \\ \|F(x_k)\| \le \frac{1}{(1+\eta)\gamma}\phi(t_k), \\ \|x_k^{(1)} - x_k\| \le s_k^{(1)} - t_k \\ \|F(x_k^{(i)})\| \le \frac{1}{(1+\eta)\gamma}\phi(s_k^{(i)}), \\ \|x_k^{(i+1)} - x_k^{(i)}\| \le s_k^{(i+1)} - s_k^{(i)},\ i = 1, 2, \cdots, m-2, \\ \|F(x_k^{(m-1)})\| \le \frac{1}{(1+\eta)\gamma}\phi(s_k^{(m-1)}), \\ \|x_{k+1} - x_k^{(m-1)}\| \le t_{k+1} - s_k^{(m-1)}. \end{cases} \tag{17}$$

We have for $k = 0$:

$$\|x_0 - x_0\| = 0 \le t_0 - t_0,$$

$$\|F(x_0)\| \le \delta \le \frac{\gamma(1 - \gamma v(t_0))(s_0^1 - t_0)}{\gamma(1+\eta)},$$

$$\|x_0^{(1)} - x_0\| \le \|I - T(\alpha; x_0)^{l_0^{(1)}}\|.\|F'(x_0)^{-1}\|\|F(x_0)\| \le (1 + \theta^{l_0^{(1)}}) < (1+\eta)\gamma\delta = s_0^{(1)}.$$

Suppose the following items hold for each $i < m-1$:

$$\begin{cases} \|F(x_0^{(i)})\| \le \frac{1}{(1+\eta)\gamma}(1 - \gamma v(t_0))(s_0^{(i+1)} - s_0^{(i)}), \\ \|x_0^{(i+1)} - x_0^{(i)}\| \le s_0^{(i+1)} - s_0^{(i)},\ i = 1, 2, \cdots, m-2. \end{cases} \tag{18}$$

We shall prove that inequalities (18) hold for $m-1$.
Using the (H) conditions, we get in turn that

$$\begin{aligned} \|F(x_0^{(m-1)})\| &\le \|F(x_0^{(m-1)}) - F(x_0^{(m-2)}) - F'(x_0)(x_0^{(m-1)} - x_0^{(m-2)}\| \\ &\quad + \|F(x_0^{(m-2)}) + F'(x_0)(x_0^{(m-1)} - x_0^{(m-2)}\| \\ &\le \|F(x_0^{(m-1)}) - F(x_0^{(m-2)}) - F'(x_0^{(m-2)})(x_0^{(m-1)} - x_0^{(m-2)})\| \\ &\quad + \|F'(x_0^{(m-2)}) - F'(x_0)\|\|(x_0^{(m-1)} - x_0^{(m-2)}\| + \eta\|F(x_0^{(m-2)}\| \\ &\le \int_0^1 w(\|(x_0^{(m-1)} - x_0^{(m-2)}\|\xi)d\xi\|\|(x_0^{(m-1)} - x_0^{(m-2)}\| \\ &\quad + w(\|x_0^{(m-2)} - x_0\|)\|(x_0^{(m-1)} - x_0^{(m-2)}\| + \eta\|F(x_0^{(m-2)})\|. \end{aligned} \tag{19}$$

Then, we also obtain that

$$\|(x_0^{(m-1)} - x_0^{(m-2)}\| \le s_0^{(m-1)} - s_0^{(m-2)},$$

$$\begin{aligned}\|x_0^{(m-2)} - x_0\| &\le \|x_0^{(m-2)} - x_0^{(m-3)}\| + \cdots + \|x_0^{(1)} - x_0\| \\ &\le (s_0^{(m-2)} - s_0^{(m-3)}) + \cdots + (s_0^{(1)} - t_0) \\ &\le s_0^{(m-2)} - t_0 = s_0^{(m-2)}\end{aligned}$$

and

$$\|F(x_0^{(m-2)}\| \le \frac{1}{(1+\eta)\gamma}(1-\gamma v(t_0))(s_0^{(m-1)} - s_0^{(m-2)}).$$

Hence, we get from inequality (19) that

$$\begin{aligned}\|F(x_0^{(m-1)})\| \le &\int_0^1 w((s_0^{(m-1)} - s_0^{(m-2)})\xi)d\xi(s_0^{(m-1)} - s_0^{(m-2)}) \\ &+ w(s_0^{(m-2)} - t_0)(s_0^{(m-1)} - s_0^{(m-2)}) + \frac{\eta(1-\gamma v(t_0))}{(1+\eta)\gamma}(s_0^{(m)} - s_0^{(m-1)}) \\ &\le \frac{1-\gamma v(t_0)}{(1+\eta)\gamma}(s_0^{(m)} - s_0^{(m-1)}).\end{aligned} \tag{20}$$

Then, we have by Equation (9) that

$$\begin{aligned}\|x_1 - x_0^{(m-1)}\| &\le \|I - T(\alpha; x_0)^{l_0^{(m)}}\|\|F'(x_0)^{-1}\|\|F(x_0^{(m-1)})\| \\ &\le (1 + ((\tau+1)\theta)^{l_0^{(m)}})\gamma\frac{1}{(1+\eta)\gamma}(1-\gamma v(t_0))(s_0^{(m)} - s_0^{(m-1)})) = t_1 - s_0^{(m-1)}\end{aligned}$$

holds, and the items (17) hold for $k = 0$. Suppose that the items (17) hold for all nonnegative integers less than k. Next, we prove the items (17) hold for k.

We get, in turn, by the induction hypotheses:

$$\begin{aligned}\|x_k - x_0\| &\le \|x_k - x_{k-1}^{(m-1)}\| + \|x_{k-1}^{(m-1)} - x_{k-1}^{(m-2)}\| + \cdots + \|x_{k-1}^{(1)} - x_{k-1}^{(0)}\| + \|x_{k-1} - x_0\|, \\ &\le (t_k - s_{k-1}^{(m-1)}) + (s_{k-1}^{(m-1)} - s_{k-1}^{(m-2)}) + \cdots + (s_{k-1}^{(1)} - t_{k-1}) + (t_{k-1} - t_0), \\ &= t_k - t_0 < r_* < r.\end{aligned}$$

In view of $x_{k-1}, x_{k-1}^{(1)}, \cdots, x_{k-1}^{(m-1)} \in \mathbb{U}(x_0, r)$, we have

$$\begin{aligned}\|F(x_k)\| \le &\|F(x_k) - F(x_{k-1}^{(m-1)}) - F'(x_{k-1})(x_k - x_{k-1}^{(m-1)})\| \\ &+ \|F(x_{k-1}^{(m-1)}) + F'(x_{k-1})(x_k - x_{k-1}^{(m-1)}\| \\ \le &\|F(x_k) - F(x_{k-1}^{(m-1)}) - F'(x_{k-1}^{(m-1)})(x_k - x_{k-1}^{(m-1)})\| \\ &+ \|F'(x_{k-1}^{(m-1)}) - F'(x_{k-1})\|\|(x_k - x_{k-1}^{(m-1)}\| + \eta\|F(x_{k-1}^{(m-1)}\| \\ \le &\int_0^1 w(\|(x_k - x_{k-1}^{(m-1)}\|\xi)d\xi\|\|(x_k - x_{k-1}^{(m-1)}\| \\ &+ w(\|x_{k-1}^{(m-1)} - x_{k-1}\|)\|x_k - x_{k-1}^{(m-1)}\| + \frac{\eta(1-\gamma v(t_{k-1}))}{(1+\eta)\gamma}(s_{k-1}^{(m)} - s_{k-1}^{(m-1)}) \\ \le &\frac{(1-\gamma v(t_k))}{(1+\eta)\gamma}(s_k^{(m)} - s_k^{(m-1)}).\end{aligned} \tag{21}$$

We also get that

$$\|x_k - x_{k-1}^{(m-1)}\| \le t_k - s_{k-1}^{(m-1)}, \tag{22}$$

$$\begin{aligned}\|x_{k-1}^{(m-1)} - x_{k-1}\| &\le \|x_{k-1}^{m-1} - x_{k-1}^{(m-2)}\| + \cdots + \|x_{k-1}^{(1)} - x_{k-1}\|,\\ &\le (s_{k-1}^{(m-1)} - s_{k-1}^{(m-2)}) + \cdots + (s_{k-1}^{(1)} - t_{k-1}),\\ &\le s_{k-1}^{(m-1)} - t_{k-1}\end{aligned} \tag{23}$$

and

$$\|F(x_{k-1}^{(m-1)})\| \le \frac{1}{1+\eta)\gamma}(1-\gamma v(t_{k-1}))(s_{k-1}^{(m)} - s_{k-1}^{(m-1)}). \tag{24}$$

It follows that

$$\begin{aligned}\|x_k^{(1)} - x_k\| &\le \|I - T(\alpha; x_k)^{l_k^{(1)}}\|\|F'(x_k)^{-1}\|\|F(x_k)\|\\ &\le (1+\theta^{l_k^{(1)}})\frac{\gamma}{1-\gamma v(t_k)}\frac{1-\gamma v(t_k)}{(1+\eta)\gamma}(s_k^{(1)} - t_k)\\ &\le s_k^{(1)} - t_k.\end{aligned}$$

Suppose that the following items hold for any positive integers less than $m-1$:

$$\begin{cases}\|F(x_0^{(i)})\| \le \frac{1}{(1+\eta)\gamma}(1-\gamma v(t_k))(s_k^{(i+1)} - s_k^{(i)}),\\ \|x_k^{(i+1)} - x_k^{(i)}\| \le s_k^{(i+1)} - s_k^{(i)},\ i = 1, 2, \cdots, m-2.\end{cases} \tag{25}$$

We will prove items (25) hold for $m-1$. As in inequality (21), we have that

$$\begin{aligned}\|F(x_k^{(m-1)})\| \le &\|F(x_k^{(m-1)}) - F(x_{k-1}^{(m-2)}) - F'(x_k)(x_k^{(m-1)} - x_k^{(m-2)})\|\\ &+ \|F(x_k^{(m-2)}) + F'(x_k)(x_k^{(m-1)} - x_k^{(m-2)})\|\\ \le&\|F(x_k^{(m-1)} - F(x_k^{(m-2)}) - F'(x_k^{(m-2)})(x_k^{(m-1)} - x_k^{(m-2)})\|\\ &+ \|F'(x_k^{(m-2)}) - F'(x_k)\|\|x_k^{(m-1)} - x_{k-1}^{(m-2)}\| + \eta\|F(x_k^{(m-2)}\|\\ \le&\frac{1}{(1+\eta)\gamma}(1-\gamma v(t_k))(s_k^{(m)} - s_k^{(m-1)}).\end{aligned} \tag{26}$$

We also get that

$$\|x_k^{(m-1)} - x_k^{(m-2)}\| \le \|s_k^{(m-1)} - s_k^{(m-2)}\|, \tag{27}$$

$$\begin{aligned}\|x_k^{(m-2)} - x_k\| &\le \|x_k^{m-2} - x_{k-1}^{(m-3)}\| + \cdots + \|x_k^{(1)} - x_k\|\\ &\le (s_k^{(m-2)} - s_k^{(m-3)}) + \cdots + (s_k^{(1)} - t_k)\\ &\le s_k^{(m-2)} - t_k,\end{aligned} \tag{28}$$

and

$$\|F(x_k^{(m-2)})\| \le \frac{1}{(1+\eta)\gamma}(1-\gamma v(t_k))(s_k^{(m-1)} - s_k^{(m-2)}). \tag{29}$$

Therefore,

$$\begin{aligned} \|x_{k+1} - x_k^{(m-1)}\| &\leq \|I - T(\alpha; x_k)^{l_k^{(m)}}\| \|F'(x_k)^{-1}\| \|F(x_k^{(m-1)})\| \\ &\leq (1 + \theta^{l_k^{(m)}}) \frac{\gamma(1 - \gamma v(t_k))(s_k^{(m)} - s_k^{(m-1)})}{(1 - \gamma v(t_k))(1 + \eta)\gamma} \\ &\leq t_{k+1} - s_k^{(m-1)} \end{aligned} \tag{30}$$

holds. The induction for items (17) is completed. The sequences $\{t_k\}$, $\{s_k\}, \cdots, s_k^{(m-1)}$ converges r^*, and

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k^{(m-1)}\| + \|x_k^{(m-1)} - x_k^{(m-2)}\| + \cdots + \|x_k^{(1)} - x_k^{(0)}\| + \|x_k - x_0\|, \\ &\leq (t_{k+1} - s_k^{(m-1)}) + (s_k^{(m-1)} - s_k^{(m-2)}) + \cdots + (s_k^{(1)} - t_k) + (t_k - t_0), \\ &= t_{k+1} - t_0 < r^* < r. \end{aligned} \tag{31}$$

Then, the sequence $\{x_k\}$ also converges to some $x \in \overline{\mathbb{U}}(x_*, r)$. By letting $k \to \infty$ in inequality (21), we get that

$$F(x_*) = 0. \tag{32}$$

□

Remark 1. *Let us specialize functions w_1, w_2, v_1, v_2 as $w_1(t) = L_1 t$, $w_2(t) = L_2 t$, $v_1(t) = K_1 t$, $v_2(t) = K_2 t$ for some positive constants K_1, K_2, L_1, L_2 and set $L = L_1 + L_2$, $K = K_1 + K_2$. Suppose that $D_0 = D$. Then, notice that*

$$K \leq L, \tag{33}$$

since

$$K_1 \leq L_1 \tag{34}$$

and

$$K_2 \leq L_2, \tag{35}$$

$$\beta_1 \leq \beta \tag{36}$$

and

$$\beta_2 \leq \beta, \tag{37}$$

where $\beta := max\{\|H(x_0)\|, \|S(x_0)\|\}$.

Notice that in [19], $K_1 = L_1$, $K_2 = L_2$. and $\beta = \beta_1 = \beta_2$. Therefore, if strict inequality holds in any of item (34), (35), (36) or (37), the present results improve the ones in [19], (see also numerical examples).

Remark 2. *The set D_0 in (H_3) can be replaced by $D_1 = D \cap U(x_1, r_0 - \|x_1 - x_0\|)$ leading to even smaller "w" and "v" functions, since $D_1 \subset D_0$.*

3. Numerical Examples

Example 1. *Suppose that the motion of an object in three dimensions is governed by system of differential equations*

$$\begin{aligned} & f_1'(x) - f_1(x) - 1 = 0, \\ & f_2'(y) - (e - 1)y - 1 = 0, \\ & f_3'(z) - 1 = 0. \end{aligned} \tag{38}$$

with x, y, $z \in D$ for $f_1(0) = f_2(0) = f_3(0) = 0$. Then, the solution of the system is given for $v = (x, y, z)^T$ by function $F := (f_1, f_2, f_3) : D \to \mathbb{R}^3$ defined by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T. \tag{39}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

Then, we have that $x_* = (0,0,0)^T$, $w(t) = w_1(t) + w_2(t)$, $v(t) = v_1(t) + v_2(t)$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$, $v_1(t) = K_1 t$, $v_2(t) = K_2 t$, *where* $L_1 = e - 1$, $L_2 = e$, $K_1 = e - 2$, $K_2 = e$, $\eta = 0.001$, $\gamma = 1$ *and* $\mu = 0.01$.

After solving the equation $h_q(t) = 0$, *we obtain the root* $r_q = 0.124067$. *Similarly, the roots of Equation (11) are:* 0.0452196 *and* 0.0933513. *So,*

$$r = min\{0.0452196, 0.0933513\} = 0.0452196.$$

Therefore,

$$r = 0.0452196 < r_q = 0.124067.$$

In addition, we have that

$$r^* = 0.0452196$$

and (see [7])

$$r_1^+ = 0.020274.$$

So,

$$\bar{r} = min\{r_1^+, r^*\} = min\{0.020274, 0.0452196\} = 0.020274.$$

It follows that sequence $\{x_k\}$ *is complete,* $\{t_k\} \to r^*$ *in D and as such it converges to* $x_* \in U(x_0, \bar{r}) = U(0, 0.020274)$.

Example 2. *Consider the system of nonlinear equation* $F(X) = 0$, *wherein* $F = (F_1, \cdots, F_n)^T$ *and* $X = (x_1, x_2, \ldots, x_n)^T$, *with*

$$F_i(X) = (3 - 2x_i)x_i^{3/2} - x_{i-1} - 2x_{i+1} + 1, \quad i = 1, 2, \ldots, n,$$

where $x_0 = x_{n+1} = 0$ *by convention. This system has a complex solution. Therefore, we consider the complex initial guess* $X_0 = (-i, -i, \ldots, -i)$. *The derivative* $F'(X)$ *is given by*

$$F'(X) = \begin{bmatrix} \frac{3}{2}(3-2x_1)\sqrt{x_1} - 2x_1^{3/2} & -2 & \cdots & 0 & 0 \\ -1 & \frac{3}{2}(3-2x_2)\sqrt{x_2} - 2x_2^{3/2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \frac{3}{2}(3-2x_n)\sqrt{x_n} - 2x_n^{3/2} \end{bmatrix}.$$

It is clear that $F'(X)$ is sparse and positive definite. Now, we solve this nonlinear problem by the Newton-HSS method (N-HSS), (see [10]), modified Newton-HSS method (MN-HSS), (see [22]), three-step modified Newton-HSS (3MN-HSS) and four-step modified Newton-HSS (4MN-HSS) method. The methods are compared in error estimates, CPU time (CPU-time) and the number of iterations. We use experimentally optimal parameter values of α for the methods corresponding to the problem dimension $n = 100, 200, 500, 1000$, see Table 1. The numerical results are displayed in Table 2. From numerical results, we observe that MN-HSS outperforms N-HSS in the sense of CPU time and the number of iterations. Note that, in this example, the results in [19] can not be applied since the operators involved are not Lipschitz. However, our results can be applied by choosing "w" and "v" functions appropriately as in Example 3.1. We leave these details to the interested readers.

Table 1. Optimal values of α for N-HSS and MN-HSS methods.

n	100	200	500	1000
N-HSS	4.1	4.1	4.2	4.1
MN-HSS	4.4	4.4	4.3	4.3
MMN-HSS	4.4	4.4	4.3	4.3

Table 2. Numerical results.

n	Method	Error Estimates	CPU-Time	Iterations
100	N-HSS	3.98×10^{-6}	1.744	5
	MN-HSS	4.16×10^{-8}	1.485	4
	3MN-HSS	8.28×10^{-5}	1.281	3
	4MN-HSS	1.12×10^{-6}	1.327	3
200	N-HSS	3.83×10^{-6}	6.162	5
	MN-HSS	5.46×10^{-8}	4.450	4
	3MN-HSS	7.53×10^{-5}	4.287	3
	4MN-HSS	9.05×10^{-7}	4.108	3
500	N-HSS	4.65×10^{-6}	32.594	5
	MN-HSS	4.94×10^{-8}	24.968	4
	3MN-HSS	7.69×10^{-5}	21.250	3
	4MN-HSS	9.62×10^{-7}	20.406	3
1000	N-HSS	4.29×10^{-6}	119.937	5
	MN-HSS	5.32×10^{-8}	98.203	4
	3MN-HSS	9.16×10^{-5}	89.018	3
	4MN-HSS	8.94×10^{-7}	91.000	3

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