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An Analytical Model for the Regeneration of Wind after Exiting a Wind Farm

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Abstract: The simplest model for an atmospheric boundary layer assumes a uniform steady wind over a certain depth, of order 1 km, with the forces of friction, pressure gradient and Coriolis in balance. A linear model is here employed for the adjustment of wind to this equilibrium, as the wake of a very wide wind farm. A length scale is predicted for the exponential adjustment to equilibrium. Calculation of this length scale is aided by knowledge of the angle for which the wind would normally cross the isobars in environmental conditions in the wake.

Keywords: wind energy; atmospheric boundary layer; wind turbine wake

1. Introduction

If wind energy is to be a significant contribution to world energy, the weather impact will be on the scale of continents [1–3]. Not only will individual wind farms be on the scale of synoptic weather systems, so will the wake. Here we define the wake as a momentum deficit over the entire depth of the atmospheric boundary layer. The wake exponentially diminishes with distance downstream of a giant wind farm, and presents a diminished resource for further downstream wind energy extraction.

This research departs from [4], which is based on a claim that "the missing momentum in the wake of an indefinitely broad wind park can only be replenished from above". In this work, we recognize the role of the large-scale pressure gradient force. The wake is defined from the momentum integrated through the depth of the boundary layer. The details of mixing of a wake within the boundary layer is unimportant, in contrast to [5]. However, even with adding a pressure gradient force that regenerates wind, we find a wake length much longer than [4] where "a wake length of 4 km for rough land surfaces and a wake length of about 18 km for smooth sea surfaces are found". Here we find a longer wake length because the momentum deficit of the entire boundary layer is designated as "wake". This designation is appropriate because the momentum of the entire boundary layer would be the resource for another downstream giant wind farm.

We use a model for a well-mixed boundary layer, widely taught in meteorological education, and easily revealed in an internet search for "geostrophic friction angle isobars". A balance of three elementary forces allows for a prediction of the angle θ for the wind vector deflecting from the isobars. We extend the model, with linearized differential equations, to predict the adjustment of the wind exiting a wind farm, and encountering lower drag conditions. We show here that simple knowledge of the climate for θ that would exist in the exit conditions (whether or not a wind farm exists upstream) provides key knowledge to find the theoretical wake length:

$$\lambda = \frac{u_0}{2f\tan\theta} \tag{1}$$

where f is the Coriolis parameter and u_0 is the boundary layer windspeed. The properties of the drag force and pressure gradient force are implicitly included in Equation (1), though they make no explicit appearance.

2. The Boundary Layer Model

The horizontal wind vector in the boundary layer is (u, v) and the height of the boundary layer is h. At the height h there is a inversion with an abrupt density decrease. The symbol g here denotes *reduced gravity*, the familiar acceleration due to gravity times the fractional density decrease. The model highlighted in the introduction is here extended to allow for disequilibrium and also a pressure gradient from the slope of h:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial P}{\partial x} - g\frac{\partial h}{\partial x} + fv + \frac{1}{h}\left(F_{x,s} - F_{x,h}\right) \quad (2)$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial P}{\partial y} - g\frac{\partial h}{\partial y} - fu + \frac{1}{h}\left(F_{y,s} - F_{y,h}\right) \quad (3)$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -h\frac{\partial u}{\partial x} - h\frac{\partial v}{\partial y} \quad (4)$$

Here $F_{x,s}$ and $F_{y,s}$ are stresses at the surface, $F_{x,h}$ and $F_{y,h}$ are stresses at $z = h$. P is a synoptic scale pressure divided by a constant density, a pressure independent of the slope of h.

Keeping the model simple, we neglect pressure perturbations that occur due to a wave response above the inversion. A review of how to include that pressure effect is in [6]. That additional term is included in the model for a wind farm application of [7,8], but not here.

As usual, the stress terms are modeled as opposing the velocity vector, here with a linear coefficient γ:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial P}{\partial x} - g\frac{\partial h}{\partial x} + fv - \frac{1}{h}\gamma u \quad (5)$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial P}{\partial y} - g\frac{\partial h}{\partial y} - fu - \frac{1}{h}\gamma v \quad (6)$$

We first find the steady equilibrium solution in the presence of a constant pressure gradient and no gradient in h:

$$-\frac{\gamma}{h_0}u_0 + fv_0 = \frac{\partial P}{\partial x} \quad (7)$$

$$-fu_0 - \frac{\gamma}{h_0}v_0 = \frac{\partial P}{\partial y} \quad (8)$$

A solution can be obtained for u_0 and v_0 in terms of P, h_0 and γ. For our purposes, we do not need those details. These properties of the equilibrium solution are useful for our endeavor:

$$-\frac{\gamma}{h_0}\left(u_0^2 + v_0^2\right) = u_0\frac{\partial P}{\partial x} + v_0\frac{\partial P}{\partial y} \quad (9)$$

$$f\left(u_0^2 + v_0^2\right) = v_0\frac{\partial P}{\partial x} - u_0\frac{\partial P}{\partial y} \quad (10)$$

Eliminating $\left(u_0^2 + v_0^2\right)$ leads to a solution for the angle θ between the pressure isobars and velocity vector:

$$\tan(\theta) = \frac{\gamma}{h_0 f} \quad (11)$$

The above equation allows for determination of γ if θ, f and h_0 are known. The value of γ will play a key role in the calculating a length scale for wake adjustment to the equilibrium.

We assume the wind farm is infinite in the y direction, and that the wind exits the wind farm at $x = 0$. All transients (in time) are assumed to have decayed away, so the adjustment to equilibrium that occurs downwind of the wind farm depends only on x. We write:

$$u = u_0 + u'(x) \tag{12}$$

$$v = v_0 + v'(x) \tag{13}$$

$$h = h_0 + h'(x) \tag{14}$$

The linearized Equations (2)–(4):

$$u_0 \frac{\partial h'}{\partial x} = -h_0 \frac{\partial u'}{\partial x} \tag{15}$$

$$u_0 \frac{\partial u'}{\partial x} = f v' - g \frac{\partial h'}{\partial x} - \frac{\gamma}{h_0} u' + \frac{\gamma u_0}{h_0^2} h' \tag{16}$$

$$u_0 \frac{\partial v'}{\partial x} = -f u' - \frac{\gamma}{h_0} v' + \frac{\gamma v_0}{h_0^2} h' \tag{17}$$

Seeking solutions that vanish as $x \to \infty$, Equation (15) yields

$$\frac{h'}{h_0} = -\frac{u'}{u_0} \tag{18}$$

So h' can be eliminated from Equations (16) and (17):

$$u_0 \frac{\partial u'}{\partial x} = f v' + \frac{g h_0}{u_0} \frac{\partial u'}{\partial x} - 2 \frac{\gamma}{h_0} u' \tag{19}$$

$$u_0 \frac{\partial v'}{\partial x} = -f u' - \frac{\gamma}{h_0} v' - \frac{\gamma v_0}{h_0 u_0} u' \tag{20}$$

2.1. Wake Solutions

Let's explore a simple case. If we take $f = 0$ and $g = 0$, we have a steady solution,

$$\frac{\gamma}{h_0} u_0 = -\frac{\partial P}{\partial x} \tag{21}$$

and Equation (19) produces the exponential decay to this solution

$$u'(x) = u'(0) \exp(-x/\lambda) \tag{22}$$

with the wake length

$$\lambda = \frac{u_0}{2\gamma} h_0 \tag{23}$$

In Equations (22) and (23), it may seem paradoxical that a wake of $u' < 0$ is removed more readily by larger values of γ. However, the reason follows from Equation (21), where we see that a larger γ is associated with a larger pressure gradient force, which regenerates the wind.

Note also using Equation (21) in Equation (23),

$$-\lambda \frac{\partial P}{\partial x} = \frac{1}{2} u_0^2 \tag{24}$$

Coincidentally, in Equation (24) we see the wake length λ is also the distance required for u_0 to have been generated by the pressure gradient.

A simple model for γ used in meteorology would use $\gamma = C_D U_{10}$ [9], where C_D is a dimensionless drag coefficient and U_{10} is the wind speed at 10 m elevation. In our well-mixed model $U_{10} = u_0$. So

$$\lambda = \frac{1}{2C_D} h_0 \tag{25}$$

With $h_0 \approx 1$ km and C_D as low as 0.001 over water [9], $\lambda \approx 500$ km. With C_D approximately 0.01 for unforested land surfaces [10], $\lambda \approx 50$ km. We visit the more complete linear solution below, and the conclusion about λ in Equation (25) will still approximately hold.

In Equation (15) we have neglected the role of entrainment. The effects of h' in Equations (15)–(17) therefore seem incomplete. So for the next level of complexity, we ignore terms with h' in Equations (16) and (17), and so can easily solve for the damped inertial oscillations. We also note the derived energy equation

$$u_0 \frac{\partial}{\partial x} \left(u'^2 + v'^2 \right) = -2 \frac{\gamma}{h_0} \left(u'^2 + v'^2 \right) \tag{26}$$

shows the perturbation energy decays with λ in Equation (23). The perturbation velocity magnitude would decay with a length scale twice that. With $f \neq 0$, Equation (11) can be used to estimate γ in Equation (23), yielding this important result:

$$\lambda = \frac{u_0}{2f \tan\theta} \tag{27}$$

Using Equation (27), an estimate for λ can be obtained without recourse to knowledge about C_D or mixing of momentum at $z = h$. Using $\tan(\theta) = 0.1$, we again obtain $\lambda = 500$ km.

For the final level of complexity, we retain the terms based on h', which leads to using Equations (19) and (20). With those, we construct an equation that resembles a balance between advection of energy (of the disturbance) and its dissipation:

$$u_0 \frac{\partial}{\partial x} \left(u'^2 + v'^2 - \frac{g h_0}{u_0^2} u'^2 \right) = -2 \frac{\gamma}{h_0} \left(2u'^2 + v'^2 + \frac{v_0}{u_0} u'v' \right) \tag{28}$$

We note the role of $u_0 h_0 / \gamma$ in Equation (28), similar to Equation (26), but exponential decay is not exactly predicted.

Let us seek solutions to Equations (19)–(20) of form $u = \widehat{u} e^{ikx}$ and $v = \widehat{v} e^{ikx}$ where $\widehat{u}$ and $\widehat{v}$ are complex constants:

$$u_0 ik\widehat{u} = f\widehat{v} + \frac{g h_0}{u_0} ik\widehat{u} - 2\frac{\gamma}{h_0}\widehat{u} \tag{29}$$

$$u_0 ik\widehat{v} = -f\widehat{u} - \frac{\gamma}{h_0}\widehat{v} - \frac{\gamma v_0}{h_0 u_0}\widehat{u} \tag{30}$$

These can be arranged as:

$$\left[k - \frac{2\gamma}{h_0 u_0 \left(1 - \frac{g h_0}{u_0^2} \right)} i \right] \widehat{u} = -i \frac{f}{u_0 - \frac{g h_0}{u_0}} \widehat{v} \tag{31}$$

$$\left(k - \frac{\gamma}{h_0 u_0} i \right) \widehat{v} = i \frac{\left(f + \gamma \frac{v_0}{h_0} \right)}{u_0} \widehat{u} \tag{32}$$

Thus, we have a classic eigenvalue problem for k, which is possibly complex: $k = k_r + ik_i$. With $e^{ikx} = e^{ik_r x} e^{-k_i x}$, we identify $1/k_i$ as the wake length λ that we seek.

There is a quadratic equation to solve for k, which offers a a variety of ways to present a solution. Here we choose to aid the algebra of finding the solution, by writing $k = K + iJ$ and solve for k in:

$$\left(k - \frac{\gamma}{h_0 u_0} i\right)\left[k - \frac{2\gamma}{h_0 u_0\left(1 - \frac{g h_0}{u_0^2}\right)} i\right] = \frac{f\left(f + \gamma \frac{v_0}{h_0}\right)}{u_0\left(u_0 - \frac{g h_0}{u_0}\right)} \tag{33}$$

The result for J is real, and thus iJ contributes to ik_i:

$$J = \frac{1}{2}\left[\frac{\gamma}{h_0 u_0} + \frac{2\gamma}{h_0 u_0\left(1 - \frac{g h_0}{u_0^2}\right)}\right] \tag{34}$$

Note J is proportional to γ, and $1/J$ provides close agreement to the estimate in Equation (23). K could be entirely real or imaginary, obtained from:

$$K^2 = \frac{f\left(f + \gamma \frac{v_0}{h_0}\right)}{u_0\left(u_0 - \frac{g h_0}{u_0}\right)} - \frac{1}{4}\left[\frac{\gamma}{h_0 u_0} - \frac{2\gamma}{h_0 u_0\left(1 - \frac{g h_0}{u_0^2}\right)}\right]^2 \tag{35}$$

When K is imaginary, it also contributes to ik_i. For example, if we assume $f = 0$ and $g = 0$, we obtain two solutions for k, both entirely imaginary. The one with nonzero $\widehat{u}$ yields the same value as the previous estimate for λ in Equation (27).

With $f \neq 0$, it may be possible to find that K is real, thus predicting a standing inertial-gravity wave in the wake. For subcritical conditions, $g h_0 > u_0^2$, but keeping $f = 0$, we cannot find a solution for a standing gravity wave in the wake of the wind farm. The simplifying approximation used here, but not in [7], causes pure gravity waves to be non-dispersive, so there is no wavelength that matches the phase speed with the flow speed.

The value of k_r does not concern us. If nonzero, the value of k_r may effect a forecast for direction of the wind when it enters a wind farm downstream. We claim here that k_i, and wake length λ, would still be consistent with the previous estimate.

3. Conclusions

We find a wind farm wake that extends through the depth of the boundary layer, with wake length about 10 times greater than those cited in [4]. Our results are not necessarily contradictory. Both [4,5] are finding replenishment of momentum at hub height by mixing from above, as does our model implicitly. However, our model considers the need to replenish the elevated source by a pressure gradient force. So we are finding a longer and deeper wake, in which that wake of [4,5] is embedded. We have shown that the wake length, and hence diminishment of the downstream resource, is predictable using elementary atmospheric dynamics. Knowledge of the angle θ Equation (11) for cross-isobaric flow, in conditions downstream of a giant wind farm, allows for easy calculation of the wake length λ Equation (27).

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References

1. Keith, D.W.; DeCarolis, J.F.; Denkenberger, D.C.; Lenschow, D.H.; Malyshev, S.L.; Pacala, S.; Rasch, P.J. The influence of large-scale wind power on global climate. *Proc. Nat. Acad. Sci. USA* **2004**, *101*, 16115–16120. [CrossRef] [PubMed]
2. Wang, C.; Prinn, R.G. Potential climatic impacts and reliability of very large-scale wind farms. *Atmos. Chem. Phys.* **2010**, *10*, 2053–2061. [CrossRef]
3. Possner, A.; Caldeira, K. Geophysical potential for wind energy over the open oceans. *Proc. Nat. Acad. Sci. USA* **2017**, *114*, 11338–11343. [CrossRef] [PubMed]
4. Stefan, E. Current issues in wind energy meteorology. *Meteorol. Appl.* **2014**, *21*, 803–819. [CrossRef]
5. Volker, P.J.H.; Badger, J.; Hahmann, A.N.; Ott, S. The Explicit Wake Parametrisation V1.0: A wind farm parametrisation in the mesoscale model WRF. *Geosci. Model Dev.* **2015**, *8*, 3715–3731. [CrossRef]
6. Jiang, Q. Applicability of Reduced-Gravity Shallow-Water Theory to Atmospheric Flow over Topography. *J. Atmos. Sci.* **2014**, *71*, 1460–1479. [CrossRef]
7. Smith, R.B. Gravity wave effects on wind farm efficiency. *Wind Energy* **2010**, *13*, 449–458. [CrossRef]
8. Allaerts, D.; Meyers, J. Gravity Waves and Wind-Farm Efficiency in Neutral and Stable Conditions. *Bound.-Layer Meteorol.* **2018**, *166*, 269–299. [CrossRef]
9. Emeis, S. *Wind Energy Meteorology: Atmospheric Physics for Wind Power Generation*; Springer: New York, NY, USA, 2012.
10. Mahrt, L.; Vickers, D.; Sun, J.; Jensen, N.O.; Jørgensen, H.; Pardyjak, E.; Fernando, H. Determination of the Surface Drag Coefficient. *Bound.-Layer Meteorol.* **2001**, *99*, 249–276. [CrossRef]