Supplementary Materials: Energy Rebound as a Potential Threat to a Low-Carbon Future: Findings from a New Exergy-Based National-Level Rebound Approach

Paul E. Brockway, Harry Saunders, Matthew K. Heun, Timothy J. Foxon, Julia K. Steinberger, John R. Barrett and Steve Sorrell

S1. Method 2 Elasticity of Energy Use with Respect to Efficiency (EEE): Extended Constant Elasticity of Substitution (CES) Function—Long-Term Rebound Derivation

S1.1. Setup

We need to develop the expressions $\frac{\tau}{Y}\frac{\partial Y}{\partial \tau}$ and $\frac{\tau}{E}\frac{\partial E}{\partial \tau}$ to deliver the output and intensity rebound elasticities (long-term) expressed in a form employing (ideally measured) parameters of the particular production function being examined:

$$\begin{aligned} \eta_{\tau}^{E_{Output}} &= \frac{\tau}{Y}\frac{\partial Y}{\partial \tau} \\ \eta_{\tau}^{E_{Intensity}} &= \frac{\tau}{E}\frac{\partial E}{\partial \tau} - \eta_{\tau}^{E_{Output}} \end{aligned} \tag{S1}$$

In this case, we're using an extended version of the CES production function, of the form:

$$Y = \gamma A \left\{ \delta \left[\delta_1 K^{-\rho_1} + \left(1-\delta_1\right) L^{-\rho_1} \right]^{\rho/\rho_1} + \left(1-\delta\right)\left(\tau E\right)^{-\rho} \right\}^{-1/\rho}; \; A = e^{\lambda(t-t_0)} \tag{S2}$$

Solving for the needed rebound elasticities requires appeal to the Implicit Function Theorem. This is because the introduction of an energy technology gain τ affects the Y and E terms in Equation (S1) in multiple complex ways, requiring setting up a series of equations. And it happens that the variables required to develop expressions for the $\frac{\partial Y}{\partial \tau}$ and $\frac{\partial E}{\partial \tau}$ terms of Equation (S1) are embedded in the equation structure in such a way that they cannot be isolated directly. The Implicit Function Theorem allows us to ask how any endogenous variables (here we mean Y and E) will change, while honoring these equations, if some exogenous variable (here we mean τ) changes.

S1.2. Equations Needed

We need three equations to describe how an economy with three factor inputs (here K, L, E) behaves when there is a change in τ.

We can construct the following three equations:

$$\begin{aligned} \psi_1 &= g\left(Y, f(K,L,\tau E)\right) = 0 \\ \psi_2 &= h\left(s_E, \frac{\partial f(K,L,\tau E)}{\partial F}\frac{E}{Y} \right) = 0 \\ \psi_3 &= k\left(s_K, \frac{\partial f(K,L,\tau E)}{\partial K}\frac{K}{Y} \right) = 0 \end{aligned} \tag{S3}$$

The first equation is essentially the production function itself, and so looks like:

$$\psi_1 = Y - \gamma A \left\{ \delta \left[\delta_1 K^{-\rho_1} + \left(1-\delta_1\right) L^{-\rho_1} \right]^{\rho/\rho_1} + \left(1-\delta\right)\left(\tau E\right)^{-\rho} \right\}^{-1/\rho} = 0 \tag{S4}$$

The second and third equations are developed from the cost shares of energy and capital:

$$
\begin{aligned}
& s_E Y = \frac{p_E}{c} E \\
& s_K Y = \frac{p_K}{c} K \\
& \Rightarrow s_E Y - \frac{p_E}{c} E = 0 \\
& \Rightarrow s_K Y - \frac{p_K}{c} K = 0 \\
& \Rightarrow Y - \frac{p_E}{c s_E} E = 0 \\
& \Rightarrow Y - \frac{p_K}{c s_K} K = 0
\end{aligned}
\tag{S5}
$$

So we choose the second and third equations to be:

$$
\begin{aligned}
& \psi_2 = Y - \frac{p_E}{c s_E} E = 0 \\
& \psi_3 = Y - \frac{p_K}{c s_K} K = 0
\end{aligned}
\tag{S6}
$$

S1.3. Implicit Function Theorem and the Jacobian

To measure rebound, we need to know how Y and E respond to changes in the energy technology gain τ. To accomplish this, we form the Jacobian matrix of $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)$, namely $J = \left[\frac{\partial \psi_i (Y_0, E_0, \tau_0)}{\partial X_j} \right]$, where $X_j = Y, E, K$. Then it will be true that:

$$
\begin{bmatrix} \frac{\partial Y}{\partial \tau} \\ \frac{\partial E}{\partial \tau} \\ \frac{\partial K}{\partial \tau} \end{bmatrix} = -J^{-1} \begin{bmatrix} \frac{\partial \psi_1}{\partial \tau} \\ \frac{\partial \psi_2}{\partial \tau} \\ \frac{\partial \psi_3}{\partial \tau} \end{bmatrix}
\tag{S7}
$$

From the terms $\frac{\partial Y}{\partial \tau}$ and $\frac{\partial E}{\partial \tau}$ we can determine the components of long-term rebound.

The Jacobian matrix is:

$$
J = \begin{pmatrix} \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial E} & \frac{\partial \psi_1}{\partial K} \\ \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial E} & \frac{\partial \psi_2}{\partial E} \\ \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial E} & \frac{\partial \psi_2}{\partial E} \end{pmatrix}
\tag{S8}
$$

S1.4. Calculating the Jacobian Elements

To develop the first row of the Jacobian, we need to calculate $\frac{\partial \psi_1}{\partial Y}, \frac{\partial \psi_1}{\partial E}, \frac{\partial \psi_1}{\partial K}$. The first element is easy: From Equation (S4) we have that $\frac{\partial \psi_1}{\partial Y} = 1$.

Calculating the second two elements is trivial as these are essentially the first-order conditions on energy and capital:

$$\frac{\partial \psi_1}{\partial E} = -\frac{\partial f(K, L, \tau E)}{\partial E} = \frac{p_F}{c}$$
$$\frac{\partial \psi_1}{\partial K} = -\frac{\partial f(K, L, \tau E)}{\partial K} = \frac{p_K}{c} \quad \text{(S9)}$$

To develop the second row of the Jacobian, we need to calculate $\frac{\partial \psi_2}{\partial Y}, \frac{\partial \psi_2}{\partial E}, \frac{\partial \psi_2}{\partial K}$.

Looking at Equations (S4) and (S6), we see that:

$$\frac{\partial \psi_2}{\partial Y} = 1$$
$$\frac{\partial \psi_2}{\partial E} = -\frac{p_E}{cs_E} \quad \text{(S10)}$$
$$\frac{\partial \psi_2}{\partial K} = 0$$

To develop the third row of the Jacobian, we need to calculate $\frac{\partial \psi_3}{\partial Y}, \frac{\partial \psi_3}{\partial E}, \frac{\partial \psi_3}{\partial K}$.

Looking at Equations (S4) and (S6), we see that:

$$\frac{\partial \psi_3}{\partial Y} = 1$$
$$\frac{\partial \psi_3}{\partial E} = 0 \quad \text{(S11)}$$
$$\frac{\partial \psi_3}{\partial K} = -\frac{p_K}{cs_K}$$

Therefore, the Jacobian matrix becomes:

$$J = \begin{pmatrix} \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial E} & \frac{\partial \psi_1}{\partial K} \\ \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial E} & \frac{\partial \psi_2}{\partial E} \\ \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial E} & \frac{\partial \psi_2}{\partial E} \end{pmatrix} = \begin{pmatrix} 1 & \frac{p_E}{c} & \frac{p_K}{c} \\ 1 & -\frac{p_E}{cs_E} & 0 \\ 1 & 0 & -\frac{p_K}{cs_K} \end{pmatrix} \quad \text{(S12)}$$

Interestingly, this matrix appears to be independent of the particular form of the production function.

S1.5. Calculating the Efficiency Gain Vector Elements

Prior to inverting the above Jacobian matrix, we need to develop the partials of the three equations with respect to the energy efficiency gain parameter, τ, as called for in Equation (S7). The three elements are $\frac{\partial \psi_1}{\partial \tau}$, $\frac{\partial \psi_2}{\partial \tau}$, and $\frac{\partial \psi_3}{\partial \tau}$. We start by invoking some substitutions to make the derivatives easier. Specifically, let $Q = \delta\left[\delta_1 K^{-\rho_1} + (1-\delta_1)L^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta)(\tau E)^{-\rho}$. Let $R = \delta_1 K^{-\rho_1} + (1-\delta_1)L^{-\rho_1}$. And let $S = \tau E$. Then, $Q = \delta R^{\rho/\rho_1} + (1-\delta)S^{-\rho}$.

S1.5.1. Partial of First Equation

So beginning with $\frac{\partial \psi_1}{\partial \tau} = Y - \gamma A\left\{\delta\left[\delta_1 K^{-\rho_1} + (1-\delta_1)L^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta)(\tau E)^{-\rho}\right\}^{-1/\rho}$ and noting that $Y = \gamma A\left[Q(S(\tau))\right]^{-1/\rho}$, from the chain rule we have:

$$\frac{\partial f(K,L,\tau E)}{\partial \tau} = \frac{\partial Y}{\partial Q}\frac{\partial Q}{\partial S}\frac{\partial S}{\partial \tau} \tag{S13}$$

The three partials are:

$$\begin{aligned}
&\frac{\partial Y}{\partial Q} = -\frac{1}{\rho}\gamma A\frac{Q^{-1/\rho}}{Q} = -\frac{1}{\rho}\frac{Y}{Q} = -\frac{1}{\rho}\frac{Y}{\left(\frac{Y}{\gamma A}\right)^{-\rho}} = -\frac{1}{\rho}(\gamma A)^{-\rho}Y^{1+\rho}\\
&\frac{\partial Q}{\partial S} = -\rho(1-\delta)\frac{S^{-\rho}}{S} = -\rho(1-\delta)\frac{(\tau E)^{-\rho}}{\tau E} = -\rho(1-\delta)\tau^{-\rho-1}E^{-\rho-1}\\
&\frac{\partial S}{\partial \tau} = E
\end{aligned} \tag{S14}$$

To get us part way, we substitute Equation (S14) into Equation (S13), yielding:

$$\begin{aligned}
\frac{\partial f(K,L,\tau E)}{\partial \tau} &= \frac{\partial Y}{\partial Q}\frac{\partial Q}{\partial S}\frac{\partial S}{\partial \tau}\\
\frac{\partial f(K,L,\tau E)}{\partial \tau} &= \left[-\frac{1}{\rho}(\gamma A)^{-\rho}Y^{1+\rho}\right]\left[-\rho(1-\delta)\tau^{-\rho-1}E^{-\rho-1}\right]E\\
\frac{\partial f(K,L,\tau E)}{\partial \tau} &= (\gamma A)^{-\rho}(1-\delta)\frac{1}{\tau^{\rho+1}}\left(\frac{Y}{E}\right)^{1+\rho}E
\end{aligned} \tag{S15}$$

So the first partial becomes:

$$\frac{\partial \psi_1}{\partial \tau} = -(\gamma A)^{-\rho}(1-\delta)\frac{1}{\tau^{\rho+1}}\left(\frac{Y}{E}\right)^{1+\rho}E \tag{S16}$$

Further simplification comes if we derive the first-order condition on energy and introduce the cost share S_E. The development is identical to Equation (S13) except for the last term:

$$\frac{\partial f(K,L,\tau E)}{\partial F} = \frac{p_F}{c} = \frac{\partial Y}{\partial Q}\frac{\partial Q}{\partial S}\frac{\partial S}{\partial E} \tag{S17}$$

where $\frac{\partial S}{\partial E} = \tau$, meaning Equation (S17) can be re-written as:

$$\begin{aligned}
\frac{\partial f(K,L,\tau E)}{\partial F} &= \frac{p_E}{c} = \left[-\frac{1}{\rho}(\gamma A)^{-\rho}Y^{1+\rho}\right]\left[-\rho(1-\delta)\tau^{-\rho-1}E^{-\rho-1}\right]\tau\\
\frac{\partial f(K,L,\tau E)}{\partial F} &= \frac{p_E}{c} = (\gamma A)^{-\rho}(1-\delta)\frac{\tau}{\tau^{\rho+1}}\left(\frac{Y}{E}\right)^{1+\rho}
\end{aligned} \tag{S18}$$

This equation can be rearranged to enable substitution into Equation (S16). That is:

$$\frac{p_E}{c\tau} = (\gamma A)^{-\rho}(1-\delta)\frac{1}{\tau^{\rho+1}}\left(\frac{Y}{E}\right)^{1+\rho} \tag{S19}$$

Substituting Equation (S19) into Equation (S16) yields:

$$\frac{\partial \psi_1}{\partial \tau} = -\frac{p_E}{c\tau}E \tag{S20}$$

But observing from the energy cost share equation that:

$$\begin{aligned}
&s_E = \frac{p_E}{c}\frac{E}{Y}\\
&\Rightarrow \frac{p_E}{c} = s_E\frac{Y}{F}
\end{aligned} \tag{S21}$$

Substituting Equation (S21) into Equation (S20) yields:

$$\begin{aligned}\frac{\partial\psi_1}{\partial\tau} &= -\frac{1}{\tau}s_E\frac{Y}{F}E \\ \frac{\partial\psi_1}{\partial\tau} &= -\frac{s_E Y}{\tau}\end{aligned} \tag{S22}$$

S1.5.2. Partial of Second Equation

The second equation is $\psi_2 = Y - \frac{p_E}{cs_E}E = 0$.

But we need to re-state this equation in a form that is explicit in τ. For this we return to the first-order condition Equation (S18):

$$\begin{aligned}(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}\left(\frac{Y}{E}\right)^{1+\rho} &= \frac{p_E}{c} \\ \left(\frac{Y}{E}\right)^{1+\rho} &= \frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}} \\ \frac{Y}{E} &= \left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}} \\ \Rightarrow Y &= \left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}}E \\ \Rightarrow Y &= \left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)}\right)^{\frac{1}{1+\rho}}\tau^{\frac{\rho}{1+\rho}}E\end{aligned} \tag{S23}$$

So ψ_2 can now be written as:

$$\psi_2 = Y - \left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)}\right)^{\frac{1}{1+\rho}}\tau^{\frac{\rho}{1+\rho}}E = 0 \tag{S24}$$

Now we can differentiate with respect to τ:

$$\begin{aligned}\frac{\partial\psi_2}{\partial\tau} &= -\left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)}\right)^{\frac{1}{1+\rho}}E\frac{\partial}{\partial\tau}\tau^{\frac{\rho}{1+\rho}} \\ \frac{\partial\psi_2}{\partial\tau} &= -\frac{\rho}{1+\rho}\left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)}\right)^{\frac{1}{1+\rho}}E\frac{\tau^{\frac{\rho}{1+\rho}}}{\tau} \\ \frac{\partial\psi_2}{\partial\tau} &= -\frac{\rho}{1+\rho}\left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}}\frac{E}{\tau}\end{aligned} \tag{S25}$$

We can simplify by invoking the cost share equation for energy:

$$\begin{aligned}s_E &= \frac{p_F}{c}\frac{E}{Y} \\ \Rightarrow \frac{Y}{E} &= \frac{p_E}{cs_E}\end{aligned} \tag{S26}$$

But from Equation (S23) we know that:

$$\frac{Y}{E} = \left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}} \tag{S27}$$

Comparing Equation (S27) with Equation (S26), we see that:

$$\left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}} = \frac{p_E}{cs_E} \tag{S28}$$

Therefore we can rewrite Equation (S25) as:

$$\frac{\partial \Psi_2}{\partial \tau} = -\frac{\rho}{1+\rho}\left(\frac{p_E}{c(\gamma A)^{-\rho}(1-\delta)\tau^{-\rho}}\right)^{\frac{1}{1+\rho}} \frac{E}{\tau}$$
$$\frac{\partial \Psi_2}{\partial \tau} = -\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau} \tag{S29}$$

S1.5.3. Partial of Third Equation

The first order of business is to derive the first-order condition on capital:

Let $Q = \delta\left[\delta_1 K^{-\rho_1} + (1-\delta_1)L^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta)(\tau E)^{-\rho}$. Let $R = \delta_1 K^{-\rho_1} + (1-\delta_1)L^{-\rho_1}$. Then $Y = \gamma A\left[Q(R(K))\right]^{-1/\rho}$, so from the chain rule:

$$\frac{\partial f(K,L,\tau E)}{\partial K} = \frac{p_K}{c} = \frac{\partial Y}{\partial Q}\frac{\partial Q}{\partial R}\frac{\partial R}{\partial K} \tag{S30}$$

Also note that $Q = \delta R^{\rho/\rho_1} + (1-\delta)(\tau E)^{-\rho}$.

Taking each component of Equation (S30) in turn:

$$\frac{\partial Y}{\partial Q} = -\frac{1}{\rho}\gamma A\frac{Q^{-1/\rho}}{Q} = -\frac{1}{\rho}\gamma A(\gamma A)^{\frac{1-\rho}{\rho}} Y^{1+\rho} = -\frac{1}{\rho}(\gamma A)^{\frac{1-\rho}{\rho}+\frac{\rho}{\rho}} Y^{1+\rho}$$
$$\frac{\partial Y}{\partial Q} = -\frac{1}{\rho}(\gamma A)^{\frac{1}{\rho}} Y^{1+\rho}$$
$$\frac{\partial Q}{\partial R} = \delta\frac{\rho}{\rho_1}\frac{R^{\rho/\rho_1}}{R} \tag{S31}$$
$$\frac{\partial R}{\partial K} = -\delta_1\rho_1\frac{K^{-\rho_1}}{K}$$

We need to express Q in terms of Y:

$$Q = \frac{Y^{-\rho}}{\gamma A}$$
$$Q^{\frac{-1}{\rho}} = \left(\frac{Y^{-\rho}}{\gamma A}\right)^{\frac{-1}{\rho}} = \left(\frac{1}{\gamma A}\right)^{\frac{-1}{\rho}} Y$$
$$\frac{Q^{\frac{-1}{\rho}}}{Q} = \frac{\left(\frac{1}{\gamma A}\right)^{\frac{-1}{\rho}} Y}{\frac{Y^{-\rho}}{\gamma A}} = \left(\frac{1}{\gamma A}\right)^{\frac{-1}{\rho}} \gamma A Y^{1+\rho} = \left(\frac{\gamma A}{1}\right)^{\frac{1}{\rho}} \gamma A Y^{1+\rho} \tag{S32}$$
$$\frac{Q^{\frac{-1}{\rho}}}{Q} = (\gamma A)^{\frac{1-\rho}{\rho}} Y^{1+\rho}$$

So the expression Equation (S30) becomes:

$$\frac{\partial f(K,L,\tau E)}{\partial K}=\left[-\frac{1}{\rho}(\gamma A)^{\frac{1}{\rho}}Y^{1+\rho}\right]\left[\delta\frac{\rho}{\rho_1}\frac{R^{\rho/\rho_1}}{R}\right]\left[-\rho_1\delta_1\frac{K^{-\rho_1}}{K}\right]$$

$$\frac{\partial f(K,L,\tau E)}{\partial K}=\frac{1}{\rho}(\gamma A)^{\frac{1}{\rho}}\delta\rho_1\delta_1\frac{\rho}{\rho_1}\left[Y^{1+\rho}\right]R^{\rho/\rho_1-\frac{\rho_1}{\rho_1}}K^{-\rho_1-1} \tag{S33}$$

$$\frac{\partial f(K,L,\tau E)}{\partial K}=(\gamma A)^{\frac{1}{\rho}}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}K^{-\rho_1-1}Y^{1+\rho}$$

We know the first-order condition on capital is:

$$\frac{\partial f(K,L,\tau E)}{\partial K}=\frac{p_K}{c} \tag{S34}$$

Therefore, from Equation (S33) we have:

$$\frac{p_K}{c}=(\gamma A)^{\frac{1}{\rho}}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}K^{-\rho_1-1}Y^{1+\rho}$$

$$\frac{p_K}{c}=(\gamma A)^{-\rho}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}\frac{Y^{\rho}}{K^{\rho_1}}\frac{Y}{K} \tag{S35}$$

We can solve this for Y in terms of K:

$$\frac{p_K}{c}=(\gamma A)^{-\rho}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}\frac{Y^{\rho+1}}{K^{\rho_1}}\frac{1}{K}$$

$$\Rightarrow Y^{\rho+1}=\frac{p_K}{c}\frac{1}{(\gamma A)^{-\rho}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}}K^{\rho_1}K \tag{S36}$$

$$\Rightarrow Y=\left(\frac{p_K}{c}\frac{1}{(\gamma A)^{-\rho}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}}K^{\rho_1}K\right)^{\frac{1}{\rho+1}}$$

We can see that the first-order condition will be a complex function of K. However, we can also see that none of the terms of Equation (S36) involve τ. K does not explicitly depend on τ. Therefore the partial derivative for the third term will be zero.

When this is used to formulate the third equation forcing the capital first-order condition to be met, it will look as follows.

$$\psi_3=Y-(\gamma A)^{-\rho}\delta\delta_1 R^{\frac{\rho-\rho_1}{\rho_1}}\frac{Y^{\rho}}{K^{\rho_1}}\frac{Y}{K}=0 \tag{S37}$$

And, from the argument above, we will have that:

$$\frac{\partial\psi_3}{\partial\tau}=0 \tag{S38}$$

S1.6. Summary to This Point

We have calculated the Jacobian matrix (but have not yet inverted it for Equation (S7). We have also calculated the vector of partials, so we have the Jacobian as:

$$J = \begin{pmatrix} 1 & \frac{p_E}{c} & \frac{p_K}{c} \\ 1 & -\frac{p_E}{cs_E} & 0 \\ 1 & 0 & -\frac{p_K}{cs_K} \end{pmatrix} \tag{S39}$$

And the efficiency gain vector of the technology partials is:

$$\Psi = \begin{bmatrix} \frac{\partial \psi_1}{\partial \tau} \\ \frac{\partial \psi_2}{\partial \tau} \\ \frac{\partial \psi_3}{\partial \tau} \end{bmatrix} = \begin{bmatrix} -\frac{s_E Y}{\tau} \\ -\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau} \\ 0 \end{bmatrix} \tag{S40}$$

Notably, the parameter ρ_1 is absent from the system of equations. In fact, the equations are identical to the equations developed by Saunders [1] for the simpler CES production function:

It seems possible that the Jacobian may be identical for any production function (CRS required, probably). For one thing, it is derived from share equations only (Equations (S5) and (S6)), which are agnostic as to production function form (the energy derivative of Equation (S4) is highly related to the energy cost share). But, unlike the Jacobian, the efficiency gain vector will depend on the functional form.

Nonetheless, the energy efficiency gain vector is the same for the current production function as for the simpler CES form in Saunders [1] (2008).

Therefore, the only real difference between the LT rebound equation in Saunders [1] (2008), and the one that applies here, is the difference in the production function specification in how it treats ρ.

Nonetheless, we take the derivation through from here to get the exact rebound equation given this function's treatment of the ρ and ρ_1 parameters and certain other parameters that differ from that used in the Saunders [1] 2008 formulation.

S1.7. Inverting the Jacobian Matrix

We need to develop the inverse matrix of the Jacobian J in Equation (S39). We do this using Cramer's rule.

Inverting J first requires calculating the determinant of J, here specified as $\Delta = \det(J)$.

This in turn requires specifying "cofactor" matrices in J associated with expansion along one row or column of J. For us, it is convenient to choose the first column of J as the selected basis. Then, the cofactors of J become:

$$\begin{gathered} J_{11} = \begin{vmatrix} -\frac{p_E}{cs_E} & 0 \\ 0 & -\frac{p_K}{cs_K} \end{vmatrix} \\ J_{21} = -\begin{vmatrix} \frac{p_E}{c} & \frac{p_K}{c} \\ 0 & -\frac{p_K}{cs_K} \end{vmatrix} \\ J_{31} = \begin{vmatrix} \frac{p_E}{c} & \frac{p_K}{c} \\ -\frac{p_E}{cs_E} & 0 \end{vmatrix} \end{gathered} \tag{S41}$$

These determinants are calculated as:

$$J_{11} = \frac{p_E}{cs_E}\frac{p_K}{cs_K}$$
$$J_{21} = \frac{p_E}{c}\frac{p_K}{cs_K} \tag{S42}$$
$$J_{31} = \frac{p_E}{cs_E}\frac{p_K}{c}$$

So the determinant is:

$$\begin{aligned}
\Delta &= 1\bullet\frac{p_E}{cs_E}\frac{p_K}{cs_K} + 1\bullet\left(\frac{p_E}{c}\frac{p_K}{cs_K}\right) + 1\bullet\frac{p_E}{cs_E}\frac{p_K}{c} \\
\Delta &= \frac{p_E}{cs_E}\frac{p_K}{cs_K} + \frac{p_E}{c}\frac{p_K}{cs_K} + \frac{p_E}{cs_E}\frac{p_K}{c} \\
\Delta &= \frac{p_E p_K}{c^2}\left(\frac{1}{s_E s_K} + \frac{1}{s_K} + \frac{1}{s_E}\right) \\
\Delta &= \frac{p_E p_K}{c^2}\left(\frac{1 + s_E + s_K}{s_E s_K}\right)
\end{aligned} \tag{S43}$$

Then, the elements of J^{-1} rely on the other cofactors:

$$\begin{aligned}
J_{12} &= -\begin{vmatrix} 1 & 0 \\ 1 & -\frac{p_K}{cs_K} \end{vmatrix} = \frac{p_K}{cs_K} \\
J_{13} &= \begin{vmatrix} 1 & -\frac{p_E}{cs_E} \\ 1 & 0 \end{vmatrix} = \frac{p_E}{cs_E} \\
J_{22} &= \begin{vmatrix} 1 & \frac{p_K}{c} \\ 1 & -\frac{p_K}{cs_K} \end{vmatrix} = -\frac{p_K}{cs_K} - \frac{p_K}{c} = -\frac{p_K}{c}\left(\frac{1}{s_K} + 1\right) = -\frac{p_K}{c}\left(\frac{1 + s_K}{s_K}\right) \\
J_{23} &= -\begin{vmatrix} 1 & \frac{p_E}{c} \\ 1 & 0 \end{vmatrix} = \frac{p_E}{c} \\
J_{32} &= -\begin{vmatrix} 1 & \frac{p_K}{c} \\ 1 & 0 \end{vmatrix} = \frac{p_K}{c} \\
J_{33} &= \begin{vmatrix} 1 & \frac{p_E}{c} \\ 1 & -\frac{p_E}{cs_E} \end{vmatrix} = -\frac{p_E}{cs_E} - \frac{p_E}{c} = -\frac{p_E}{c}\left(\frac{1}{s_E} + 1\right) = -\frac{p_E}{c}\left(\frac{1 + s_E}{s_E}\right)
\end{aligned} \tag{S44}$$

The inverse of J is then:

$$J^{-1} = \frac{1}{\Delta}\begin{pmatrix} J_{11} & J_{21} & J_{31} \\ J_{12} & J_{22} & J_{32} \\ J_{13} & J_{23} & J_{33} \end{pmatrix} \tag{S45}$$

So plugging in the values from Equations (S42) and (S44), the inverse becomes:

$$J^{-1}=\frac{1}{\Delta}\begin{pmatrix}\frac{p_E}{cs_E}\frac{p_K}{cs_K} & \frac{p_E}{c}\frac{p_K}{cs_K} & \frac{p_K}{c}\frac{p_E}{cs_E}\\ \frac{p_K}{cs_K} & -\frac{p_K}{c}\left(\frac{1+s_K}{s_K}\right) & \frac{p_K}{c}\\ \frac{p_E}{cs_E} & \frac{p_E}{c} & -\frac{p_E}{c}\left(\frac{1+s_E}{s_E}\right)\end{pmatrix} \tag{S46}$$

S1.8. Solution

The Solution Vector is now:

$$\begin{bmatrix}\frac{\partial Y}{\partial\tau}\\ \frac{\partial E}{\partial\tau}\\ \frac{\partial K}{\partial\tau}\end{bmatrix}=-\frac{1}{\Delta}\begin{bmatrix}\frac{p_E}{cs_F}\frac{p_K}{cs_K} & \frac{p_E}{c}\frac{p_K}{cs_K} & \frac{p_K}{c}\frac{p_E}{cs_E}\\ \frac{p_K}{cs_K} & -\frac{p_K}{c}\left(\frac{1+s_K}{s_K}\right) & \frac{p_K}{c}\\ \frac{p_E}{cs_E} & \frac{p_E}{c} & -\frac{p_E}{c}\left(\frac{1+s_E}{s_E}\right)\end{bmatrix}\begin{bmatrix}-\frac{s_E Y}{\tau}\\ -\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau}\\ 0\end{bmatrix} \tag{S47}$$

Substituting in Δ,

$$\begin{bmatrix}\frac{\partial Y}{\partial\tau}\\ \frac{\partial E}{\partial\tau}\\ \frac{\partial K}{\partial\tau}\end{bmatrix}=-\frac{c^2 s_E s_K}{p_E p_K(1+s_E+s_K)}\begin{bmatrix}\frac{p_E}{cs_E}\frac{p_K}{cs_K} & \frac{p_E}{c}\frac{p_K}{cs_K} & \frac{p_K}{c}\frac{p_E}{cs_E}\\ \frac{p_K}{cs_K} & -\frac{p_K}{c}\left(\frac{1+s_K}{s_K}\right) & \frac{p_K}{c}\\ \frac{p_E}{cs_E} & \frac{p_E}{c} & -\frac{p_E}{c}\left(\frac{1+s_E}{s_E}\right)\end{bmatrix}\begin{bmatrix}-\frac{s_E Y}{\tau}\\ -\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau}\\ 0\end{bmatrix} \tag{S48}$$

For the first equation we need to remove E from the second element of the efficiency vector (but we'll need it in this form later). Noting that $s_E=\frac{p_E}{c}\frac{E}{Y}$ and substituting this into the second element of the efficiency gain vector yields:

$$-\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau}=-\frac{\rho}{1+\rho}\frac{Y}{E}\frac{E}{\tau}=-\frac{\rho}{1+\rho}\frac{Y}{\tau} \tag{S49}$$

So the first equation becomes:

$$\begin{aligned}\frac{\partial Y}{\partial\tau}&=-\frac{c^2 s_E s_K}{p_E p_K(1+s_E+s_K)}\left(-\frac{p_E}{cs_E}\frac{p_K}{cs_K}\frac{s_E Y}{\tau}-\frac{p_F}{c}\frac{p_K}{cs_K}\frac{\rho}{1+\rho}\frac{Y}{\tau}\right)\\ \Rightarrow\frac{\tau}{Y}\frac{\partial Y}{\partial\tau}&=-\frac{c^2 s_E s_K}{p_E p_K(1+s_E+s_K)}\left(-\frac{p_E}{cs_E}\frac{p_K s_F}{cs_K}-\frac{p_F}{c}\frac{p_K}{cs_K}\frac{\rho}{1+\rho}\right)\\ \Rightarrow\frac{\tau}{Y}\frac{\partial Y}{\partial\tau}&=\frac{c^2 s_E s_K}{(1+s_E+s_K)}\left(\frac{1}{cs_E}\frac{s_E}{cs_K}+\frac{1}{c}\frac{1}{cs_K}\frac{\rho}{1+\rho}\right)\\ \Rightarrow\frac{\tau}{Y}\frac{\partial Y}{\partial\tau}&=\frac{s_E}{(1+s_E+s_K)}\left(1+\frac{\rho}{1+\rho}\right)\\ \Rightarrow\frac{\tau}{Y}\frac{\partial Y}{\partial\tau}&=\frac{s_E}{(1+s_E+s_K)}\left(\frac{1+2\rho}{1+\rho}\right)\end{aligned} \tag{S50}$$

For the second equation, we need to remove Y from the first element of the efficiency vector. As before, noting that $s_E=\frac{p_E}{c}\frac{E}{Y}$ and substituting this into the first element of the efficiency gain vector yields:

$$-\frac{s_E Y}{\tau} = -\frac{p_E}{c}\frac{E}{Y}\frac{Y}{\tau} = -\frac{p_E}{c}\frac{E}{\tau} \tag{S51}$$

So the second equation becomes:

$$\begin{aligned}
\frac{\partial E}{\partial \tau} &= -\frac{c^2 s_E s_K}{p_E p_K(1+s_F+s_K)}\left(-\frac{p_K}{cs_K}\frac{p_E}{c}\frac{E}{\tau} + \frac{p_K}{c}\left(\frac{1+s_K}{s_K}\right)\frac{\rho}{1+\rho}\frac{p_E}{cs_E}\frac{E}{\tau}\right) \\
\Rightarrow \frac{\tau}{E}\frac{\partial E}{\partial \tau} &= -\frac{c^2 s_E s_K}{(1+s_E+s_K)}\left(-\frac{1}{cs_K}\frac{1}{c} + \frac{1}{c}\left(\frac{1+s_K}{s_K}\right)\frac{\rho}{1+\rho}\frac{1}{cs_E}\right) \\
\Rightarrow \frac{\tau}{E}\frac{\partial E}{\partial \tau} &= \frac{s_E}{(1+s_E+s_K)}\left(1 - \left(\frac{1+s_K}{1}\right)\frac{\rho}{1+\rho}\frac{1}{s_E}\right) \\
\Rightarrow \frac{\tau}{E}\frac{\partial E}{\partial \tau} &= \frac{1}{(1+s_E+s_K)}\left(\frac{(1+\rho)s_E - \rho(1+s_K)}{(1+\rho)}\right) \\
\Rightarrow \frac{\tau}{E}\frac{\partial E}{\partial \tau} &= \frac{1}{(1+s_E+s_K)}\left(\frac{\rho(s_E - s_K - 1)+s_E}{(1+\rho)}\right)
\end{aligned} \tag{S52}$$

Thus the long-term rebound equation from Equation (S52) is:

$$\mathrm{Re} = 1 + \eta_\tau^E \quad = 1 + \frac{\tau}{E}\frac{\partial E}{\partial \tau} = 1 + \frac{1}{(1+s_E+s_K)}\left(\frac{\rho(s_E - s_K - 1)+s_E}{(1+\rho)}\right) \tag{S53}$$

$$Re = \frac{(1+s_E+s_K)(1+\rho) + (\rho(s_E - s_K - 1)+s_E)}{(1+s_E+s_K)(1+\rho)} \tag{S54}$$

In addition, from Equations (S50) and (S52) we can state the elasticity components for rebound calculations as follows:

$$\eta_\tau^{E_{Output}} = \frac{\tau}{Y}\frac{\partial Y}{\partial \tau} = \frac{s_E}{(1+s_E+s_K)}\left(\frac{1+2\rho}{1+\rho}\right) \tag{S55}$$

$$\eta_\tau^{E_{Intensity}} = \frac{\tau}{E}\frac{\partial E}{\partial \tau} - \eta_\tau^{E_{Output}}$$

$$\eta_\tau^{E_{Intensity}} = \frac{(1+s_E+s_K)(1+\rho) + (\rho(s_E - s_K - 1)+s_E)}{(1+s_E+s_K)(1+\rho)} - \frac{s_E}{(1+s_E+s_K)}\left(\frac{1+2\rho}{1+\rho}\right)$$

$$\eta_\tau^{E_{Intensity}} = \frac{1}{(1+\rho)} \tag{S56}$$

Reference

1. Saunders, H.D. Fuel conserving (and using) production functions. *Energy Econ.* **2008**, *30*, 2184–2235.