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A Stochastic Model of Anomalously Fast Transport of Heat Energy in Crystalline Bodies

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Abstract: In this work, a new method for constructing the infinite-dimensional Ornstein–Uhlenbeck stochastic process is introduced. The constructed process is used to perturb the harmonic system in order to model anomalously fast heat transport in one-dimensional nanomaterials. The introduced method made it possible to obtain a transition probability function that allows for a different approach to the analysis of equations with such a disturbance. This creates the opportunity to relax assumptions about temporal correlations for such a process, which may lead to a qualitatively different model of energy transport through vibrations of the crystal lattice and, as a result, to obtain the superdiffusion equation on a macroscopic scale with an order of the fractional Laplacian different from the value of 3/4 obtained so far in stochastic models. Simulations confirming these predictions are presented and discussed.

Keywords: Ornstein–Uhlenbeck process; heat transfer; thermal conductivity; interacting particle systems

Citation: Stępień, Ł.; Łagodowski, Z.A. A Stochastic Model of Anomalously Fast Transport of Heat Energy in Crystalline Bodies. *Energies* **2023**, *16*, 7117. https://doi.org/10.3390/en16207117

Academic Editor: Xiaohu Yang

Received: 11 September 2023
Revised: 10 October 2023
Accepted: 11 October 2023
Published: 17 October 2023

1. Introduction

Thermal conductivity in nanomaterials is an extremely interesting research area in the field of nanotechnology and materials science. In our work, we deal with modeling heat flow in one-dimensional crystalline bodies, for example nanowires. So, let us focus our attention on such materials and briefly describe the possibilities of their use. Metallic nanowires have unique thermal conductivity properties due to their small size and structure. Research on metallic nanowires and their properties such as electrical and thermal conductivity began in full force at the turn of the 20th and 21st centuries, and many research works and experiments have been conducted since then. The study of thermal conductivity in metallic nanowires is important for both scientific and practical reasons. They could help design more efficient thermoelectric materials that convert heat into electricity, and develop advanced thermal materials used in electronics, nanotechnologies, energy technology, and more.

The phenomenon of thermal superconductivity in model of one-dimensional crystalline body was first observed by Lepri, Livi, and Politi, and described in [1]. In this paper, the system of equations describing the vibrations of coupled nonlinear oscillators was solved numerically and the divergence of thermal conductivity was demonstrated. The importance of this research is underlined in review work [2], where the authors write: "This marked the beginning of a research endeavor that, over more than two decades, has been devoted to understanding the mechanisms giving rise to anomalous transport in low-dimensional systems. Far from being a purely academic exercise, this research has unveiled the possibility of observing such peculiar effects in nanomaterials, such as nanotubes, nanowires, or graphene [3,4]". Since then, research into physical and mathematical models has continued [5–11]. A significant voice in the ongoing discussion was the article [12], in which a stochastic disturbance preserving energy and momentum was introduced into the harmonic model. The authors pointed out the important role of the

law of momentum conservation in the anomalous thermal conductivity. The next step was made in the work [13], where the authors have shown that the microscopic dynamics, introduced in [12], satisfy the linear Boltzmann equation for phonons on the mesoscopic scale. In turn, the works [14] and [15] show that the solution of the Boltzmann equation scaled to the macroscopic scale satisfies the superdiffusion equation with an order of the fractional Laplacian equal to $3/4$. Subsequently, in [16], a direct transition from the microscopic model to the macroscopic superdiffusion equation was made. In [17], the same authors modified the stochastic perturbation, introduced in [12], by replacing the Gaussian noise with the Ornstein–Uhlenbeck process, then obtained the linear Boltzmann equation at the mesoscopic scale.

All stochastic models discussed led to the same superdiffusion equation. This work paves the way to obtain the superdiffusion equation with a different order of fractional Laplacian. Namely, we construct the stochastic disturbance in a different way and present a proposal, supported by simulations, on how to transform the microscopic model into a macroscopic model to achieve the intended research goal.

Despite the utilization of highly advanced mathematical techniques, our intention is to communicate this research to a community primarily focused on experimental investigations. Our underlying belief is that fostering an exchange of research methods between these two groups of researchers may yield mutual benefits, potentially leading to collaborations that unveil new perspectives and avenues of exploration.

In the following part of the introduction, we discuss the construction of the model, introducing the reader to the topic.

1.1. Stochastic Models of Heat Transport

In insulating solids thermal energy is carried out by lattice vibrations propagating through the material. This is also the main medium of heat transfer in semiconductors [18]. On a microscopic scale, energy is transmitted by the interactions of neighboring atoms. We focus on the one-dimensional model. The equilibrium positions of atoms are denoted by integers x. The deflection from the equilibrium position of the atom at site x is denoted by $\mathfrak{q}_x$ and its momentum by $\mathfrak{p}_x$. Its total energy $\mathfrak{e}_x$ is

$$\mathfrak{e}_x := \frac{1}{2}\mathfrak{p}_x^2 + W(\mathfrak{q}_x) + \frac{1}{2}\sum_{|x-x'|=1} V(\mathfrak{q}_x - \mathfrak{q}_{x'}), \tag{1}$$

where $\frac{1}{2}\mathfrak{p}_x^2$ is the kinetic energy, and W, V are potentials, wherein V depends only on relative positions of adjacent atoms [19,20]. The atom transmits energy to its neighbors through elastic bonds—see Figure 1.

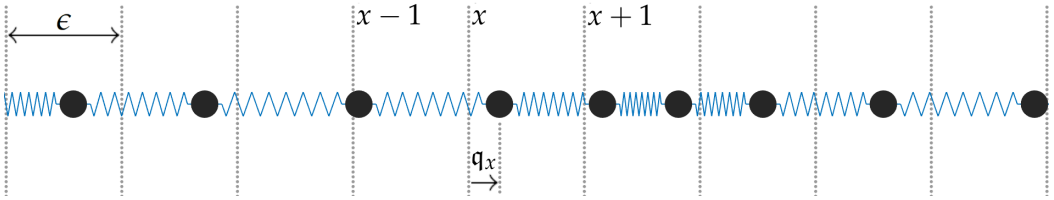

Figure 1. The model of vibrational energy transport through a one-dimensional crystalline lattice—linear chain of atoms oscillating around their equilibrium positions. Bonds between neighboring atoms are drawn as springs ([18], p. 1317).

The wave propagating along the chain can be decomposed into sinusoidal oscillations with a specific direction of propagation—called *normal modes*. Each normal mode is labeled by number k, which belongs to the interval $[-1/2, 1/2]$. Endpoints $-1/2$ and $1/2$ are identified, as they represent the same normal mode—the sinusoidal wave with wavelength equal to 2ϵ, where ϵ is the distance between equilibrium positions of two adjacent oscillators (see Figure 1). In this normal mode, every pair of two adjacent atoms oscillate in the

counterphase. If $0 < |k| < 1/2$, then k and $-k$ denote waves of the same space length $\epsilon/|k|$, but traveling in the opposite directions along the chain (Figure 2).

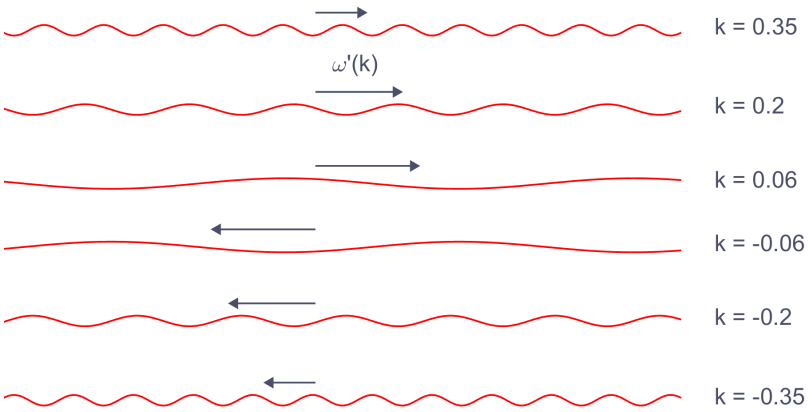

Figure 2. Energy carriers—sinusoidal oscillations, $\omega'(k)$—group velocity of mode k.

Total energy of the system $\mathrm{E} = \sum_x \mathfrak{e}_x$ decomposes into normal mode energies $\mathfrak{e}(k)$

$$\mathrm{E} = \int_{-1/2}^{1/2} \mathfrak{e}(k)\, dk.$$

We will be always assuming that potential W in (1) is zero, as this is the case when anomalous heat transport appears in dimension one. Let us for a moment assume that V is quadratic, $V(q) = aq^2$ for some $a > 0$. The model is then linear and oscillators are harmonic. For convenience, we set $a = 1$. Now, (1) takes the form

$$\mathfrak{e}_x := \frac{1}{2}\mathfrak{p}_x{}^2 + \frac{1}{2}\sum_{|x-x'|=1} (\mathfrak{q}_x - \mathfrak{q}_{x'})^2. \tag{2}$$

Hamilton's equations derived from (2) are

$$\begin{cases} \dfrac{d\mathfrak{q}_x}{dt} = \mathfrak{p}_x \\ \dfrac{d\mathfrak{p}_x}{dt} = \mathfrak{q}_{x+1} - 2\mathfrak{q}_x + \mathfrak{q}_{x-1} \end{cases} \qquad x \in \mathbb{Z}, \tag{3}$$

see [21] on p. 6. In this case, each normal mode k has the *phase velocity*

$$\omega(k) = 2|\sin(\pi k)|,$$

see [22] on p. 68-69 for the derivation. The function $\omega(k)$ is called *dispersion relation*. The *group velocity* for a *wave packet* (Figure 3) is specified by the derivative $\omega'(k)$, see [23] on p. 47.

After proper space-time rescaling of the dynamics of energy propagation given by (3), we obtain the linear transport equation at the mesoscopic scale (see Figure 4)

$$\partial_t u(t,x,k) + \omega'(k)\partial_x u(t,x,k) = 0. \tag{4}$$

Here, the function $u(t,x,k)$ has physical interpretation of the energy density of the normal mode $k \in [-1/2, 1/2]$ at space coordinate $x \in \mathbb{R}$ at time instant t. Heuristically, according to (4), if a quantum of energy is carried by normal mode k, then it travels ballistically along the chain at constant velocity $\omega'(k)$. It also stays in this mode forever.

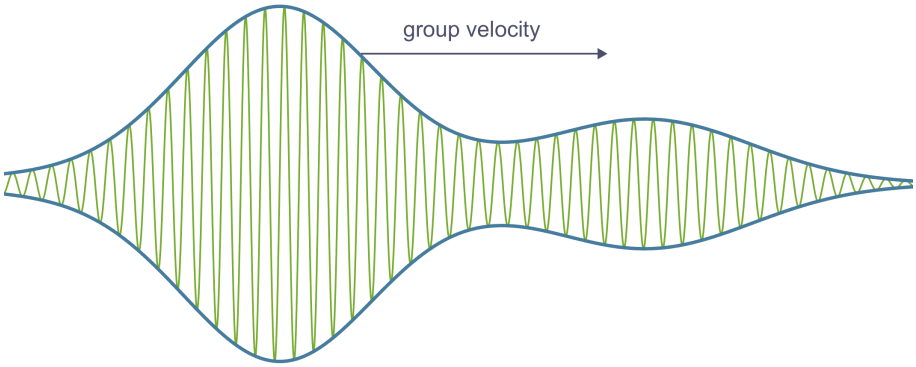

Figure 3. A wave packet—superposition of normal modes—propagates through the medium with the *group velocity*, which is determined by the derivative ω' of the dispersion relation.

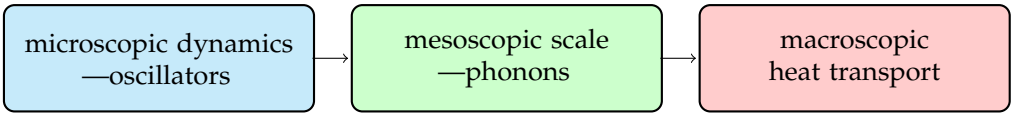

Figure 4. Different scales of dynamics for a model of heat energy transport.

If we further rescale the dynamics in space and time in attempt to obtain to the macroscopic scale, the speed of energy propagation approaches infinity, which is nonsense in the context of heat propagation, thus the assumption that oscillators are harmonic (V is quadratic) does not work in modeling heat transfer. On the other hand, if the potential V has the form

$$V(q) = aq^2 + V_h(q), \tag{5}$$

where $V_h(q)$ is a nonzero non-quadratic term (see i.g. classical FPU model [24,25]) then the study of the model encounters major analytical difficulties. Therefore, mathematical physicists are interested in probabilistic models in which $V_h \equiv 0$, but the dynamics of harmonic oscillators is randomly disturbed. Stochastic, time-dependent processes are introduced into the equations to mimic the chaos occurring in deterministic non-linear models. With this approach, the dynamics (3) is altered in the following manner:

$$\begin{cases} \dfrac{d\mathfrak{q}_x}{dt} = \mathfrak{p}_x \\ \dfrac{d\mathfrak{p}_x}{dt} = \mathfrak{q}_{x+1} - 2\mathfrak{q}_x + \mathfrak{q}_{x-1} + \dfrac{d\zeta_x}{dt}, \end{cases} \tag{6}$$

where $\zeta_x = \{\zeta_x(t) : t \geq 0\}$, $x \in \mathbb{Z}$ is a sequence of stochastic processes. Such models result in the *linear Boltzmann equation* at the mesoscopic scale ([13], p. 172). Compare it with Equation (4).

$$\partial_t u(t,x,k) + \omega'(k)\partial_x u(t,x,k) = \mathcal{L}u(t,x,k). \tag{7}$$

Here, $\mathcal{L}$ is a *scattering operator* acting on the variable k. In a heuristic description the appearance of $\mathcal{L}$ means, that—in opposition to the harmonic case—different normal modes interact and exchange energy. A quantum of energy absorbed by a normal mode is in this context called *phonon*. We will say that phonon is *in the state* k_0 when it is absorbed by mode k_0. Being in the state k_0, the phonon propagates with velocity $\omega'(k_0)$ until it is intercepted (at a random time instant) by a (random) mode k_1 and changes its velocity to $\omega'(k_1)$—and so on. In this description, we perceive the thermal energy of the medium as a cloud of quasiparticles—phonons—traveling in this way. The dynamics of this cloud at the macroscopic scale gives us the heat equation.

We recall the classical heat (diffusion) equation:

$$\partial_t u(t,x) = c\,\Delta_x u(t,x), \tag{8}$$

where c is a positive constant, t, x are time and space coordinates, and Δ_x is the Laplacian operator acting on x. Microscopic mathematical models try to capture the fact that observed heat propagation does not always satisfy the above equation. The phenomenon of anomalous, super-diffusive heat flow is observed numerically and experimentally in one dimensional nanomaterials. It is described by superdiffusion equation

$$\partial_t u(t,x) = -c\big(-\Delta_x\big)^{\alpha/2} u(t,x) \tag{9}$$

where $0 < \alpha < 2$ is a parameter and $-\big(-\Delta_x\big)^{\alpha/2}$ is the fractional Laplacian. In recent years, many articles have been published devoted to understanding such unusual heat propagation.

1.2. Classical and Anomalous Heat Transport

As we indicated above, the Boltzmann Equation (7) is related to a stochastic process describing the motion of a phonon traveling with the velocity $\omega'(k)$, wherein its state k, and consequently its velocity, changes randomly at random time instants. More specifically, the scattering operator $\mathcal{L}$ has the form

$$\mathcal{L}f(k) = \int_{-1/2}^{1/2} R(k,k')[f(k') - f(k)]dk'.$$

The function $R(k,k')$, called *collision kernel*, determines the probability that a phonon in a state k will go to a state k', and the average time the particle stays in any state k. We emphasize the following facts:

(a) The shape of the kernel, hence the parameters of phonon trajectories, emerges from the stochastic perturbation put into microscopic dynamics (6);
(b) The shape of the kernel determines type of heat transport at the macroscopic scale.

We elaborate point (b): if the trajectories of phonons are asymptotically diffusive, then the heat transport satisfies classical heat Equation (8). However, we can deal with models in which some normal modes weakly interact with other modes, which has an effect on a macroscopic scale. A phonon that has fallen into such a normal mode travels in ballistic motion for a relatively long time before it falls out of it and its trajectory becomes diffusive again, and these ballistic parts of the trajectory make heat transfer anomalously fast.

In the model introduced in [12,19] the stochastic perturbation of microscopic dynamics is spatially and temporally uncorrelated—it is a Gaussian noise, i.e., ζ_x, $x \in \mathbb{Z}$ of (6) is a family of independent Brownian motions. This model reproduces anomalous heat transport in the one dimensional chain, which agrees with observations. The collision kernel obtained for this one-dimensional model is

$$R(k,k') = \frac{1}{2}\big(R_+(k,k') + R_-(k,k')\big) \tag{10}$$

where

$$R_\pm(k,k') := 16\sin^2(\pi k)\sin^2(\pi k')\sin^2(\pi(k \mp k')). \tag{11}$$

The heat transport related to the model satisfies Equation (9) with $\alpha = 3/2$. Mathematically, this value of α results from the fact that the function $R(k)$, defined as

$$R(k) = \int R(k,k')dk',$$

is asymptotically similar to k^2 as $k \to 0$. The model was later modified in [17] by replacing Gaussian noise with a Ornstein–Uhlenbeck (OU) type random field, which is space-time stationary Gaussian, Markovian and self-correlated. In the Ornstein–Uhlenbeck process, the temporal and spatial correlations are determined by two functions, denoted by $\sigma(k)$

and $\gamma(k)$, wherein $\gamma(k)$ determines the length of time correlations for the normal mode k. The collision kernel depends on γ and σ. It reads

$$R(k,k') := \frac{2\sigma(k+k')\gamma(k+k')R_+(k,k')}{\gamma^2(k+k') + \left[\omega(k') + \omega(k)\right]^2} + \frac{2\sigma(k-k')\gamma(k-k')R_-(k,k')}{\gamma^2(k-k') + \left[\omega(k') - \omega(k)\right]^2}, \tag{12}$$

where R_+ and R_- are given by (11). In this case, it still holds $R(k) = \int R(k,k')dk' \sim k^2$ as $k \to 0$, and the only predictable outcome on the macroscopic scale is superdiffusion with $\alpha = 3/2$ as in the model with Gaussian noise. On the other hand, one-dimensional numerical models show different rates of thermal conductivity divergence [2,26]. Therefore, it is important to look for such models that lead to a superdiffusion equation with fractional Laplacian of order $\alpha/2$ different from $3/4$ ($\alpha \neq 3/2$). If γ is separated from zero, the only predictable outcome is superdiffusion with $\alpha = 3/2$. Analyzing the formula (12), it can be predicted that desirable results can be obtained at the macroscopic scale if we allow the time correlations to become indefinitely long, i.e., if

$$\gamma(k) \to 0 \text{ for normal modes } k \to 0. \tag{13}$$

However in [17], this assumption is not possible for mathematical reasons: γ must be separated from zero because of the way the process is constructed. Our approach based on the innovative construction of the OU process via the transition probability function opens the possibility of meeting the assumption (13). To substantiate our hypothesis, we employ simulations, the details of which are elaborated upon in Section 3.

2. Construction of Ornstein–Uhlenbeck Process

2.1. Mathematical Preliminaries

Let H be a separable Hilbert space over the real field with the inner product $\langle \cdot, \cdot \rangle_H$ and the Borel σ-algebra $\mathcal{B}(H)$. By $L(H)$ we denote the space of all bounded linear operators of H into itself, and by $L^+_{tc}(H)$ the space of all $Q \in L(H)$ which are

- *Symmetric*, i.e., $\langle Qu, v\rangle_H = \langle u, Qv\rangle_H$, $u, v \in H$;
- *Positive*, i.e., $\langle Qu, u\rangle_H \geq 0$ for all $u \in H$;
- *Of trace-class*, i.e., satisfying

$$\mathrm{tr}(Q) := \textstyle\sum_n \langle Qe_n, e_n\rangle_H < \infty$$

for some (and hence every) complete orthonormal system $\{e_n\}$ of H.

Given arbitrary $Q \in L^+_{tc}(H)$ and $m \in H$, the Gausian probability measure $\mu_{m,Q}$ on $(H, \mathcal{B}(H))$ with the mean m and the covariance Q is defined as the measure with characteristic function

$$\widehat{\mu_{m,Q}}(u) = \exp\left\{-i\langle m, u\rangle_H - \frac{1}{2}\langle Qu, u\rangle_H\right\}, \; u \in H.$$

In the case of m being zero, we will write μ_Q instead of $\mu_{0,Q}$ and call the measure centered Gaussian. Assume that $G \in L^*_{tc}(H)$, and $\{T_t : t \geq 0\} \subset L(H)$ is a strongly continuous semigroup, wherein T_t are symmetric, commute with G and for some $M > 0$, $\gamma > 0$, the following estimate holds

$$\|T_t\| \leq Me^{-\gamma t}, \quad t \geq 0.$$

Denote $R_t := I - T_{2t}$, $t \geq 0$. Then, $R_tG \in L^+_{tc}(H)$ and the formula

$$P_tf(x) = \int_H f(T_tx + y)\mu_{R_tG}(dy), \tag{14}$$

$x \in H$, $t \geq 0$ defines a Markov semigroup of operators P_t, $t \geq 0$ on the space $B(H)$ of Borel bounded functions equipped with supremum norm, called the *Ornstein–Uhlenbeck semigroup* ([27], p. 115). In particular, the formula

$$p(t, x, A) = P_t \mathbb{1}_A(x), \tag{15}$$

$(t, x, A) \in [0, \infty) \times H \times \mathcal{B}(H)$, defines the transition probability function for a family of Markov processes in H related to the semigroup $\{P_t\}$ (cf. [28]). For any probability measure η on $\mathcal{B}(H)$, there exists a Markov process $\zeta_\eta = \{\zeta_\eta(t) : t \geq 0\}$ with transition probability (15), and with the initial distribution, the law of $\zeta_\eta(0)$, being η. Measure μ_G is the unique invariant probability measure for the semigroup (14). For every $p \geq 1$, the semigroup extends uniquely to the Markov semigroup on $L_p(\mu_G)$ (see Theorem 8.20 and Proposition 8.21 in [27]). If the initial law is the stationary measure, then the covariance of the process reads

$$\mathbb{E}\langle \zeta_G(t), u\rangle \langle \zeta_G(t+s), v\rangle = \langle T_s G u, v\rangle.$$

The OU process we investigate below is a coarser case of Gaussian, Markovian, and time-homogeneous random field $\xi_t = \{(\xi_t)_y : y \in \mathbb{Z}\}, t \geq 0$. It exists in a Hilbert space, but has no transition probability function in the strict sense and no Markov semigroup on a space of functions defined pointwise. In [17], it is defined by the covariance function

$$\mathbb{E}(\xi_t)_y (\xi_{t+s})_z = \int_{-1/2}^{1/2} \mathrm{e}^{-\gamma(k)s} \mathrm{e}^{-2\pi i k(y-z)} \sigma(k) dk, \tag{16}$$

cf. [29] p. 37. Functions $\gamma(\cdot)$ and $\sigma(\cdot)$ are positive, continuous, and even. It follows that ξ is Markovian and lives in a weighed l_2 space H. Let us denote by π the law of ξ_0 on $\mathcal{B}(H)$. The Ornstein–Uhlenbeck semigroup $\{P_t : t \geq 0\}$ acting on $L_2(\pi)$ related to ξ can be constructed with use of the Wiener chaos decomposition—details can be read in [30]. In what follows, we obtain it by the formula analogous to (14). Either way, $P_t f(x)$ cannot be meaningfully determined for every $x \in H$, and our semigroup still exists in $L_2(\pi)$, but we obtain the equality $P_t P_s f(x) = P_{t+s} f(x)$ almost surely in x with respect to a family of reference measures. It can also be established, for all $t, s \geq 0$, pointwise on a dense linear subspaces of H invariant in mappings playing role of T_t in formula (14). As a result, the function $P(t, x, A) := P_t \mathbb{1}_A(x)$, although not being full blown transition probability (not well defined for every $x \in H$), can be used to construct non-stationary Markov processes related to the Ornstein–Uhlenbeck semigroup $\{P_t : t \geq 0\}$ for a class of initial distributions. It suffices that candidate for the initial law is absolutely continuous with respect to the stationary measure, but starting from a single point x, is also possible with additional assumptions on x.

2.2. Push-Forward Mappings and Related Abstract Wiener Measures

Assume that B is a real separable Banach space and H is a real separable Hilbert space continuously and densely embedded in B, i.e., there exists a continuous mapping $\iota : H \to B$, such that $\iota(H)$ is dense in B. Denote by ι^* the dual mapping of ι, $\iota^* : B^* \to H^*$. If μ is a Gaussian measure on B with characteristic function

$$\widehat{\mu}(u) = \exp\left\{-\frac{\|\iota^* u\|^2_{H^*}}{2}\right\}, \quad u \in B^*,$$

then the triple (B, H, μ) is called the *abstract Wiener space* (cf. [31]).

Let us fix a sequence $\lambda = \{\lambda_z, z \in \mathbb{Z}\}$ of positive numbers satisfying $\sum_{z\in\mathbb{Z}} \lambda_z < \infty$. We introduce the following Hilbert spaces over the real field:

- H_λ—the space of sequences $x = \{x_z : z \in \mathbb{Z}\}$ with finite norm

$$\|x\|_\lambda = \Big(\sum_{z\in\mathbb{Z}} \lambda_z (x_z)^2 \Big)^{1/2}$$

and the inner product

$$\langle x, y\rangle_\lambda = \sum_{z\in\mathbb{Z}} \lambda_z x_z y_z;$$

- H_λ^*—the dual of H_λ represented by sequences $u = \{u_z : z \in \mathbb{Z}\}$ with the inner product and norm defined by

$$\langle u, v\rangle_{\lambda^*} = \sum_{z\in\mathbb{Z}} \lambda_z^{-1} u_z v_z, \quad \|u\|_{\lambda^*}^2 = \sum_{z\in\mathbb{Z}} \lambda_z^{-1} (u_z)^2.$$

The dual relation between $x \in H_\lambda$ and $u \in H_\lambda^*$ is given by

$$(x, u) = \sum_{z\in\mathbb{Z}} x_z u_z.$$

We denote by $\mathbb{T}$ the interval $[-1/2, 1/2]$ with topology of a circle, i.e., with endpoints $-1/2$ and $1/2$ identified, and we define the following set of real functions

$$\mathcal{E} := C(\mathbb{T}) \cap \big\{F(\cdot) : \forall_{k\in\mathbb{T}}\, F(k) > 0 \wedge F(k) = F(-k)\big\},$$

where $C(\mathbb{T})$ is the space of all real continuous functions on $\mathbb{T}$. By $\overline{u}$, we denote complex conjugation of complex number u. For fixed $F(\cdot) \in \mathcal{E}$, the space of functions

$$L_2^e(F) := \Big\{ g(\cdot) : g(-k) = \overline{g(k)},\ k \in \mathbb{T}, \\ \text{and } \int_{\mathbb{T}} |g(k)|^2 F(k) dk < \infty \Big\}$$

is Hilbert space with the inner product

$$\langle f, g\rangle_F = \int_{\mathbb{T}} f(k) g(-k) F(k) dk.$$

The discrete Fourier transform $\widehat{u} = \mathcal{F}u$ of a sequence $u = \{u_z : z \in \mathbb{Z}\} \in l_1(\mathbb{Z})$ is defined by the formula

$$\widehat{u}(k) = \sum_{z\in\mathbb{Z}} u_z \mathrm{e}^{-2\pi i k z}, \quad k \in \mathbb{T}, \tag{17}$$

and, for $u \in l_2(\mathbb{Z})$, it is defined as the extension of (17) to the isometry between $l_2(\mathbb{Z})$ and $L_2^e(\mathbb{1})$, here $\mathbb{1} \equiv 1$ on $\mathbb{T}$. The inverse Fourier transform $\mathcal{F}^{-1}f$ of a function f integrable on $\mathbb{T}$ will be denoted by $\tilde{f}$:

$$\tilde{f}_z = \int_{\mathbb{T}} \mathrm{e}^{2\pi i k z} f(k) dk, \quad z \in \mathbb{Z}.$$

All norms $\|\cdot\|_F$, $F \in \mathcal{E}$, are equivalent. The image of $L_2^e(F)$ under the inverse Fourier transform $\mathcal{F}^{-1}$ is $l_2(\mathbb{Z})$, which is dense in H_λ. Any $L_2^e(F)$ generates a centered Gaussian measure μ_F on H_λ with the characteristic function

$$\widehat{\mu_F}(u) = \exp\left\{ -\frac{\|\widehat{u}\|_F^2}{2} \right\}, \quad u \in H_\lambda^*.$$

The covariance of μ_F reads

$$\int_{H_\lambda} (x, u)(x, v) \mu_F(dx) = \langle \widehat{u}, \widehat{v}\rangle_F, \quad u, v \in H_\lambda^*.$$

For any measurable mapping $\mathcal{J} : H_\lambda \to H_\lambda$ and probability measure μ on $\mathcal{B}(H_\lambda)$, we denote by $\mathcal{J}\mu$ the push-forward measure of μ under $\mathcal{J}$ defined by

$$\mathcal{J}\mu(A) := \mu\{x : \mathcal{J}x \in A\}, \quad A \in \mathcal{B}(H_\lambda).$$

By $\delta_y = \{(\delta_y)_z : z \in \mathbb{Z}\}$ we denote the Kronecker delta: $(\delta_y)_y = 1$ and $(\delta_y)_z = 0$ if $z \neq y$.

Lemma 1. *Let $X = \{X_z : z \in \mathbb{Z}\}$ be a sequence of independent identically distributed (i.i.d.) standard Gaussian random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We claim the following.*

(1) For every $u \in l_2(\mathbb{Z})$ and $F \in \mathcal{E}$, the series $\sum_{z\in\mathbb{Z}} X_z(J_F u)_z$, where

$$(J_F u)_z := \int_{\mathbb{T}} \mathrm{e}^{2\pi i k z} \widehat{u}(k) F(k) dk,$$

is convergent in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ and almost surely (a.s.) on Ω.

(2) Let us denote by $\mathcal{J}_F u(X)$, for $u \in l_2(\mathbb{Z})$, a real-valued random variable almost surely given by convergent series of point (1)

$$\mathcal{J}_F u(X) \overset{a.s.}{=} \sum_{z\in\mathbb{Z}} X_z (J_F u)_z. \tag{18}$$

The sequence $\{\mathcal{J}_F \delta_z(X) : z \in \mathbb{Z}\}$, as the $\mathbb{R}^{\mathbb{Z}}$-valued function of $\omega \in \Omega$, is almost surely H_λ-valued. If $\tilde{\mathcal{J}}_F(X)$ is a H_λ-valued random variable satisfying

$$\tilde{\mathcal{J}}_F(X) \overset{a.s.}{=} \{\mathcal{J}_F \delta_z(X) : z \in \mathbb{Z}\},$$

then the distribution of $\tilde{\mathcal{J}}_F(X)$ is μ_{F^2}.

Assume that ξ_G, $G \in \mathcal{E}$, is a H_λ-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with the distribution μ_G.

(3) If $u \in l_2(\mathbb{Z})$, $F \in \mathcal{E}$, then the series $\sum_{z\in\mathbb{Z}} (\xi_G)_z (J_F u)_z$ is convergent in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ and almost surely on Ω.

(4) Let a real-valued random variable $\mathcal{J}_F u(\xi_G)$, for $u \in l_2(\mathbb{Z})$, be such that

$$\mathcal{J}_F u(\xi_G) \overset{a.s.}{=} \sum_{z\in\mathbb{Z}} (\xi_G)_z (J_F u)_z. \tag{19}$$

The sequence $\{\mathcal{J}_F \delta_z(\xi_G) : z \in \mathbb{Z}\}$ almost surely belongs to H_λ. If $\tilde{\mathcal{J}}_F(\xi_G)$ is a H_λ-valued random variable satisfying

$$\tilde{\mathcal{J}}_F(\xi_G) \overset{a.s.}{=} \{\mathcal{J}_F \delta_z(\xi_G) : z \in \mathbb{Z}\},$$

then $\tilde{\mathcal{J}}_F(\xi_G)$ has the distribution $\mu_{F^2 G}$.

Proof. Let us prove (1). At first we note that, for $u, v \in l_2(\mathbb{Z})$, $\sum_{z\in\mathbb{Z}} (J_F u)_z (J_G v)_z = \langle \widehat{u}, \widehat{v} \rangle_{FG}$. This is the Parseval's identity. For $i < j$, we obtain

$$\mathbb{E} \Big| \sum_{z=i}^{j} X_z (J_F u)_z \Big|^2 = \sum_{z=i}^{j} |(J_F u)_z|^2 \leq \|\widehat{u}\|^2_{F^2} \tag{20}$$

and we conclude that the series $\sum_{z\in\mathbb{Z}} X_z (J_F u)_z$ is convergent in $L_2(\Omega)$. By the Doob martingale convergence theorem, the uniform boundedness of partial sums in (20) implies that the series converges not only in $L_2(\Omega)$ but also $\mathbb{P}$-a.s.

Now, we prove (2). A real valued random variable $\mathcal{J}_F u(X)$ satisfying (18), as L_2-limit of centered Gaussians, is centered Gaussian. By (20), $\mathbb{E}|\mathcal{J}_F u(X)|^2 = \|\widehat{u}\|^2_{F^2}$ and

$$\mathbb{E}\sum_{|z|\le N}\lambda_z|\mathcal{J}_F\delta_z(X)|^2 = \sum_{|z|\le N}\lambda_z\|\widehat{\delta}_z\|^2_{F^2} \le \sum_{z\in\mathbb{Z}}\lambda_z\int_{\mathbb{T}}F^2(k)dk < \infty.$$

It follows that the series $\sum_{z\in\mathbb{Z}}\lambda_z|\mathcal{J}_F\delta_z(X)|^2$ is finite $\mathbb{P}$-almost surely. Hence, there exist a H_λ-valued Gaussian random variable $\tilde{\mathcal{J}}_F(X)$ which equals $\{\mathcal{J}_F\delta_z(X) : z\in\mathbb{Z}\}$ on a set A of probability 1. For $u\in H^*_\lambda$, $\omega\in A$, $i<j$

$$\sum_{i\le z\le j}u_z\sum_{y\in\mathbb{Z}}X_y(\omega)(J_F\delta_z)_y = \sum_{y\in\mathbb{Z}}X_y(\omega)\int_{\mathbb{T}}\mathrm{e}^{2\pi iky}\sum_{i\le z\le j}u_z\mathrm{e}^{-2\pi ikz}F(k)dk.$$

Let $i\to-\infty$, $j\to\infty$. The left hand side is convergent to $(\tilde{\mathcal{J}}_F(X(\omega)),u)$. The right hand side, having $L_2(A,\mathcal{F},\mathbb{P})$—norm squared equal to $\int_{\mathbb{T}}\left|\sum_{i\le z\le j}u_z\mathrm{e}^{-2\pi ikz}\right|^2F^2(k)dk$, converges in L_2 to $\sum_{y\in\mathbb{Z}}X_y(J_Fu)_y$. We conclude that

$$(\tilde{\mathcal{J}}_F(X),u)\overset{a.s.}{=}\mathcal{J}_Fu(X).$$

The covariance of the centered Gaussian $\tilde{\mathcal{J}}_FX$ is

$$\mathbb{E}(\tilde{\mathcal{J}}_FX,u)(\tilde{\mathcal{J}}_FX,v) = \langle\widehat{u},\widehat{v}\rangle_{F^2},\ u,v\in H^*_\lambda,$$

so it has the law μ_{F^2}.

Let us now show (3) and (4). We calculate

$$\mathbb{E}\Big|\sum_{i\le z\le j}(\xi_G)_z(J_Fu)_z\Big|^2 = \int_{\mathbb{T}}\Big|\sum_{i\le z\le j}\mathrm{e}^{-2\pi ikz}(J_Fu)_z\Big|^2G(k)dk.$$

In $L^e_2(G)$, we have $\sum_{z\in\mathbb{Z}}\mathrm{e}^{-2\pi i(\cdot)z}(J_Fu)_z = \widehat{u}(\cdot)F(\cdot)$, so the series $\sum_{z\in\mathbb{Z}}(\xi_G)_z(J_Fu)_z$ is convergent in $L_2(\Omega)$, and by the Doob martingale convergence theorem it also converges almost surely. In the above considerations, we can replace ξ_G with $\tilde{\mathcal{J}}_{\sqrt{G}}X$ as both have the same distribution μ_G. Now, we consider the following almost sure equality

$$\sum_{i\le z\le j}(\tilde{\mathcal{J}}_{\sqrt{G}}X)_z(J_Fu)_z \overset{a.s.}{=} \sum_{y\in\mathbb{Z}}X_y\sum_{i\le z\le j}(J_{\sqrt{G}}\delta_z)_y(J_Fu)_z,\quad i<j. \tag{21}$$

$(J_{\sqrt{G}}\delta_z)_y = (J_{\sqrt{G}}\delta_y)_z$ holds for all $y,z\in\mathbb{Z}$. Series $\sum_{z\in\mathbb{Z}}(J_{\sqrt{G}}\delta_z)_y(J_Fu)_z$ is convergent, equals $\langle\widehat{\delta}_y,\widehat{u}\rangle_{F\sqrt{G}} = (J_{F\sqrt{G}}u)_y$, and $\sum_{y\in\mathbb{Z}}X_y(J_{F\sqrt{G}}u)_y$ belongs to $L_2(\Omega)$. We pass to the $L_2(\Omega)$-limits on both sides of (21). We obtain that

$$\mathcal{J}_Fu(\tilde{\mathcal{J}}_{\sqrt{G}}X)\overset{a.s.}{=}\sum_{z\in\mathbb{Z}}(\tilde{\mathcal{J}}_{\sqrt{G}}X)_z(J_Fu)_z\overset{a.s.}{=} \overset{a.s.}{=}\sum_{y\in\mathbb{Z}}X_y(J_{F\sqrt{G}}u)_y\overset{a.s.}{=}\mathcal{J}_{F\sqrt{G}}u(X).$$ By this,

$$\mathbb{E}\sum_{|z|\le N}\lambda_z|\mathcal{J}_F\delta_z(\xi_G)|^2=\mathbb{E}\sum_{|z|\le N}\lambda_z\left|\mathcal{J}_{F\sqrt{G}}\delta_z(X)\right|^2$$
$$=\sum_{|z|\le N}\lambda_z\|\widehat{\delta_z}\|^2_{F^2G}\le\sum_{z\in\mathbb{Z}}\lambda_z\int_{\mathbb{T}}F^2(k)G(k)dk<\infty,$$

and it follows that the series $\sum_{z\in\mathbb{Z}}\lambda_z|\mathcal{J}_F\delta_z(\xi_G)|^2$ is finite $\mathbb{P}$-almost surely. Denote by $\tilde{\mathcal{J}}_F\xi_G$ a H_λ-valued Gaussian random variable such that

$$\tilde{\mathcal{J}}_F\xi_G\overset{a.s.}{=}\{\mathcal{J}_F\delta_z(\xi_G):z\in\mathbb{Z}\}.$$

We have

$$(\tilde{\mathcal{J}}_F\xi_G,u)\overset{a.s.}{=}\sum_{z\in\mathbb{Z}}u_z\sum_{y\in\mathbb{Z}}(\xi_G)_y\int_{\mathbb{T}}\mathrm{e}^{2\pi ik(y-z)}F(k)dk$$
$$\overset{a.s.}{=}\sum_{y\in\mathbb{Z}}(\xi_G)_y(J_Fu)_y\overset{a.s.}{=}\mathcal{J}_Fu(\xi_G).$$

The covariance of $\tilde{\mathcal{J}}_F\xi_G$ is

$$\mathbb{E}(\tilde{\mathcal{J}}_F\xi_G,u)(\tilde{\mathcal{J}}_F\xi_G,v)=\langle\widehat{u},\widehat{v}\rangle_{F^2G},\ \ u,v\in H^*_\lambda.$$

□

Definition 1. *We denote by $\mathfrak{A}_F$ the set of all $x\in H_\lambda$ for which the series $\sum_{y\in\mathbb{Z}}x_y(J_F\delta_z)_y$ converges for every $z\in\mathbb{Z}$, and the sequence $\left\{\sum_{y\in\mathbb{Z}}x_y(J_F\delta_z)_y:z\in\mathbb{Z}\right\}$ belongs to H_λ.*

Remark 1. *By Lemma 1, for fixed u in H^*_λ the series $\sum_{z\in\mathbb{Z}}x_z(J_Fu)_z$ is convergent on a set of measure μ_G equal to 1. Also by the Lemma, $\mu_G(\mathfrak{A}_F)=1$ for every $G\in\mathcal{E}$.*

Definition 2. *For any $F\in\mathcal{E}$, we define mapping $\mathcal{J}_F:H_\lambda\to H_\lambda$ by*

$$(\mathcal{J}_Fx)_z:=\begin{cases}\sum\limits_{y\in\mathbb{Z}}x_y(J_F\delta_z)_y, & x\in\mathfrak{A}_F,\\ \mathbf{0}, & x\in H_\lambda\backslash\mathfrak{A}_F,\end{cases}$$

$z\in\mathbb{Z}$, here $\mathbf{0}=(\dots0,0,0\dots)$.

Setting zero on negligible set in above definition is arbitrary and made only to have $\mathcal{J}_F$ well defined on the whole of H_λ. We note that $\mathcal{J}_F$ is not linear on H_λ, but $\mathcal{J}_F|_{\mathfrak{A}_F}$ is linear, which is crucial.

By $\mu*\nu$, we denote the convolution of probability measures μ and ν, i.e., for any measurable set A

$$\mu*\nu(A)=\iint_{H^2_\lambda}\mathbb{1}_A(x+y)\mu(dx)\nu(dy).$$

Proposition 1. *For arbitrary $F,G,H\in\mathcal{E}$:*

(a) The measure μ_{FG} is the push-forward measure of μ_F under $\mathcal{J}_{\sqrt{G}}$:

$$\mu_{FG}(A)=\mu_F\{x:\mathcal{J}_{\sqrt{G}}x\in A\},\ A\in\mathcal{B}(H_\lambda);$$

(b) $\mathcal{J}_F\mathcal{J}_G\overset{a.s.}{=}\mathcal{J}_{FG}$ with respect to μ_H;

(c) $\mu_F*\mu_G=\mu_{F+G}$.

Proof. Proving (a), it is enough to notice that, by Lemma 1, the law of $\mathcal{J}_{\sqrt{G}}$ considered on probability space $(H_\lambda, \mathcal{B}(H_\lambda), \mu_F)$ is μ_{FG}. Now, we prove (b). By (4) of Lemma 1, both $\mathcal{J}_{FG}$ and $\mathcal{J}_F\mathcal{J}_G$ have the law $\mu_{F^2G^2H}$ when considered on $(H_\lambda, \mathcal{B}(H_\lambda), \mu_H)$ and

$$\begin{aligned}&\mu_H(x : \mathcal{J}_F\mathcal{J}_G x = 0) = \mu_H(x : \mathcal{J}_{FG}x = 0)\\ &= \mu_{F^2G^2H}(\{0\}) = 0.\end{aligned}$$

Let us denote by $\mathfrak{C}_{FG}$ the set of $x \in H_\lambda$ for which $\mathcal{J}_F\mathcal{J}_G x \neq 0$ and $\mathcal{J}_{FG}x \neq 0$. Then, $\mu_H(\mathfrak{C}_{FG}) = 1$ and for $z \in \mathbb{Z}$, $x \in \mathfrak{C}_{FG}$

$$\begin{aligned}(\mathcal{J}_{FG}x)_z &= \sum_{y\in\mathbb{Z}} x_y (J_{FG}\delta_z)_y =\\ &= \sum_{y\in\mathbb{Z}} x_y \sum_{y'\in\mathbb{Z}} (J_G\delta_y)_{y'}(J_F\delta_z)_{y'},\end{aligned}$$

$$\text{and}\quad (\mathcal{J}_F\mathcal{J}_G x)_z = \sum_{y'\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} x_y (J_G\delta_{y'})_y (J_F\delta_z)_{y'}.$$

We recall that $(J_G\delta_y)_{y'} = (J_G\delta_{y'})_y$. By (3) of Lemma 1, the series $x \mapsto \sum_{y\in\mathbb{Z}} x_y (J_G\delta_{y'})_y$ belongs to $L_2(H_\lambda, \mathcal{B}(H_\lambda), \mu_H)$ for every $y' \in \mathbb{Z}$, and so does the finite linear combination of such series

$$x \mapsto \sum_{y\in\mathbb{Z}} x_y \sum_{i\le y'\le j} (J_G\delta_y)_{y'}(J_F\delta_z)_{y'}. \tag{22}$$

As $i \to -\infty$ and $j \to \infty$, (22) converges in $L_2(H_\lambda, \mathcal{B}(H_\lambda), \mu_H)$ to $(\mathcal{J}_{FG}(\cdot))_z$. On the other hand, on $x \in \mathfrak{C}_{FG}$ the value of (22) can be rewritten as

$$\sum_{i\le y'\le j} (\mathcal{J}_G x)_{y'}(J_F\delta_z)_{y'}. \tag{23}$$

Combining $L_2(\mu_H)$-convergence of (22) to $(\mathcal{J}_{FG}(\cdot))_z$, convergence of (23) to $(\mathcal{J}_F\mathcal{J}_G x)_z$ for $x \in \mathfrak{C}_{FG}$, and equality of both on $\mathfrak{C}_{FG}$ we obtain (b). Regarding (c), it suffices to note that $\widehat{\mu_F}\widehat{\mu_G} = \widehat{\mu_{F+G}}$. □

2.3. Nonstationary OU Processes

Having push-forward mappings constructed, we can now bring back the OU semigroup and define the quasi-transition probability $P(t, x, A)$ with formulas analogous to (14) and (15). Let $\gamma(\cdot), \sigma(\cdot) \in \mathcal{E}$ and let $e_t(\cdot), h_t(\cdot) \in \mathcal{E}$ be given by

$$e_t(k) := \mathrm{e}^{-\gamma(k)t},\ h_t(k) := 1 - \mathrm{e}^{-2\gamma(k)t},$$

$k \in \mathbb{T}, t \ge 0$.

Lemma 2. *Let $f : H_\lambda \to \mathbb{R}$ be integrable with respect to μ_σ. For $t \ge 0$ let $P_t f$ be given by*

$$P_t f(x) = \int_{H_\lambda} f(\mathcal{J}_{e_t}x + y)\mu_{h_t\sigma}(dy),\ x \in H_\lambda. \tag{24}$$

Then:

(a) If $f \in L_p(\mu_\sigma)$ for some $p \ge 1$, then does $P_t f$ and $\|P_t f\|_{L_p(\mu_\sigma)} \le \|f\|_{L_p(\mu_\sigma)}$;

(b) $P_t \mathbb{1}_{H_\lambda} = \mathbb{1}_{H_\lambda}$;

(c) Given that $x \in \mathfrak{A}_{e_s} \cap \mathfrak{A}_{e_{t+s}}$, $\mathcal{J}_{e_s}x \in \mathfrak{A}_{e_t}$ and $\mathcal{J}_{e_{t+s}}x = \mathcal{J}_{e_t}\mathcal{J}_{e_s}x$, it holds

$$P_t P_s f(x) = P_{t+s} f(x).$$

In particular, $P_t P_s f = P_{t+s} f$ μ_F-a.s. for every $F \in \mathcal{E}$.

Proof. Using Jensen inequality, properties listed in Proposition 1 and identity $e_t^2 + h_t = 1$, we calculate

$$\begin{aligned}&\int_{H_\lambda} |P_t f(x)|^p \mu_\sigma(dx) \leq \\ &\iint_{H_\lambda^2} |f(\mathcal{J}_{e_t} x + y)|^p \mu_\sigma(dx) \mu_{h_t\sigma}(dy) = \\ &\int_{H_\lambda} |f(z)|^p \mu_{e_t^2\sigma} * \mu_{h_t\sigma}(dz) = \int_{H_\lambda} |f(z)|^p \mu_\sigma(dz).\end{aligned}$$

This proves (a). (b) is obvious. Under assumptions on x in (c), which hold μ_F-almost surely, $F \in \mathcal{E}$, we have $P_t P_s f(x) =$

$$\begin{aligned}&= \iint f\big(\mathcal{J}_{e_t}(\mathcal{J}_{e_s} x + y) + z\big) \mu_{h_s\sigma}(dy) \mu_{h_t\sigma}(dz) \\ &= \iint f\big(\mathcal{J}_{e_{t+s}} x + y + z\big) \mu_{e_t^2 h_s\sigma}(dy) \mu_{h_t\sigma}(dz) \\ &= \int f\big(\mathcal{J}_{e_{t+s}} x + w\big) \mu_{(e_t^2 h_s + h_t)\sigma}(dw) = P_{t+s} f(x).\end{aligned}$$

□

It follows from Lemma 2 that $\{P_t : t \geq 0\}$, given on integrable functions by (24), constitutes a Markov semigroup on every $L_p(\mu_\sigma)$, $p \geq 1$. Now, let us define

$$P(t, x, A) := P_t \mathbb{1}_A(x) = \int_{H_\lambda} \mathbb{1}_A(\,\mathcal{J}_{e_t} x + y\,) \mu_{h_t\sigma}(dy),$$

for $x \in H_\lambda$, $t \geq 0$, $A \in \mathcal{B}(H_\lambda)$. This is deficient transition probability function, because it is not properly defined for $x \notin \mathfrak{A}_{e_t}$. But the Chapman–Kolmogorov equation, which can be expressed as $P_t P_s \mathbb{1}_A(x) = P_{t+s} \mathbb{1}_A(x)$, holds almost surely in x for every μ_F, $F \in \mathcal{E}$. We can also identify dense subsets of H_λ consisting of x for which assumptions in (c) of Lemma 2 are satisfied for every $t, s \geq 0$. It follows that we can construct nonstationary Markov processes related to the semigroup $\{P_t\}$ for a class of initial distributions.

Theorem 1. *Let η be a probability measure on $\mathcal{B}(H_\lambda)$ satisfying at least one of the following two conditions:*

(i) η is absolutely continuous with respect to some measure μ_F, $F \in \mathcal{E}$;
(ii) $\eta(\mathfrak{D}) = 1$ for a set $\mathfrak{D}$ satisfying $\mathfrak{D} \subseteq \mathfrak{A}_{e_t}$ for all $t > 0$, $x \in \mathfrak{D} \Rightarrow \mathcal{J}_{e_t} x \in \mathfrak{D}$ for all $t > 0$, and $x \in \mathfrak{D} \Rightarrow \mathcal{J}_{e_t} \mathcal{J}_{e_s} x = \mathcal{J}_{e_{t+s}} x$ for all $t, s > 0$.

For arbitrary $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_n$ and $A_0, A_1, \ldots, A_n \in \mathcal{B}(H_\lambda)$ denote

$$\begin{aligned}&\mathbb{P}_{t_1, t_2, \ldots, t_n}(A_0 \times A_1 \times A_2 \times \cdots \times A_n) := \\ &\int_{A_0} P_{s_1}[\mathbb{1}_{A_1} P_{s_2}[\ldots P_{s_{n-1}}[\mathbb{1}_{A_{n-1}} P_{s_n} \mathbb{1}_{A_n}] \ldots]](x)\, \eta(dx),\end{aligned}$$

where $s_k = t_k - t_{k-1}$, $k = 1, \ldots, n$. There exists Markov process $\xi_\eta = \{\xi_\eta(t) : t \geq 0\}$ in H_λ with finite-dimensional distributions given by

$$\begin{aligned}&\mathbb{P}[\xi_\eta(0) \in A_0, \xi_\eta(t_1) \in A_1, \ldots, \xi_\eta(t_n) \in A_n] \\ &\qquad = \mathbb{P}_{t_1, \ldots, t_n}(A_0 \times A_1 \times \cdots \times A_n).\end{aligned}$$

Proof. $\mathbb{P}_{t_1, \ldots, t_n}$ defines a probability measure on H_λ^{n+1}. For $k = 2, \ldots, n-1$, denote

$$R_k := \mathbb{1}_{A_k} P_{s_{k+1}}[\mathbb{1}_{A_{k+1}} \ldots P_{s_{n-1}}[\mathbb{1}_{A_{n-1}} P_{s_n} \mathbb{1}_{A_n}] \ldots],$$

and $R_n := \mathbb{1}_{A_n}$, so for $k \in \{1, \dots, n-1\}$, we can write

$$\mathbb{P}_{t_1,\dots,t_n}[A_0 \times A_1 \times \cdots \times A_n] = \int_{A_0} P_{s_1}[\mathbb{1}_{A_1} P_{s_2}[\dots \mathbb{P}_{s_k}[\mathbb{1}_{A_k} P_{s_{k+1}} R_{k+1}] \dots]](x)\eta(dx).$$

If $A_k = H_\lambda$ for some $k \in \{1, \dots, n-1\}$ then $P_{s_k}[\mathbb{1}_{A_k} P_{s_{k+1}} R_{k+1}](y) = P_{s_k} P_{s_{k+1}} R_{k+1}(y)$, $y \in H_\lambda$, and by (c) of Lemma 2 it holds

$$\begin{aligned}\mathbb{P}_{t_1,\dots,t_n}[A_0 \cdots A_{k-1} \times H_\lambda \times A_{k+1} \cdots A_n] &= \\ = \int_{A_0} P_{s_1}[\mathbb{1}_{A_1} P_{s_2}[\dots P_{s_k+s_{k+1}}[R_{k+1}] \dots]](x)\eta(dx) &\\ = \mathbb{P}_{t_1,\dots t_{k-1},t_{k+1},\dots,t_n}[A_0 \cdots A_{k-1} \times A_{k+1} \cdots A_n]. &\end{aligned}$$

Also, trivially $\mathbb{P}_{t_1,\dots,t_n}[A_0 \times \cdots \times A_{n-1} \times H_\lambda]$ equals $\mathbb{P}_{t_1,\dots,t_{n-1}}[A_0 \times \cdots \times A_{n-1}]$. Thus, the consistency conditions of the Kolmogorov extension theorem (cf. [32]) are satisfied for the family of measures $\mathbb{P}_{t_1,\dots,t_n}$, and the process ξ_η exists. □

Remark 2. *Examples of a set $\mathfrak{D}$ satisfying assumption (ii) of Theorem 1 are:*

- ◇ *Space $l_2(\mathbb{Z})$. If $x \in l_2(\mathbb{Z})$, then $\mathcal{J}_F x$ is the inverse Fourier transform of $\widehat{x}F \in L_2^e(\mathbb{1})$.*
- ◇ *If $\gamma(\cdot)$ is of bounded variation, then also is $\mathrm{e}^{-\gamma(\cdot)t}$ for every $t > 0$. Now, let $N \in \mathbb{N}$ and assume that $F(\cdot) \in \mathcal{E}$ is of bounded variation. It follows that the series $S(N, F, y, w) :=$*

$$\frac{1}{N} \sum_{z \in \mathbb{Z}} \int_{-N/2}^{N/2} \mathrm{e}^{-2\pi i z l} \mathrm{e}^{2\pi i (y-w) l/N} F(l/N) dl$$

is convergent ([33], p. 156). *If $x = \{x_z : z \in \mathbb{Z}\}$ is periodic with period N, then*

$$\begin{aligned}\sum_{w=0}^{N-1} x_w S(N, F, y, w) &= \\ = \sum_{z \in \mathbb{Z}} \sum_{w=0}^{N-1} x_{w+Nz} \int_{\mathbb{T}} \mathrm{e}^{2\pi i k (y-w-Nz)} F(k) dk &\\ = \sum_{z \in \mathbb{Z}} x_z \int_{\mathbb{T}} \mathrm{e}^{2\pi i k (y-z)} F(k) dk, &\end{aligned}$$

and this sum, as a function of $y \in \mathbb{Z}$, is periodic with period N as well. In particular it belongs to H_λ, so $x \in \mathfrak{A}_F$. We conclude that if $\gamma(\cdot)$ is of bounded variation, then the set of all periodic sequences satisfies assumptions made on $\mathfrak{D}$.

- ◇ *Assume that the inverse Fourier transform of $\gamma(\cdot)$ belongs to $l_1(\mathbb{Z})$. Then, γ belongs to the* Wiener algebra *$A(\mathbb{T})$ defined as the linear normed space of complex functions on $\mathbb{T}$ with the norm $\|F\|_{A(\mathbb{T})} := \sum_{z \in \mathbb{Z}} |(\tilde{F})_z|$. $A(\mathbb{T})$ is a Banach algebra* ([34], p. 32), *so $e_t(\cdot) = \mathrm{e}^{-\gamma(\cdot)t}$ belongs to $A(\mathbb{T})$ for every $t > 0$. If $x \in H_\lambda$ is bounded, then*

$$\Big| \sum_{y \in \mathbb{Z}} x_y (J_{e_t} \delta_z)_y \Big| \le \max_{y \in \mathbb{Z}} |x_y| \cdot \|e_t\|_{A(\mathbb{T})}, \quad z \in \mathbb{Z},$$

so $x \in \mathfrak{A}_{e_t}$ and $\mathcal{J}_{e_t} x$ is bounded. Hence, the set of all bounded sequences satisfies assumptions on $\mathfrak{D}$. By theorem of Bernstein ([34], p. 33), *a sufficient condition for γ to be in $A(\mathbb{T})$ is*

$$\sup_{k \in \mathbb{T},\, h \neq 0} \frac{|\gamma(k+h) - \gamma(k)|}{|h|^\alpha} < \infty$$

for some $\alpha > 1/2$.

If $(\mathcal{J}_{e_t}x,u)(\mathcal{J}_{e_{t+s}}x,v)$, $u,v\in H_\lambda^*$, is η-integrable in x, then $\mathbb{E}(\xi_\eta(t),u)(\xi_\eta(t+s),v)=$

$$\begin{aligned}&\iiint(y,u)(z,v)P(s,y,dz)P(t,x,dy)\eta(dx)=\\&\int_{H_\lambda}(\mathcal{J}_{e_t}x,u)(\mathcal{J}_{e_{t+s}}x,v)\eta(dx)\\&-\int_{\mathbb{T}}\mathrm{e}^{-\gamma(k)(2t+s)}\widehat{u}(k)\widehat{v}(-k)\sigma(k)dk\\&+\int_{\mathbb{T}}\mathrm{e}^{-\gamma(k)s}\widehat{u}(k)\widehat{v}(-k)\sigma(k)dk.\end{aligned}$$

2.4. Equation of Heat Energy Transport on The Microscopic Scale

In the previous section, we constructed a Ornstein–Uhlenbeck stochastic process. The covariance function of this process is given by:

$$\mathbb{E}\xi_y(t)\xi_z(s)=\int_{\mathbb{T}}\mathrm{e}^{-2\pi ik(y-z)}\mathrm{e}^{-\gamma(k)|t-s|}\sigma(k)dk.$$

With this process, we disturb the dynamics in the thermal energy transport model on a microscopic scale, replacing the Gaussian noise in the model introduced in [12] with the OU process. Now, the thermal energy transport equations at the microscopic scale have the following form:

$$\begin{cases}\dfrac{d\mathfrak{q}_y(t)}{dt}=\mathfrak{p}_y(t)\\[2ex]\dfrac{d\mathfrak{p}_y(t)}{dt}=-(\alpha*\mathfrak{q}(t))_y+\sqrt{\epsilon}\displaystyle\sum_{k=-1,0,1}(Y_{y+k}\mathfrak{p}_y(t))\,\xi_{y+k}(t),\quad y\in\mathbb{Z}.\end{cases}\tag{25}$$

We will use Boltzmann equations for simulation. The first of them are those obtained in [13], where the vibrations were disturbed by Gaussian noise. The second one can be obtained by rescaling Equation (25) to the mesoscopic scale, and its form is the same as in [17].

3. Numerical Simulations

We present numerical simulations that support our predictions. Consider the Boltzmann linear equation at the mesoscopic scale with an initial condition

$$\begin{aligned}&\partial_t u(t,x,k)+\omega'(k)\partial_x u(t,x,k)=\mathcal{L}u(t,x,k),\\&u(0,x,k)=u_0(x,k).\end{aligned}\tag{26}$$

We assume that the initial condition is constant on a long interval of the space coordinate x (energy is uniformly distributed on the one-dimensional rod). With this assumption $\partial_x u(t,x,k)=0$. Thus, we can omit dependence on x and instead of $u(t,x,k)$, we can write $u(t,k)$, which is the distribution of energy over normal modes at time t. The initial value problem (26) reduces to

$$\begin{aligned}&\partial_t u(t,k)=\mathcal{L}u(t,k),\\&u(0,k)=u_0(k).\end{aligned}\tag{27}$$

We impose the initial condition $u_0(k)$ highly unevenly distributed over normal modes (see the graph for $t=0$ on Figure 5), and we solve (27) numerically. For the model with Gaussian noise, i.e., with collision kernel (10) we obtain the following picture of $u(t,k)$, at chosen time instants, seen on Figure 5.

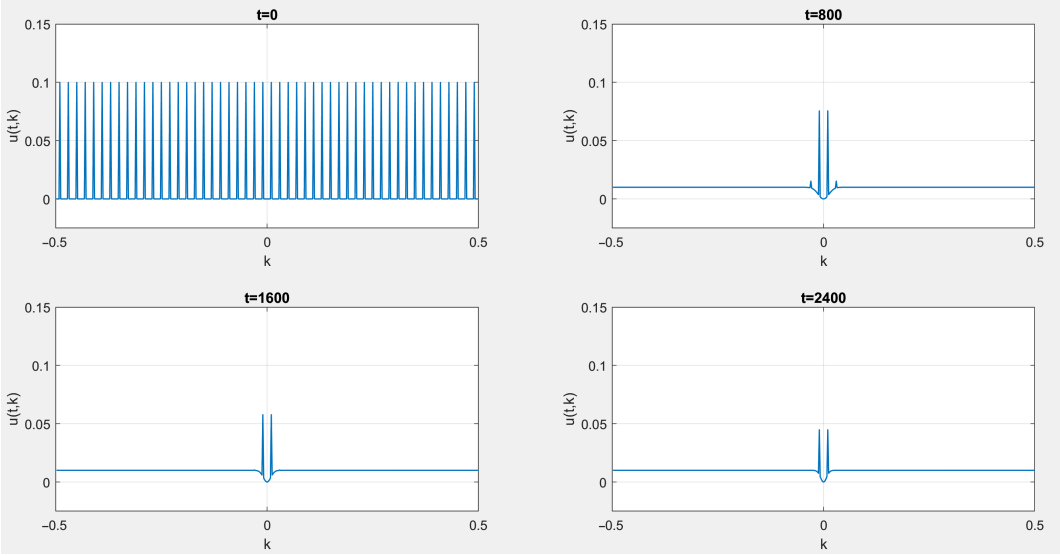

Figure 5. Evolution of energy distribution on normal modes under the regime of linear Boltzmann equation related to Gaussian noise perturbation at the microscopic scale. The initial distribution does not depend on space variable x.

Initially, the energy is set concentrated on modes $k = \ldots -0.03, -0.01, 0.01, 0.03, 0.05\ldots$ After 2400 time steps, we observe that energy is uniformly distributed over all modes outside a small vicinity of 0; however, the spikes closest to $k = 0$ are still visible. This corresponds to the fact that long-wave energy is not susceptible to dissipation, which results in superdiffusive heat flow. Now, let us consider the scattering rates obtained from linear dynamics with Ornstein–Uhlenbeck perturbation, i.e., determined by the kernel (12). We made both models comparable by normalizing both kernels, i.e., by dividing each $R(k, k')$ by $\sum_k \sum_{k'} R(k, k')$. If the scattering rates are given by (12) with γ and σ being a positive constant functions (and therefore separated from zero), the plots of energy distribution are as seen on Figure 6.

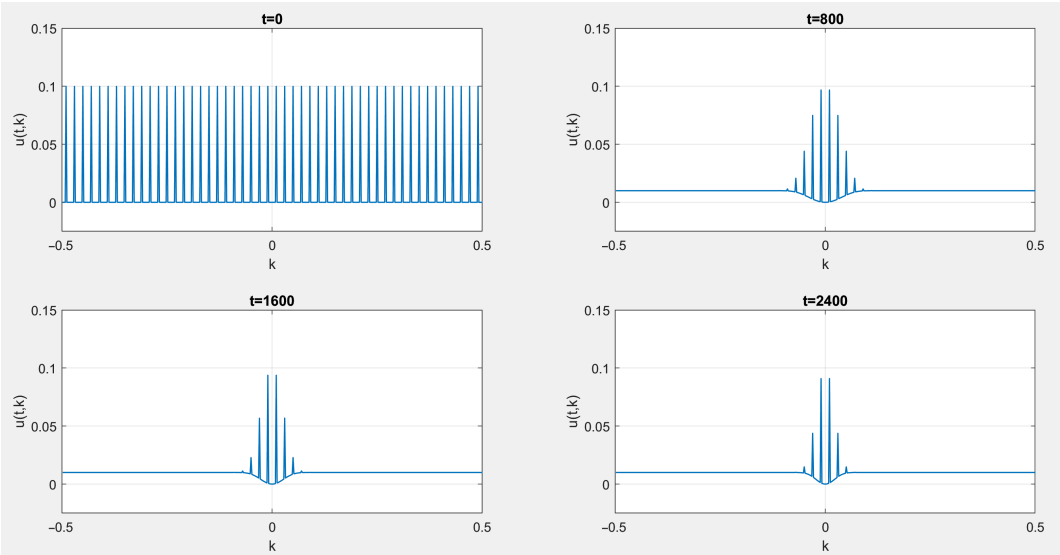

Figure 6. Evolution of energy distribution on normal modes under the regime of linear Boltzmann equation related to Ornstein–Uhlenbeck perturbation at the microscopic scale, with $\gamma \equiv \frac{1}{4}$ and $\sigma \equiv 1$. The initial distribution does not depend on space variable x.

The tendency of long waves to conserve energy has strengthened. However, as we indicated in the introductory chapters, the superdiffusion coefficient $\alpha = 3/2$ still emerges

from such dynamics. The way to change the situation is to let $\gamma(k) \to 0$ when $k \to 0$. Letting $\gamma(k) = \frac{1}{5}|\sin(2\pi k)|^{1/4}$, $k \in [-1/2, 1/2]$, we observe a radical change in the picture of the evolution of the energy distribution with respect to normal mode, see Figure 7.

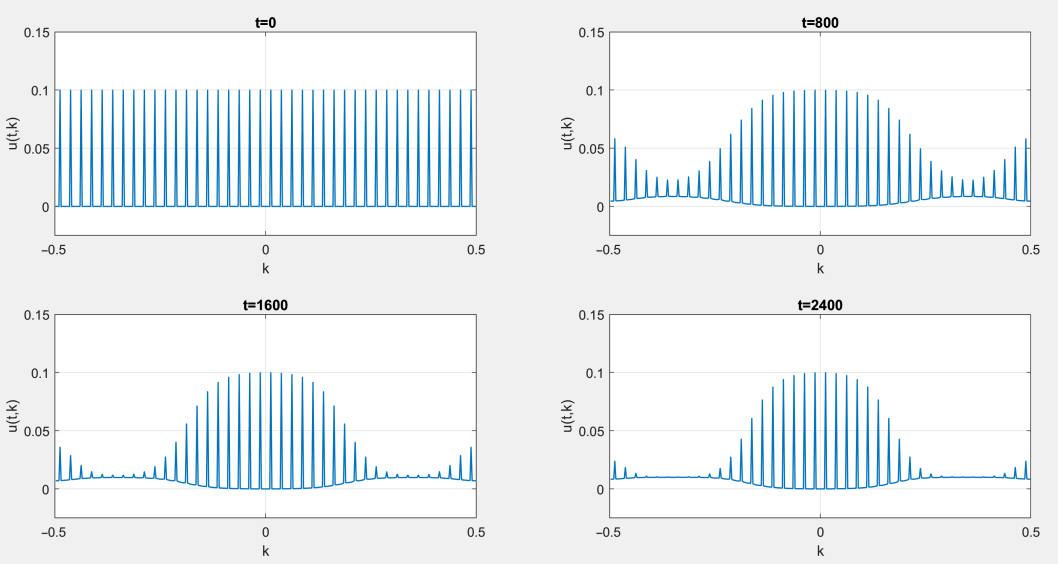

Figure 7. Evolution of energy distribution on normal modes under the regime of linear Boltzmann equation related to Ornstein–Uhlenbeck perturbation at the microscopic scale, with $\gamma(k) = \frac{1}{5}(\sin(2\pi k))^{1/4}$ and $\sigma \equiv 1$. The initial distribution does not depend on space variable x.

The tendency of long waves to conserve energy increased significantly, and interestingly, we also see that the modes related to short wavelengths—in neighborhood of $1/2$ $(-1/2)$—became conservative. The picture of energy evolution when we admit indefinitely long time correlation is so different from the first two simulations for superdiffusion with the α parameter equal to $3/2$ that it may confirm the hypothesis about the assumption leading to the superdiffusion model with α different from $3/2$. The final confirmation should be made by mathematical considerations discussed in the next section.

4. Discussion and Future Work

The Ornstein–Uhlenbeck random field $\xi = (\xi_x(t))$, in contrast to the Gaussian noise, has non-zero correlations over time and thus appears to be more natural for modeling a physical process which has some inertia. In [17], this Ornstein–Uhlenbeck process was defined as stationary Gaussian random field with postulated covariance function (16). The appropriate theorem guarantees its existence, see Theorem 8.2 in [30]. It has properties of a Markov process; however, to our knowledge, there is no transition probability function for it that meets the definition given in [35] p. 156. In the present work, we proceed a different path of construction of ξ. We construct the function $\mathbb{P}(t, x, A)$, which has a properties of transition probability (let us call it quasi-transition probability function) that allows constructing Markov processes whose stationary representative is the Ornstein–Uhlenbeck process introduced in [17]. The construction of the function $\mathbb{P}$ was the most difficult task. The quasi-transition probability function allows for a different approach to the analysis of equations with such a disturbance. With this approach, we can think of weakening the assumptions on temporal correlations for such a process, which may be of great importance for the heat flow model. The collision kernel (12) depends on the time correlation $\gamma(k)$ of OU perturbation, and we would like to meet the condition (13). This can be fulfilled by making the function $\gamma(k)$ dependent on the scaling parameter $\epsilon > 0$, which tends to zero while approaching the macroscopic scale. Namely, we define the function $\gamma(k) = \epsilon + \tilde{\gamma}(k)$, where $\tilde{\gamma}(k)$ is assumed to satisfy (13). While the parameter ϵ tends to zero, $\gamma(k) \to \tilde{\gamma}(k)$ and we asymptotically approach the situation when time correlations are

indefinitely long. The presented methodology is a very extensive task that goes beyond the scope of current project. Therefore, it will be further pursued in our subsequent research. Consequently, our objective is to derive a superdiffusive heat equation with an α coefficient that differs from 3/2.

Author Contributions: Conceptualization, Z.A.Ł.; validation, Z.A.Ł. and Ł.S.; formal analysis, Ł.S.; writing—review and editing, Z.A.Ł. and Ł.S.; visualization, Ł.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Lublin University of Technology, grant numbers FN-17, FD-20/IT-3/011 and FD-20/IT-3/030. The APC was funded by Lublin University of Technology.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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