A Pseudo-Bayesian Model for Stock Returns In Financial Crises

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ABSTRACT

Recently, there has been a considerable interest in the Bayesian approach for explaining investors' behaviorial biases by incorporating conservative and representative heuristics when making financial decisions, (see, for example, Barberis, Shleifer and Vishny (1998)). To establish a quantitative link between some important market anomalies and investors' behaviorial biases, Lam, Liu, and Wong (2010) introduced a pseudo-Bayesian approach for developing properties of stock returns, where weights induced by investors' conservative and representative heuristics are assigned to observations of the earning shocks and stock prices. In response to the recent global financial crisis, we introduce a new pseudo-Bayesian model to incorporate the impact of a financial crisis. Properties of stock returns during the financial crisis and recovery from the crisis are established. The proposed model can be applied to investigate some important market anomalies including short-term underreaction, long-term overreaction, and excess volatility during financial crisis. We also explain in some detail the linkage between these market anomalies and investors' behavioral biases during financial crisis.

KEYWORDS: Bayesian model, Representative and conservative heuristics, Underreaction, Overreaction, Stock price, Stock return, financial crisis.

1 INTRODUCTION

In the past two decades, market excess volatility, overreaction, and underreaction have been important market anomalies which pose a major challenge to economists and financial researchers. Advocates of market rationality, such as Fama and French (1996) and Brav and Heaton (2002), postulated that overreaction and underreaction can be well explained under the efficient market paradigm. Promoters of behavioral biases, see, for example, Lam, Liu, and Wong (2010, 2011, henceforth LLW) and the references therein developed several behavioral models to explain the overreaction and underreaction phenomena. In contrast with the efficient market theory (Lam, Wong, and Wong, 2006; Lean, McAleer, and Wong, 2010), these models suggest that sophisticated investors may earn superior returns by taking advantage of underreaction and overreaction without assuming extra risk.

Most behavioral models adopt the bounded rationalism approach which violate some assumptions under rational expectations in the classical asset-pricing theory. Basically, there are five ingredients in the classical asset-pricing model, namely, an economic structure for asset price dynamics, rational agents' beliefs on asset prices, the structure of market information, rational agents' predictions by updating their views based on available information and rational agents' investment decisions. Meanwhile, the following four assumptions are imposed in the rationality paradigm: 1) the agents' knowledge of the economic structure for asset price dynamics is correct, Sargent (1993); 2) agents can process the immediately and homogenously distributed information; 3) investors update their beliefs using a version of the Bayes' rule; and 4) investors' choices are determined by Savage's notion of subjective expected utility.

In a typical behavioral model, one of the four assumptions mentioned above is violated as investors' behavioural bias would induce a deviation from the rational paradigm. For example, Barberis, Shleifer, and Vishny (1998, henceforth BSV) supposed that while earning announcements follow a random walk, investors adopting conservative and representative heuristics believe that the announcements fall into either a trending regime or a mean-reverting regime, which may be described by a two-state Markov chain. They then deduced that such violation against Assumption 1 leads to both short-term underreaction and long-term overreaction in the market. Through deviation from Assumption 2, Daniel, Hirshleifer, and Subrahmanyam (1998, henceforth DHS) demonstrated that the market exhibits short-term underreaction and long-term overreaction since some investors having private information are overconfident. Grinblatt and Han (2005) believed that investors refuse to sell in a falling market since they are unwilling to admit their mistakes. This leads to formation of market momentum, Their approach violates Assumption 4 since they adopted the prospect theory, (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Broll, Egozcue, Wong, and Zitikis, 2010; Egozcue, Fuentes García, Wong, and Zitikis, 2011), where the maximization of investors' final wealth may not be the sole criterion for making investment decisions. Gervais and Odean (2001) developed a behavioral model, where insider traders assign excessive weights to their past successful predictions of a security's dividend via a learning bias factor and use an updated probability that is larger than that derived from the Bayes' rule. This model violates Assumption 3.

Behavioral finance, which aims at combining psychological phenomena with finance theories to explain market anomalies, may be traced to the early work of Slovic (1972). Several psychologists have observed that when new information emerges, people are too conservative and slow in changing their prior beliefs. Furthermore, Edwards (1968) established a Bayesian model to underweigh useful statistical evidence, but to put more weight on investors' priors since he observes that conservative investors might pay little attention, or even no attention, to the recent earnings announcements and still hold their prior beliefs based on past earnings in their valuation of shares. Similarly, Grether (1980) documented that individuals who exhibit conservatism update their beliefs too slowly in the face of new evidence. Nonetheless, Klein (1990), Mendenhall (1991), and Abarbanell and Bernard (1992) argued that investors tend to underreact to new information.

On the other hand, representative heuristic, the bias in which the individuals expect key population parameters to be "represented" in any recent sequence of data, has been shown in many experimental studies, see, for example, DeBondt and Thaler (1985), Lakonishok, Shleifer, and Vishny (1994), Barberis, Shleifer, and Vishny (1998). Tversky and Kahneman (1971) suggested that local representativeness is a belief in the "law of small numbers," and investors may find that even small samples are highly representative of the populations. In addition, Kahneman and Tversky (1973) found that a person following this heuristic evaluates the probability of an uncertain event, by the degree to which the essential properties resemble to its parent population and reflects the salient features of the process by which it is generated.

Griffin and Tversky (1992) reconciled conservatism with representativeness by assuming that people update their beliefs based on the "strength," the salient and extreme aspects of the evidence; and "weight," the statistical information, such as sample sizes. In this setup, when revising their forecasts, people overemphasize on the strength of the evidence and de-emphasize on its weight. Conservatism would follow when facing evidence with high weight but low strength whereas overreaction occurs in a manner consistent with representativeness when the evidence has high strength but a low weight. Furthermore, Shefrin and Statman (1995) found that investors rely on representative heuristics in forming expectations because they tend to regard good stocks as the stocks of large companies with low book-to-market ratios.

In this paper, we adopt the LLW's model for weight assignments to develop certain important characteristic of stock returns during financial crisis and its subsequent recovery. More specifically, an asset pricing model based on a modified random walk model for the earnings announcement of an asset is proposed. The modified random walk model can incorporate the impact of a financial crisis on the earnings of a firm. To provide a quantitative description for investors' representative and conservative heuristics, we follow the LLW's pseudo-Bayesian approach and assume that the likelihood function for earning shocks of the stock in a Bayesian paradigm is weighted by investors' behavioral biases. The degree of deviation of weight from the standard Bayesian approach, which assigns equal weights to data, could quantitatively reflect investors' level of behavioral biases. Then the price dynamics of the asset are derived using the pseudo-Bayesian approach which will then be used to develop some characteristics of stock returns. The proposed model can be used to study and explain some market anomalies including short-term underreaction, long-term overreaction, and excess volatility during financial crisis.

The rest of the paper is organized as follows. In the next section, we present the asset pricing theory based on the modified random walk model for the earnings announcement of an asset. Section three first discusses the pseudo-Bayesian framework to update information about earning shocks in the asset pricing theory. Then we derive the price dynamics and develop some properties for the stock returns based on the pseudo-Bayesian asset pricing theory. Section four is devoted to discussing how cognitive biases are reflected in the weight assignment schemes in Section three. The final section gives some concluding remarks. The proofs of the results are placed in Appendix.

2 ASSET PRICING MODEL WITH FINANCIAL CRISIS

In BSV, a model for market sentiment was considered, where a representative investor observes the earnings of an asset and updates his/her belief to price the asset. It was assumed that N_t, the earnings announcement of the asset at time t, follows the following random walk:

$$N_t = N_{t-1} + y_t \ , \tag{1}$$

where y_t is an earnings shock at time t.

Using a discounting model based on rational expectation, (see, for example, Thompson and Wong (1991, 1996) and Wong and Chan (2004)), the asset is priced at time t as P_t which is given by:

$$\begin{aligned} P_t &= E_t\left[\frac{N_{t+1}}{1+r} + \frac{N_{t+2}}{(1+r)^2} + \cdots\right] \\ &= \frac{N_t}{r} + \frac{1+r}{r} \times \left[\frac{E_t[y_{t+1}]}{1+r} + \frac{E_t[y_{t+2}]}{(1+r)^2} + \cdots\right] \ , \end{aligned} \tag{2}$$

where r is the discount rate, or the investor's anticipated return, which is assumed to be a given positive constant.

In (2), $E_t[\cdot]$ represents the investor's conditional expectation given the information set Ω_t describing all information available to the investor at time t. Here we assume that y_t is Ω_t-measurable, (i.e., the value of y_t is known exactly given information Ω_t up to and including time t). Consequently, by definition, both N_t and P_t are Ω_t-measurable.

Instead of using the random walk model in (1), we consider here a modified random walk model which can incorporate the impact of a financial crisis on the dynamics of the earnings announcement. The modified random walk model is presented in the sequel.

To illustrate the main idea of the proposed model, we first consider a simplified situation that there are two states of an economy, namely, a normal economic condition and an economic condition under a financial crisis. When the economic condition is normal, we assume that the earnings announcement of the asset follows a random walk model. Unfortunately, if the financial crisis occurs at time t_0, then the time after t_0, the earnings announcement of the asset also follows another random walk model which is discounted by a factor $\delta_0 < 0$ such that

$$N_t = \begin{cases} N_{t-1} + y_t, & t < t_0 \ , \\ \delta_0 + N_{t-1} + y_t, & t \geq t_0 \ . \end{cases} \tag{3}$$

In other words, the one-step, conditional expected earning of the asset is reduced by the amount of $|\delta_0|$ in each period after the crisis has occurred at time t_0. The model

(3) reflects the situation where the market is dominated by some pessimistic investors who think that the economy will never recover and that the crisis is the end of the world.

More generally, we consider the situation where the economy has three states, namely, a normal economic condition, an economic condition under a financial crisis and the condition under recovery. Again we suppose that when the economy is in a normal condition, the earnings announcement of the asset follows a random walk model. If the economy is experiencing a financial crisis starting at time t_0, the earnings announcement of the asset after time t_0 is the modified random walk model where the one-period conditional expected earning is discounted by the amount of $|\delta_0|$ with $\delta_0 < 0$. During recovery, the earnings announcement of the asset will follow a random walk with drift $\delta_1 > 0$ such that

$$N_t = \begin{cases} N_{t-1} + y_t, & t < t_0 \quad , \quad t \geq t_2 \;\; ; \\ \delta_0 + N_{t-1} + y_t, & t_0 \leq t < t_1 \;\; ; \\ \delta_1 + N_{t-1} + y_t, & t_1 \leq t < t_2 \; ; \end{cases} \tag{4}$$

for some $t_0 < t_0 + 1 \leq t_1 < t_1 + 1 \leq t_2$.

The rationale of considering this more complicated model is to incorporate the impact of recovery from a crisis on the earnings announcement of the asset. Specifically, when the economy goes into a crisis at time t_0, the stock market starts to crash at that time. At time t_1, economy is expected to recover, and thus the stock market starts to rise. Whereas, at time t_2, the economy becomes stable and the stock market follows the random walk without drift again.

From now on, we impose the following standard assumptions which are modification of those in Lam, Liu, and Wong (2010, 2011) so as to adapt them to the pseudo-Bayesian framework to be presented in the next section.

Assumption 1: The earnings announcement process $\{N_t\}$ can follow a random walk model (1), the random walk model with drift δ_0 in (3), or the random walk model with drifts δ_0 and δ_1 in (4). Furthermore, the earnings shocks $\{y_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with a common distribution being a Gaussian distribution with constant mean μ and variance σ_y^2.

Assumption 2: The representative agent knows the nature of the random walk model, except that the mean μ is unknown. The agent estimates μ using observations about the earning shocks $\{y_t\}$. To simplify our discussion, we assume that the agent knows σ_y^2.

Assumption 3: The agent uses a "biased" statistical method to update his or her belief in such a way that the agent's behavioral biases are reflected.

3 A PSEUDO-BAYESIAN APPROACH AND PROPERTIES OF STOCK RETURNS

Before we present the pseudo-Bayesian approach adopted by a behaviorally biased agent, we first describe the standard (or rational) Bayesian approach to update information on the mean level of the earnings shock. To simplify our discussion, we consider a vague, or improper, prior for the unknown mean μ [1]. That is,

$$P_0(\mu) \propto 1 \ ,$$

see, for example, DeGroot (1970), Matsumura, Tsui and Wong (1990), Wong and Bian (2000) for related discussions.

The likelihood function of μ given the observed earning shocks $\{y_t\}$ is:

$$L(y_1, y_2, \cdots, y_t|\mu) = \prod_{i=1}^{t} L\big(y_{t-i+1}|\mu\big) \ . \tag{5}$$

It is well-known that by applying the Bayes' formula, the posterior distribution of μ given $\{y_1, y_2, \cdots, y_t\}$ is:

$$P\big(\mu \mid y_1, y_2, \cdots, y_t\big) \propto \prod_{i=1}^{t} L\big(y_{t-i+1} \mid \mu\big) \ . \tag{6}$$

In this standard Bayesian approach, an equal weight is placed on each observation in $y_1, y_2, \cdots, y_t$. In consistent with the predictions of traditional efficient markets, this rational expectations asset-pricing theory assumes that investors can have access both to the correct specification of the "true" economic model and to unbiased estimators of the model parameters, (see, for example, Friedman (1979)). Obviously, if the rational investor is endowed with an objectively correct prior and the correct likelihood

[1]Note that in a Bayesian statistical paradigm, an unknown parameter is viewed as a random quantity and a prior distribution is then assigned to this random quantity.

function, he/she will obtain the rational expectation equilibrium. Consequently, any structural irrationally induced financial anomaly would disappear. Attainment of such structural knowledge on convergence to a rational expectation solution has been studied widely in the literature. For example, Blume and Easley (1982) and Bray and Kreps (1987) observed that investors have to recognize and incorporate how their beliefs about the unknown essential features of an economy influence the structural model of the economy. However, the extreme knowledge required in these models is implausible. Blume and Easley (1982) showed that if investors do not recognize the effect of learning on prices to obtain equilibrium, that convergence of beliefs is not guaranteed within a general equilibrium learning model.

Nonetheless, as evidence has mounted against the traditional Bayesian model, theories of financial anomalies have to be developed by relaxing some of those assumptions imposed in the standard theories. One approach is to assume that investors are plagued with cognitive biases, (see, for example, Slovic (1972)), and they may incorrectly assign different weights to different observations. To model such behavioral biases, we assume that they place weight ω_1 on the most recent observation y_t, ω_2 on the second most recent observation y_{t-1} , and so on, with the possibility that ω_i's may not equal to 1. Then we consider the following weighted likelihood function associated with the vector of weights $\omega := (\omega_1, \omega_2, \cdots, \omega_t)$:

$$L^{\omega}(y_1, y_2, \cdots, y_t|\mu) = \prod_{i=1}^{t} L\big(y_{t-i+1}|\mu\big)^{\omega_i} \ . \tag{7}$$

where L^{ω} represents the weighted likelihood function depending on the subjective weighted ω. Then, by the Bayes' formula, the posterior distribution of μ given $\{y_t\}$ is:

$$P\big(\mu \mid y_1, \cdots, y_t\big) \propto \prod_{i=1}^{t} L\big(y_{t-i+1} \mid \mu\big)^{\omega_i} \ . \tag{8}$$

Consequently, the posterior mean and posterior variance of the unknown mean μ can be obtained from the posterior distribution of μ. Using these results, we can derive the price and return dynamics of the stock under the behavioral model. We present these dynamics in Proposition 1.

Proposition 1 ***(Price and return dynamics in the pseudo-Bayesian approach)*** *Under a pseudo-Bayesian approach with a vague prior, the random walk N_t as stated in (1), (3), or (4) and an incorrect likelihood $L^{\omega}(\mu)$ as stated in (7), for any $k \geq 1$ the predictive mean $E_t[y_{t+k}]$ of the future earning shock y_{t+k} given*

$\{y_1, y_2, \cdots, y_t\}$, *and the posterior variance* σ_t^2 *of* μ *given* $\{y_1, y_2, \cdots, y_t\}$ *are, respectively, given by:*

$$\begin{aligned} E_t[y_{t+k}] &= \frac{\omega_t y_1 + \cdots + \omega_1 y_t}{s_t} := d_t \ , \\ \sigma_t^2 &= \frac{\sigma_y^2}{s_t} \ , \end{aligned} \tag{9}$$

where $s_t = \Sigma_{i=1}^t \omega_i$.

1. *If the random walk* $\{N_t\}$ *follows (1), then the price at time* t *using the rational expectation pricing model in (2) becomes:*

$$E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] = \frac{N_t}{r(1+r)^k} + \frac{[(1+k)r+1]d_t}{r^2(1+r)^k} \ . \tag{10}$$

2. *If the random walk* $\{N_t\}$ *follows (3), then the price at time* t *using the rational expectation pricing model in (2) is given by:*

$$\begin{aligned} E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] &= \frac{N_t}{r(1+r)^k} + \frac{[(1+k)r+1]d_t}{r^2(1+r)^k} + \\ & \begin{cases} \dfrac{\delta_0}{r^2(1+r)^{a_t-2}} & \text{if} \ \ t < t+k < t_0 \ , \\ \frac{[(k+2-a_t)r+1]\delta_0}{r^2(1+r)^{k+2}} & \text{if} \ \ t < t_0 \le t+k \ , \\ \frac{[(1+k)r+1]\delta_0}{r^2(1+r)^k} & \text{if} \ \ t_0 \le t < t+k \ , \end{cases} \end{aligned} \tag{11}$$

where $a_t = \max(\lceil t_0 - t \rceil, 0)$.

3. *If the random walk* $\{N_t\}$ *follows (4), then the price at time* t *using the rational*

expectation pricing model in (2) is given by:

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & \frac{N_t}{r(1+r)^k} + \frac{d_t[(k+1)r+1]}{r^2(1+r)^k} + \\
& \begin{cases}
\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-a_t}-1\right] + \frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right) & \text{if} \quad t<t+k<t_0 \\
\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-k-2}([k+2-a_t]r+1)-1\right] + \frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right) & \text{if} \quad t<t_0\le t+k<t_1 \\
\frac{(b_t-a_t)\delta_0}{r(1+r)^k} + \frac{\delta_1}{r^2(1+r)^{c_t-2}}\left[(r+1)^{c_t-k-2}([k+2-b_t]r+1)-1\right] & \text{if} \quad t<t_0, t_1\le t+k<t_2 \\
\frac{(b_t-a_t)\delta_0+(c_t-b_t)\delta_1}{r(1+r)^k} & \text{if} \quad t<t_0, t_2\le t+k \\
\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-k-2}([k+1]r+1)-1\right] + \frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right) & \text{if} \quad t_0\le t<t+k<t_1 \\
\frac{\delta_0(b_t-1)}{r(1+r)^k} + \frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-k-2}([k+2-b_t]r+1)-1\right) & \text{if} \quad t_0\le t<t_1\le t+k<t_2 \\
\frac{\delta_0(b_t-1)+\delta_1(c_t-b_t)}{r(1+r)^k} & \text{if} \quad t_0\le t<t_1, t_2\le t+k \\
\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-2-k}([k+1]r+1)-1\right) & \text{if} \quad t_1\le t<t+k<t_2 \\
\frac{\delta_1(c_t-1)}{r(1+r)^k} & \text{if} \quad t_1\le t<t_2\le t+k \\
0 & \text{if} \quad t_2\le t<t+k\ ,
\end{cases}
\end{aligned}
\tag{12}
$$

where $a_t = \max\{\lceil t_0-t\rceil, 0\}$, $b_t = \max\{\lceil t_1-t\rceil, 0\}$, $c_t = \max\{\lceil t_2-t\rceil, 0\}$, *and* $d_t = \frac{\omega_t y_1+\cdots+\omega_1 y_t}{s_t}$.

The proof of Proposition 1 is in the appendix.

In whatever case, we can have $E_t[P_{t+k}] = \frac{N_t}{r} + \frac{[r(1+k)+1]d_t}{r^2} + c_k$ where c_k may be different in different situations as described above. For P_t, we can have $P_t = \frac{N_t}{r} + \frac{[r+1]d_t}{r^2} + c_0$, where c_0 may also be different in different situations. Consequently, the k-period conditional expected absolute return

$$R_{t,t+k} = \frac{kd_t}{r} + c_k - c_0.$$

Since the constant doesn't effect the variance, so the market volatility will be

$$Var(R_{t,t+k}) = \frac{k^2\sum_{i=1}^t \omega_i^2}{r^2(\sum_{i=1}^t \omega_i)^2}\sigma_y^2.$$

Using Cauchy inequality, we can have $(\sum_{i=1}^t \omega_i)^2 \le t\sum_{i=1}^t \omega_i^2$. On the other hand, since $0 \le \omega_i \le 1$ for $\forall i$, we can obtain that $\sum_{i=1}^t \omega_i^2 \le \sum_{i=1}^t \omega_i$. So, for the market volatility, we can have

$$\frac{k^2}{r^2 t}\sigma_y^2 \le Var(R_{t,t+k}) \le \frac{k^2}{r^2\sum_{i=1}^t \omega_i}\sigma_y^2.$$

As a result, if $s_t = \sum_{i=1}^t \omega_i \to \infty$, we can have $Var(R_{t,t+k}) \equiv 0$. If $s_t = \sum_{i=1}^t \omega_i \to$

$s_\infty < \infty$, we can have

$$Var(R_{t,t+k}) \le \frac{k^2}{r^2 s_\infty}\sigma_y^2.$$

$R_{t,t+1} = \frac{d_t}{r} + c_1 - c_0$, where c_1 is some constance and may be different in different situations. Let $Z = \frac{y_t-\mu}{\sigma_y}$, $D_1(s) = E(Z|Z > s)$ and $D_2(s) = E(Z|Z < -s)$

$$\begin{aligned}
U_+ &= E\Big[R_{t+1}|y_t > \mu + s\sigma_y, \cdots, y_{t-j+1} > \mu + s\sigma_y\Big] \\
&= \frac{1}{r}E\Big[d_t|y_t > \mu + s\sigma_y, \cdots, y_{t-j+1} > \mu + s\sigma_y\Big] + (c_1 - c_0) \\
&= \frac{1}{rs_t}E\Big[\omega_j y_{t-j+1} + \cdots + \omega_1 y_t|y_t > \mu + s\sigma_y, \cdots, y_{t-j+1} > \mu + s\sigma_y\Big] + (c_1 - c_0) \\
&= \frac{\sum_{i=1}^{j}\omega_i}{rs_t}D_1(s) + (c_1 - c_0).
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
U_- &= E\Big[R_{t+1}|y_t < \mu - s\sigma_y, \cdots, y_{t-j+1} < \mu - s\sigma_y\Big] + (c_1 - c_0) \\
&= \frac{\sum_{i=1}^{j}\omega_i}{rs_t}D_2(s) + (c_1 - c_0).
\end{aligned}$$

Consequently,

$$U_t(s,j) = U_+ - U_- = \frac{\sum_{i=1}^{j}\omega_i}{rs_t}(D_1(s) - D_2(s)).$$

Consequently, we can have $U_t(s,j) > 0, \partial U_t(s,j)/\partial s > 0$. **There is no under-reaction or over-reaction!**

Brief Proof. Let $h(s) = \mu + s\sigma_y, g(s) = \mu - s\sigma_y$, we can have

$$\begin{aligned}
D_1(s) &= \frac{\int_{h(s)}^{\infty} yf(y)dy}{\int_{h(s)}^{\infty} f(y)dy \cdot \sigma_y} - \frac{\mu}{\sigma_y} \\
D_2(s) &= \frac{\int_{0}^{g(s)} yf(y)dy}{\int_{0}^{g(s)} f(y)dy \cdot \sigma_y} - \frac{\mu}{\sigma_y}.
\end{aligned}$$

Then, we can obtain that

$$
\begin{aligned}
D_1'(s) &= \frac{f(h(s))h'(s)\Big[\int_{h(s)}^{\infty} yf(y)dy - h(s)\int_{h(s)}^{\infty} f(y)dy\Big]}{(\int_{h(s)}^{\infty} f(y)dy)^2 \cdot \sigma_y} \\
&= \frac{f(h(s))h'(s)\int_{h(s)}^{\infty} f(y)dy(\xi - h(s))}{(\int_{h(s)}^{\infty} f(y)dy)^2 \cdot \sigma_y}
\end{aligned}
$$

where $h(s) < \xi < \infty$, the last equation follows from mean value theorem of integrals. Then we can have $D_1'(s) > 0$

Using a similar argument, we can have

$$
\begin{aligned}
D_2'(s) &= \frac{f(g(s))g'(s)\Big[g(s)\int_0^{g(s)} f(y)dy - \int_0^{g(s)} yf(y)dy\Big]}{(\int_0^{g(s)} f(y)dy)^2 \cdot \sigma_y} \\
&= \frac{f(g(s))g'(s)\int_0^{g(s)} f(y)dy(g(s) - \zeta)}{(\int_0^{g(s)} f(y)dy)^2 \cdot \sigma_y}
\end{aligned}
$$

where $0 < \zeta < g(s)$, note that $g'(s) = -\sigma_y < 0$, so we can have $D_2'(s) < 0$. As a result, we can have $\partial U_t(s,j)/\partial s > 0$. This completes the proof of Proposition 5.

On the other hand, if we define the 1-period return $R_{t,t+1} = P_{t+1} - P_t$, from the above discussion, we can know that we can obtain the same conclusion as Lam, Liu and Wong(2009)('A Pseudo-Bayesian Model in Financial Decision Making with Implications to Market Volatility, Under- and Overreaction'). This can be explained in the following way. Nor matter which model N_t, the earnings announcement of the asset at time t follows, only the mean term will change. That's there is some shift if N_t follows some change-point model as model (2) or (3) instead of model 1. But as for the variance term, no change will occur.

From Proposition 1, we observe that the conditional expected present value of the asset at time $t+k$ given information Ω_t up to and including the current time t depends on the current and the future earning shocks. For example, in the simplest random walk model, the current earning announcement only depend on (1) the current earnings announcement and (2) the current as well as the expected future earning shocks. However, it is generally believed that the price of the asset also depends on the economic situation. In the view of economic cycles, the economy will experience an expansion period after suffering from a period of financial recession. Furthermore, in a long run, the economy will go back to the normal status. The present value of the asset is, therefore, proportional to the predictive mean of the future earning shocks, the current earnings announcement, the duration of economic recovery and the recovery rate of the economy. It is also inversely proportional to the risk-free interest rate, the duration of economic downturn and the deteriorating rate under economic crisis.

To fully describe this situation, we may consider the random walk $\{N_t\}$ follows (4). Under this circumstance, when the current economy is in the state of just before the economic downturn or under the recession period, the price of asset not only depends on the current earnings announcement, the predictive mean of the future earning shocks and the risk-free interest rate, but also depends on how long and serious of the impact of both the economic turmoil and the economic expansion on the price of the asset. If the duration of the economic turmoil is longer or the effect of the economic turmoil is more serious, the price of the asset becomes lower. This is reflected in the coefficients of δ_0 and δ_1, respectively. Similarly, the coefficient of δ_1 is determined by the duration of the economic expansion while the value of δ_1 depends on the level of the economic expansion. In particular, when we are at the economic expansion period, (i.e., $t_1 \leq t < t_2$), the effect of the term δ_0 becomes vanish, and only the term of δ_1 reflects the effect of the economy shift. Furthermore, if the bad days and the good days of the economy are all gone, (i.e., $t_2 \leq t$), the estimation on the price of the asset is the same as that obtained from the random walk, $\{N_t\}$, from (1).

To be more precise, under a different future economic condition, the effect of the current earnings announcement and the expected future earning shocks, $E_t(y_{t+j})$, $j = 1, 2, \cdots$, on the asset price are the same in all scenarios. It is because these factors represent the cash flows of the asset in the absence of the impact from the status of the economy. The impact of the economic condition on the asset price is described by both δ_0 and δ_1 as well as their coefficients. To begin with, we may consider four different cases under the same condition that the current economy is in the state of just before the economic downturn. Also, we have the following inequalities:

$$\frac{b-a}{(1+r)^k} < \frac{\left[(r+1)^{b-a}-1\right]}{r(1+r)^{b-2}} \ ,$$
$$\frac{1}{r(1+r)^{b-2}}[(r+1)^{b-k-2}([k+2-a]r+1)-1] \leq \frac{b-a}{(1+r)^k} \ , \text{ and}$$
$$(r+1)^{b-k-2}([k+2-a]r+1) < (1+r)^{b-a} \ ,$$

for $k > b > a$.

In the first case, when the economic environment at time $t+k$ is in the state of just before the economic downturn, the expected present value of the asset at time $t+k$ depends on the coefficient of δ_0 and δ_1, (i.e., the duration of both economic recession and economic expansion, and the level of the impact under both economic conditions on the asset. Therefore, the deviation of the asset price from its price under the stable economic condition depends on the actual effect from the change of the economic condition.

In the second case, if the economic environment at time $t+k$ is now in the economic recession, the coefficient of δ_0 is lower than that in the first case. It implies that if the current economy is just before the economic recession, the expected present value of the asset at time $t+k$ is lower in the first case, If the economy at time $t+k$ is just before the recession and it is higher than that in the second case, the economy at time $t+k$ is in downturn period. It is because if the economy at time $t+k$ is in the recession period, only part of the asset loss is realized before the time $t+k$. However, if the economy at time $t+k$ is just before the recession, the total negative impact of the economic recession is counted on the cash flow of the asset at time $t+k$.

In the third case, if the economic environment at time $t+k$ is just after the economic recession and is now in the state of the economic expansion, the present value of the asset may be higher or lower than that of the first two cases. We observe that the coefficient of δ_1 is lower than those in the first two cases while the coefficient of δ_0 is higher than that in the second case. This implies that the asset value at the economic expansion period is even lower than that in the economic recession period. The main reason is that even though we experience the economic expansion at time $t+k$, the additional amount of asset return is required to discount back to the current time by going through the whole economic recession period, and this devaluation of the asset price is significant in this recession period, which is in between time t an time $t+k$.

In the fourth case, when the economic environment at time $t+k$ is just after the economic expansion period, the coefficients of both δ_0 and δ_1 are smaller than those in the first case. However, the coefficient of δ_0 and the coefficient of δ_1 is higher and lower than those in the second case, respectively. This implies that the present value in the fourth case is lower than that in the second case. Comparing with the third case, the coefficient of δ_1 in the fourth case is larger. Therefore, the present value of the asset is highest when the economy at time $t+k$ is in the recession period, medium when the economy at time $t+k$ is after the expansion period and lowest when the economy at time $t+k$ is in the expansion period.

We now consider three different cases, (i.e., fifth to seventh), under the same condition that the current economy is in the state of economic downturn. In the fifth case, when the economic environment at time $t+k$ is in the state of economic downturn, we may compare the expected present value of the asset at time $t+k$ in this case to that in the first case, (i.e., the present asset value when the current economy is in the state of the economy just before the downturn while the economic environment at time $t+k$ is still in the state of economic downturn. We observe that the present asset value at time $t+k$ in this case is higher than that in the case where the current economy is in the state of the economy just before the downturn. It may be attributed to the fact that part of the negative effect from the economic downturn has already been digested when the current economy is in the state of economic downturn. The impact of economic downturn on the current scenario is relatively small. In the sixth case, when the economic environment at time $t+k$ is in the state of economic expansion, the coefficients of both δ_0 and δ_1 are smaller than those in the fifth case. In the seventh case, when the economic environment at time $t+k$ is in the state of just after the economic expansion, the coefficient of δ_1 is larger than that in the sixth case. Consequently, the present asset value in the economy at time $t+k$ after the expansion period is higher than that in the expansion period. It is because if the the economy at time $t+k$ is after the expansion period, then the expansion period is closer to the time t and when the present value is under consideration, the asset price will be higher. On the other hand, when the economy at time $t+k$ is in the expansion period, the time interval between the current time and the economic expansion period becomes larger and it makes the present value smaller.

We can also consider another two different cases, (i.e., eighth to ninth), under the same condition that the current economy is in the state of economic expansion. In this situation, the economy at time $t+k$ is either in the expansion period or after the expansion period. Similarly to the previously obtained results, the present asset value is higher if the economy at time $t + k$ is after the expansion period.

Finally, if the current economy is just after the economy expansion, then there is no impact from the structural changes in economic conditions.

On the other hand, some of the pessimistic investors may believe that the economic downturn will unavoidably occur. In their opinion, the price of the asset is eventually reduced by a future economic turmoil. Once an economic downturn occurs, the effect of the economic turmoil will sustain forever. In this situation, when the investor believes that the economy at time $t + k$ is just before the economic downturn, the present value of the asset at time $t + k$ is the highest. However, when the investor believes that the economy at time t is just before the economic downturn while the economy at time $t + k$ is in the economic downturn, the present value of the asset at time $t + k$ will be reduced. Finally, when the investor believes that the economy at time t is in the economic downturn period, the present value of the asset at time $t+k$ is smallest. The main reason is that in the last scenario, the pessimistic investor believes that the economic downturn occurs from the present time and will last forever.

4 HOW COGNITIVE BIASES ARE REFLECTED IN THE WEIGHT ASSIGNMENT SCHEMES?

In the model developed in the last section, we incorporate general weights on observations into a simple asset-pricing setup. This allows us to examine the price formation process under a rational expectation approach with biased weights. This approach enables practitioners and academic researchers to compare ways in which investors, with or without cognitive biases, incorporate their prior beliefs into the historical data to estimate the valuation-relevant parameters that can lead to anomalous asset-price behavior. We note that the idea of using different weights on evidence is not new in the finance literature. For example, Brav and Heaton (2002) considered weights given by:

$$\omega_1 = \cdots = \omega_{\frac{t}{2}} = 1 \quad \text{and} \quad \omega_{\frac{t}{2}} = \cdots = \omega_t = 0 \ ,$$

where t is an even number.

Under this weighting scheme, investors simply ignore the distant half of the available data. Also, it is common in the psychological literature to assume that investors calculate the posterior mean, which is a weighted average rather than a simple average as suggested by a correct Bayesian approach. In this paper, we follow LLW to use a more general assumption that investors may use weights, ω_1, ω_2, $\cdots$, satisfying

$0 \leq \omega_i \leq 1$ for all i. By allowing more flexibility in the choice of weights, investors' various behavioral biases can be represented quantitatively. Specifically, in (A), (B), and (C) below, we spell out three weight assignment schemes to characterize the conservative and/or representative heuristics.

(A) Investors using a conservative heuristic assign weights as: $0 \leq \omega_1 < \omega_2 < \cdots < \omega_{n_0} = \omega_{n_0+1} = \cdots = 1$ for some integers $n_0 \geq 1$,

(B) investors using a representative heuristic assign weights as: $1 = \omega_1 = \omega_2 = \cdots = \omega_{m_0} > \omega_{m_0+1} > \omega_{m_0+2} > \cdots \geq 0$ where m_0 is a positive integer, and

(C) investors using both conservative and representative heuristics assign weights as: $0 \leq \omega_1 < \omega_2 < \cdots < \omega_{n_0} = \omega_{n_0+1} = \cdots = \omega_{m_0} = 1 > \omega_{m_0+1} > \cdots \geq 0$ for $1 \leq n_0 \leq m_0$.

Note that the weight assignment scheme of $\omega_1 < \omega_2 < \cdots < \omega_{n_0} = 1$ is consistent with the psychological literature on conservative heuristics as reviewed in the introduction. Basically, people are overconservative in that they underweigh recent information and overweigh prior information. The parameter n_0 reflects the conservative heuristic that most recent n_0 observations are underweighted. If Edwards (1968) is right in that it takes two to five observations to do one observation's worth of work in inducing a subject to change his/her opinions, $\omega_1, \omega_2, \cdots, \omega_{n_0}$ can be substantially less than 1 for $n_0 \leq 5$. The smaller are the weights, the more conservative are the investors. Thus, the magnitudes of the weights $\omega_1, \omega_2, \cdots, \omega_{n_0}$ can be used to measure the degree of conservatism. The evidence suggests that underreaction reflects the uncertainty regarding possible structural change in the data and a lack of knowledge that a change occurred. This will result in a failure to fully incorporate the price implications of this change into the estimation of the valuation-relevant parameters.

The weight assignment in Scheme B is consistent with the psychological literature on the representative heuristic, as reviewed in the introduction. The representative heuristic in behavioral finance is often described as the tendency of experimental subjects to overweigh recent clusters of observations and underweigh older observations that would otherwise moderate beliefs. Heavy weights on recent data could be a reaction to concern with structural changes. Whenever such changes occur, the weights placed on recent data will be very high, or similarly, the weights placed on the older data will be very low, which will result in a pattern of overreaction caused by the representative heuristic. The representative heuristic is characterized by a parameter m_0 showing that the investor underweight the observations beyond the most recent

m_0 data points. Here, the parameter m_0 arises from the "law of small numbers," (see Tversky and Kahneman 1971), in the mind of the investor. Because of their representative heuristic, investors have the tendency to treat a small sample size, like m_0, as if it were large enough to represent the whole population. Consequently, they assign weights much smaller than 1 for observations beyond the most recent m_0 observations. Put it differently, the weights $\omega_{m_0+1}, \omega_{m_0+2}, \cdots$ are assigned to be much smaller than 1. Furthermore, we assume here that $\sum_{i=m_0+1}^{\infty} \omega_i$ is infinite, (i.e., $\sum_{i=m_0+1}^{\infty} \omega_i = \infty$), because if the sum equals to infinity, the law of large numbers is still at work. For a genuine belief in the law of small numbers, the sum should be finite, meaning that the small sample of the most recent observations can play an overwhelming role in the inference process.

Our model formulation asserts that investors are influenced by the conservative and representativeness heuristics simultaneously. This is different from the regime switching formulation in BSV in which investors are under the influence of one heuristic and then suddenly shift to another regime of being influenced by another heuristic. In other words, conservatism and representativeness are not mutually exclusive and investors can be simultaneously influenced by both heuristics at any point in time. When the investor is under the influence of both heuristics, the model has two parameters n_0 and m_0 as described above. Here, conservatism is reflected by the existence of $n_0 > 0$ and the smallness of the sum $\omega_1 + \omega_2 + \cdots + \omega_{n_0-1}$, and representativeness is reflected by the existence of $m_0 < \infty$ and the smallness of the sum $\omega_{m_0+1} + \omega_{m_0+2} + \cdots$. Notice that a type (C) investors degenerate into a type (A) investors when $m_0 = \infty$ and degenerate into a type (B) investors when $n_0 = 0$. Also when $m_0 = \infty$ and $n_0 = 0$, all weights are equal to 1 and the investor has no behavioral bias. In this sense, the third type of investors embraces all of the other types. Therefore, it suffices to consider investors of the third type.

In general, the k-period conditional expected absolute return, $R_{t,t+k}$ is given by the conditional expectation of $P_{t+k} - P_t$ given the current observable information Ω_t, where P_t is Ω_t-measurable. Then we have the following corollary for the value of $R_{t,t+k}$ at time t.

Corollary 1 ***(Expected Returns in the pseudo-Bayesian approach)***
Under the pseudo-Bayesian approach with a vague prior and an incorrect likelihood $L(\mu)$ as stated in (7),

1. *if the random walk $\{N_t\}$ follows (1), then the expected k-period return, $R_{t,t+k}$, from time t to time $t+k$ under the rational expectation pricing model in (2) is*

given by:

$$\begin{aligned} R_{t,t+k} &= E_t(P_{t+k} - P_t) \\ &= \frac{N_t}{r(1+r)^k} + \frac{[(1+k)r+1]d_t}{r^2(1+r)^k} - P_t \ . \end{aligned} \tag{13}$$

2. *if the random walk $\{N_t\}$ follows (3), then the expected k-period return, $R_{t,t+k}$, from time t to time $t+k$ under the rational expectation pricing model in (2) is given by:*

$$\begin{aligned} R_{t,t+k} &= E_t(P_{t+k} - P_t) \\ &= E_t(P_{t+k}) - P_t \ , \end{aligned} \tag{14}$$

where $E_t(P_{t+k})$ is defined in (11).

3. *if the random walk $\{N_t\}$ follows (4), then the expected k-period return, $R_{t,t+k}$, from time t to time $t+k$ under the rational expectation pricing model in (2) is given by:*

$$\begin{aligned} R_{t,t+k} &= E_t(P_{t+k} - P_t) \\ &= E_t(P_{t+k}) - P_t \ , \end{aligned} \tag{15}$$

where $E_t(P_{t+k})$ is defined in (12).

5 CONCLUDING REMARKS

Barberis, Shleifer and Vishny (1998) and others have developed Bayesian models to explain investors' behavioral biases using conservative heuristics and representative heuristics in making decisions under uncertainty. To extend their work, Lam, Liu, and Wong (2010) posited that some investors possess conservative and/or representative heuristics that lead them to underweigh recent observations and/or underweigh past observations of the earnings shocks of corporations. They introduced a quantitative pseudo-Bayesian approach to model such investors' behavior. Compared with other behavioral models where investors possess either conservative heuristics at one time or representative heuristics at another time but not both at the same time, this specification captures the essential feature of either conservative or representative biases in a parsimonious model that allows investors to possess conservative or representative heuristics at the same time. Our current paper extended their work further by developing properties for stock returns by incorporating the impacts of a financial crisis and recovery from the crisis, which could then be used to study some market anomalies including short-term underreaction, long-term overreaction, and the excess volatility during the crisis.

Understanding investors' behavior will be useful in making decisions about investments. The information on companies, (Thompson and Wong, 1991, 1996), the economic and financial environment (Broll, Wahl and Wong, 2006; Fong, Lean, and Wong, 2008), technical analysis (Wong, Chew, and Sikorski, 2001; Wong, Manzur, and Chew, 2003; Kung and Wong, 2009) could be used to make better investment decisions. Academic researchers and practitioners could incorporate the theory developed in this paper with the mean-variance rule (Wong, 2007; Wong and Ma, 2008; Bai, Wang, and Wong, 2011), CAPM statistics (Leung and Wong, 2008), VaR rule (Ma and Wong, 2010), portfolio optimization (Bai, Liu, and Wong, 2009, 2011; Egozcue and Wong, 2010), or other advanced econometric techniques (Wong and Miller, 1990; Li and Lam, 1995; So, Li and Lam, 1997; So, Lam and Li, 1998; Tiku, Wong, Vaughan, and Bian, 2000; Wong and Bian, 2005; Bai, Wong, and Zhang, 2010; Bai, Li, Liu, and Wong, 2011) to make better investment decisions. Another extension to improve investment decision-making is to study behaviors of different types of investors (Wong and Li, 1999; Li and Wong, 1999; Wong and Chan, 2008) or to incorporate stochastic dominance criteria (Gasbarro, Wong and Zumwalt, 2007; Post, 2003; Wong, Phoon, and Lean, 2008) to study investors' conservative and representative heuristics. For example, Fong, Wong, and Lean (2005), Wong, Thompson, Wei, and Chow (2006), and Sriboonchitta, Wong, Dhompongsa, and Nguyen, (2009) find that it is winners dominate losers in the sense of the second order ascending stochastic dominance while losers dominate winners in the sense of the second order descending stochastic dominance, inferring that risk averters will prefer to invest in winners whereas risk seekers will prefer to invest in losers. This finding could explain why the momentum profit could still exist after discovery.

In addition, we note that Lam, Liu, and Wong (2008) have developed three new test statistics, including two ordered tests and a rank correlation test, and apply them to examining the under- and overreaction hypothesis in global markets consisting of 25 national market indices. They find evidence to support the existence of the magnitude effect in the under- and overreaction hypothesis. Their findings could also support the first part of the underreaction and overreaction hypothesis in DeBondt and Thaler (1985), Jegadeesh and Titman (1993), and others that there exist momentum profits in short periods and contrarian profits in long periods.

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Appendix

Proof of Proposition 1:

Since the exact likelihood for the observation y_i is given by $L(y_i \mid \mu) \propto e^{-\frac{(\mu-y_i)^2}{2\sigma_y^2}}$. The pseudo likelihood in (7) becomes

$$L^{\omega}(\mu) \propto L(y_1 \mid \mu)^{\omega_t} \cdots L(y_t \mid \mu)^{\omega_1} \propto e^{-\omega_t \frac{(\mu-y_1)^2}{2\sigma_y^2}} \cdots e^{-\omega_1 \frac{(\mu-y_t)^2}{2\sigma_y^2}} \propto e^{-\frac{(\mu-\frac{\omega_t y_1+\cdots+\omega_1 y_t}{\omega_1+\cdots+\omega_t})^2}{2(\frac{\sigma_y^2}{\omega_1+\cdots+\omega_t})}} . \tag{16}$$

One could easily obtain the posterior mean and posterior variance as stated in (1) by maximizing the likelihood function of μ as stated in (16).

Readers may refer to Lam, Liu, and Wong (2010) for the proof of P_t when the earning announcements N_t follow the random walk model stated in (1).

Let $a_t = \max\{\lceil t_0 - t \rceil, 0\}$, $b_t = \max\{\lceil t_1 - t \rceil, 0\}$, $c_t = \max\{\lceil t_2 - t \rceil, 0\}$ $(a_t \leq b_t \leq c_t)$, and $d = E_t[y_{t+k}]$, we have the following results:

$$\begin{aligned}
\sum_{i=j}^{z} \left(\frac{1}{1+r}\right)^i &= \frac{1}{r}\left[\left(\frac{1}{1+r}\right)^{j-1} - \left(\frac{1}{1+r}\right)^z\right] , \\
\sum_{i=1}^{z} \frac{i}{(1+r)^i} &= \frac{1}{r^2(1+r)^z}\left[(1+r)^{z+1} - (z+1)r - 1\right] , \quad \text{and} \\
\sum_{i=j}^{z} \frac{i}{(1+r)^i} &= \frac{1}{r^2(1+r)^z}[(1+r)^{z-j+1}(jr+1) - (z+1)r - 1] ,
\end{aligned}$$

for $t_0 < t_0 + 1 \leq t_1 < t_1 + 1 \leq t_2$.

Now we turn to derive the price P_{t+k} under the rational expectations pricing model in (2) with the earning announcements N_t follow the random walk with drifts model stated in (4).
For $t < t + k < t_0$, we have

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}} + \frac{N_{t+k+2}}{(r+1)^{k+2}} + \cdots\right] \qquad (17) \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}} + \frac{N_{t+k+2}}{(r+1)^{k+2}} + \cdots + \frac{N_{t+a_t-1}}{(r+1)^{a_t-1}}\right] + \left[\frac{N_{t+a_t}}{(r+1)^{a_t}} + \cdots\right] \\
= \quad & E_t\left[\frac{N_t + \sum_{i=1}^{k+1} y_{t+i}}{(r+1)^{k+1}} + \frac{N_t + \sum_{i=1}^{k+2} y_{t+i}}{(r+1)^{k+2}} + \cdots + \frac{N_t + \sum_{i=1}^{a_t-1} y_{t+i}}{(r+1)^{a_t-1}}\right] + \\
& E_t\left[\frac{N_t + \sum_{i=1}^{a_t} y_{t+i} + \delta_0}{(r+1)^{a_t}} + \frac{N_t + \sum_{i=1}^{a_t+1} y_{t+i} + 2\delta_0}{(r+1)^{a_t+1}} + \cdots + \frac{N_t + \sum_{i=1}^{v} y_{t+i} + (v - a_t + 1)\delta_0}{(r+1)^{v}} + \cdots +\right] \\
= \quad & \left[\frac{N_t}{(r+1)^{k+1}} + \frac{N_t}{(r+1)^{k+2}} + \cdots\right] + \left[\frac{(k+1)d_t}{(r+1)^{k+1}} + \frac{(k+2)d_t}{(r+1)^{k+2}} + \cdots\right] + \\
& \left[\frac{\delta_0}{(r+1)^{a_t}} + \frac{2\delta_0}{(r+1)^{a_t+1}} + \cdots + \frac{(v - a_t + 1)\delta_0}{(r+1)^{v}} + \cdots +\right] \\
= \quad & \frac{N_t}{r(1+r)^k} + \frac{[(1+k)r+1]d_t}{r^2(1+r)^k} + \frac{\delta_0}{r^2(1+r)^{a_t-2}} \; .
\end{aligned}
$$

For $t < t_0 \leq t + k$, from (17) we get

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & E_t\left[\frac{N_{t+a_t-1} + \sum_{i=a_t}^{k+1} y_{i+t} + (k+1-a_t+1)\delta_0}{(r+1)^{k+1}} + \frac{N_{t+a_t-1} + \sum_{i=a_t}^{k+2} y_{i+t} + (k+2-a_t+1)\delta_0}{(r+1)^{k+2}} + \cdots\right] \\
= \quad & E_t\left[\frac{N_{t+a_t-1} + \sum_{i=1}^{k+1} y_{i+t} + (k+2-a_t)\delta_0}{(r+1)^{k+1}} + \frac{N_{t+a_t-1} + \sum_{i=1}^{k+2} y_{i+t} + (k+3-a_t)\delta_0}{(r+1)^{k+2}} + \cdots\right] \\
= \quad & N_t\left(\frac{1}{(1+r)^{k+1}} + \frac{1}{(1+r)^{k+2}} + \cdots\right) + d_t\left(\frac{(k+1)}{(1+r)^{k+1}} + \frac{(k+2)}{(1+r)^{k+2}} + \cdots\right) + \left(\frac{(k+2-a_t)}{(1+r)^{k+1}} + \frac{(k+3-a_t)}{(1+r)^{k+2}} + \cdots\right) \\
= \quad & N_t\left(\frac{1}{r(1+r)^k}\right) + d_t\left(\frac{(k+1)r+1}{(1+r)^k r^2}\right) + \delta_0(1-a_t)\frac{1}{r^2}\left([(k+2-a_t)r+1](1+r)^{-(k+2-a_t)+1}\right) \\
= \quad & \frac{N_t}{r(1+r)^k} + \frac{[(k+1)r+1]d_t}{(1+r)^k r^2} + \frac{\delta_0(1-a_t)[(k+2-a_t)r+1]}{r^2(1+r)^{k+1-a_t}} \; .
\end{aligned}
$$

For $t_0 \leq t < t+k$, from (17) we obtain

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & E_t\left[\frac{N_t+\sum_{i=1}^{k+1} y_{i+t}+(k+1)\delta_0}{(r+1)^{k+1}}+\frac{N_t+\sum_{i=1}^{k+2} y_{i+t}+(k+2)\delta_0}{(r+1)^{k+2}}+\cdots\right] \\
= \quad & \frac{N_t}{r(1+r)^k}+\frac{[(1+k)r+1](d_t+\delta_0)}{r^2(1+r)^k}\ .
\end{aligned}
$$

Similarly, one could easily obtain the price P_t when the earning announcements N_t follow the random walk with drift model stated in (3).

For $t < t+k < t_0$, from (17) one could easily get

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}}+\frac{N_{t+k+2}}{(r+1)^{k+2}}+\cdots+\frac{N_{t+a_t-1}}{(r+1)^{a_t-1}}\right]+E_t\left[\frac{N_{t+a_t}}{(r+1)^{a_t}}+\cdots+\frac{N_{t+b_t-1}}{(r+1)^{b_t-1}}\right] \\
& +E_t\left[\frac{N_{t+b_t}}{(r+1)^{b_t}}+\cdots+\frac{N_{t+c_t-1}}{(r+1)^{c_t-1}}\right]+E_t\left[\frac{N_{t+c_t}}{(r+1)^{c_t}}+\cdots\right] \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}}+\cdots+\frac{N_{t+a_t-1}}{(r+1)^{a_t-1}}\right]+E_t\left[\frac{N_{t+a_t}}{(r+1)^{a_t}}+\cdots+\frac{N_{t+b_t-1}}{(r+1)^{b_t-1}}\right] \\
& +E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{b_t} y_{t+i}+\delta_1}{(r+1)^{b_t}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{b_t+1} y_{t+i}+2\delta_1}{(r+1)^{b_t+1}}+\cdots+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t-1} y_{t+i}+(c_t-b_t)\delta_1}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t+1} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t+1}}\cdots\right] \qquad (18) \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}}+\cdots+\frac{N_{t+a_t-1}}{(r+1)^{a_t-1}}\right]+E_t\left[\frac{N_{t+a_t-1}+y_{t+a_t}}{(r+1)^{a_t}}+\cdots+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{b_t-1} y_{t+i}}{(r+1)^{b_t-1}}\right] \\
& +E_t\left[\frac{N_{t+a_t-1}+\sum_{i=a_t}^{b_t} y_{t+i}}{(r+1)^{b_t}}+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{b_t+1} y_{t+i}}{(r+1)^{b_t+1}}+\cdots+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t-1} y_{t+i}}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t} y_{t+i}}{(r+1)^{c_t}}+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t+1} y_{t+i}}{(r+1)^{c_t+1}}\cdots\right] \\
& +\delta_0\left[\left(\frac{1}{(r+1)^{a_t}}+\cdots+\frac{(b_t-a_t-1)}{(r+1)^{b_t-2}}\right)+(b_t-a_t)\left(\frac{1}{(r+1)^{b_t-1}}+\frac{1}{(r+1)^{b_t}}+\cdots\right)\right] \\
& +\delta_1\left[\frac{1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right)\right] \\
= \quad & \frac{N_t}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}+\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-a_t}-1\right]+\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right)
\end{aligned}
$$

For $t < t_0 \leq t+k < t_1$, from (18) then

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \;& E_t\left[\frac{N_t+\sum_{i=1}^{k+1} y_{t+i}}{(r+1)^{k+1}}+\cdots+\frac{N_t+\sum_{i=1}^{a_t-1} y_{t+i}}{(r+1)^{a_t-1}}\right]+E_t\left[\frac{N_t+\sum_{i=1}^{a_t} y_{t+i}}{(r+1)^{a_t}}+\cdots+\frac{N_t+\sum_{i=1}^{b_t-1} y_{t+i}}{(r+1)^{b_t-1}}\right] \\
& +E_t\left[\frac{N_t+\sum_{i=1}^{b_t} y_{t+i}}{(r+1)^{b_t}}+\frac{N_t+\sum_{i=1}^{b_t+1} y_{t+i}}{(r+1)^{b_t+1}}+\cdots+\frac{N_t+\sum_{i=1}^{c_t-1} y_{t+i}}{(r+1)^{c_t-1}}\right]+E_t\left[\frac{N_t+\sum_{i=1}^{c_t} y_{t+i}}{(r+1)^{c_t}}+\frac{N_t+\sum_{i=1}^{c_t+1} y_{t+i}}{(r+1)^{c_t+1}}\cdots\right] \\
& +\delta_0\left[\left(\frac{k+2-a_t}{(r+1)^{k+1}}+\cdots+\frac{(b_t-a_t-1)}{(r+1)^{b_t-2}}\right)+(b_t-a_t)\left(\frac{1}{(r+1)^{b_t-1}}+\frac{1}{(r+1)^{b_t}}+\cdots\right)\right]+\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right) \\
= \;& \frac{N_t}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}+\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-k-2}([k+2-a_t]r+1)-1\right]+\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right)
\end{aligned}
$$

For $t < t_0$ and $t_1 \leq t+k < t_2$, from (18) we get

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \;& E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{k+1} y_{t+i}+(k+2-b_t)\delta_1}{(r+1)^{k+1}}+\cdots+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t-1} y_{t+i}+(c_t-b_t)\delta_1}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t+1} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t+1}}\cdots\right] \\
= \;& E_t\left[\frac{N_{t+a_t-1}+\sum_{i=a_t}^{k+1} y_{t+i}+(b_t-a_t)\delta_0}{(r+1)^{k+1}}+\cdots+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t-1} y_{t+i}+(b_t-a_t)\delta_0}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t} y_{t+i}+\delta_0(b_t-a_t)}{(r+1)^{c_t}}+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{c_t+1} y_{t+i}+\delta_0(b_t-a_t)}{(r+1)^{c_t+1}}+\cdots\right] \\
& +\delta_1\left[\left(\frac{k+2-b_t}{(r+1)^{k+1}}+\cdots+\frac{(c_t-b_t-1)}{(r+1)^{c_t-2}}\right)+(c_t-b_t)\left(\frac{1}{(r+1)^{c_t-1}}+\frac{1}{(r+1)^{c_t}}+\cdots\right)\right] \\
= \;& \frac{N_t+(b_t-a_t)\delta_0}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}+\frac{\delta_1}{r^2(1+r)^{c_t-2}}\left[(r+1)^{c_t-k-2}([k+2-b_t]r+1)-1\right]
\end{aligned}
$$

For $t < t_0$ and $t_2 \leq t+k$, from (18) then

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \;& E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{k+1} y_{t+i}+(c_t-b_t)\delta_1}{(r+1)^{k+1}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{k+2} y_{t+i}+(c_t-b_t)\delta_1}{(r+1)^{k+2}}+\cdots\right] \\
= \;& E_t\left[\frac{N_{t+a_t-1}+\sum_{i=a_t}^{k+1} y_{t+i}+(c_t-b_t)\delta_1+(b_t-a_t)\delta_0}{(r+1)^{k+1}}+\frac{N_{t+a_t-1}+\sum_{i=a_t}^{k+2} y_{t+i}+(c_t-b_t)\delta_1+(b_t-a_t)\delta_0}{(r+1)^{k+2}}+\cdots\right] \\
= \;& \frac{N_t+(b_t-a_t)\delta_0+(c_t-b_t)\delta_1}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}
\end{aligned}
$$

For $t_0 \le t < t+k < t_1$, from (18) one could easily obtain

$$
\begin{aligned}
& E_t\Big[\frac{P_{t+k}}{(1+r)^k}\Big] \\
& +E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t+1} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t+1}}\cdots\right] \\
= \; & E_t\left[\frac{N_t+\sum_{i=1}^{k+1} y_{t+i}+\delta_0(k+1)}{(r+1)^{k+1}}+\cdots+\frac{N_t+\sum_{i=1}^{b_t-1} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{b_t-1}}\right] \\
& +E_t\left[\frac{N_t+\sum_{i=1}^{b_t} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{b_t}}+\frac{N_t+\sum_{i=1}^{b_t+1} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{b_t+1}}+\cdots+\frac{N_t+\sum_{i=1}^{c_t-1} y_{t+i}+(b_t-1)\delta_0}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_t+\sum_{i=1}^{c_t} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{c_t}}+\frac{N_t+\sum_{i=1}^{c_t+1} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{c_t+1}}\cdots\right] \\
& +\delta_1\left[\left(\frac{1}{(r+1)^{b_t}}+\cdots+\frac{(c_t-b_t-1)}{(r+1)^{c_t-2}}\right)+(c_t-b_t)\left(\frac{1}{(r+1)^{c_t-1}}+\frac{1}{(r+1)^{c_t}}+\cdots\right)\right] \\
= \; & \frac{N_t}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k} \\
& +\frac{\delta_0}{r^2(1+r)^{b_t-2}}\left[(r+1)^{b_t-k-2}([k+1]r+1)-1\right]+\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-b_t}-1\right)
\end{aligned}
$$

For $t_0 \le t < t_1 \le t+k < t_2$, from (18) then

$$
\begin{aligned}
& E_t\Big[\frac{P_{t+k}}{(1+r)^k}\Big] \\
= \; & E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{k} y_{t+i}+(k-b_t)\delta_1}{(r+1)^{k+1}}+\cdots+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t-1} y_{t+i}+(c_t-b_t)\delta_1}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t}}+\frac{N_{t+b_t-1}+\sum_{i=b_t}^{c_t+1} y_{t+i}+\delta_1(c_t-b_t)}{(r+1)^{c_t+1}}\cdots\right] \\
= \; & E_t\left[\frac{N_t+\sum_{i=1}^{k+1} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{k+1}}+\cdots+\frac{N_t+\sum_{i=1}^{c_t-1} y_{t+i}+(b_t-1)\delta_0}{(r+1)^{c_t-1}}\right] \\
& +E_t\left[\frac{N_t+\sum_{i=1}^{c_t} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{c_t}}+\frac{N_t+\sum_{i=1}^{c_t+1} y_{t+i}+\delta_0(b_t-1)}{(r+1)^{c_t+1}}\cdots\right] \\
& +\delta_1\left[\left(\frac{k-b_t}{(r+1)^{k+1}}+\cdots+\frac{(c_t-b_t-1)}{(r+1)^{c_t-2}}\right)+(c_t-b_t)\left(\frac{1}{(r+1)^{c_t-1}}+\frac{1}{(r+1)^{c_t}}+\cdots\right)\right] \\
= \; & \frac{N_t+\delta_0(b_t-1)}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}+\frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-k-2}([k+2-b_t]r+1)-1\right)
\end{aligned}
$$

For $t_0 \le t < t_1, t_2 \le t+k$, from (17) we have

$$
\begin{aligned}
& E_t\Big[\frac{P_{t+k}}{(1+r)^k}\Big] \\
= \; & E_t\left[\frac{N_{t+c_t-1}+\sum_{i=c_t}^{k+1} y_{t+i}}{(r+1)^{k+1}}+\frac{N_{t+c_t-1}+\sum_{i=c_t}^{k+2} y_{t+i}}{(r+1)^{k+2}}\cdots\right] \\
= \; & \frac{N_t+\delta_0(b_t-1)+\delta_1(c_t-b_t)}{r(1+r)^k}+\frac{d_t[(k+1)r+1]}{r^2(1+r)^k}
\end{aligned}
$$

For $t_1 \leq t < t+k < t_2$, from (18) one could get

$$
\begin{aligned}
& E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] \\
= \quad & E_t\left[\frac{N_{t+k+1}}{(r+1)^{k+1}} + \cdots + \frac{N_{t+c_t-1}}{(r+1)^{c_t-1}}\right] + E_t\left[\frac{N_{t+c_t-1} + y_{t+c_t}}{(r+1)^{c_t}} + \frac{N_{t+c_t-1} + \sum_{i=c_t}^{c_t+1} y_{t+i}}{(r+1)^{c_t+1}} \cdots\right] \\
= \quad & E_t\left[\frac{N_t + \sum_{i=1}^{k+1} y_{t+i} + (k+1)\delta_1}{(r+1)^{k+1}} + \cdots + \frac{N_t + \sum_{i=1}^{c_t-1} y_{t+i} + (c_t-1)\delta_1}{(r+1)^{c_t-1}}\right] \\
& + E_t\left[\frac{N_t + \sum_{i=1}^{c_t} y_{t+i} + \delta_1(c_t-1)}{(r+1)^{c_t}} + \frac{N_t + \sum_{i=1}^{c_t+1} y_{t+i} + \delta_1(c_t-1)}{(r+1)^{c_t+1}} \cdots\right] \\
= \quad & \frac{N_t}{r(1+r)^k} + \frac{d_t[(k+1)r+1]}{r^2(1+r)^k} + \frac{\delta_1}{(r+1)^{c_t-2}r^2}\left((r+1)^{c_t-2-k}([k+1]r+1) - 1\right)
\end{aligned}
$$

For $t_1 \leq t < t_2 \leq t+k$, from (18) then

$$
\begin{aligned}
E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] &= E_t\left[\frac{N_{t+c_t-1} + \sum_{i=c_t}^{k+1} y_{t+c_t}}{(r+1)^{k+1}} + \frac{N_{t+c_t-1} + \sum_{i=c_t}^{k+2} y_{t+c_t}}{(r+1)^{k+2}} + \cdots\right] \\
&= \frac{N_t + \delta_1(c_t-1)}{r(1+r)^k} + \frac{d_t[(k+1)r+1]}{r^2(1+r)^k}
\end{aligned}
$$

For $t_2 \leq t < t+k$, from (17) we obtain

$$
\begin{aligned}
E_t\left[\frac{P_{t+k}}{(1+r)^k}\right] &= E_t\left[\frac{N_t + \sum_{i=1}^{k+1} y_{t+c_t}}{(r+1)^{k+1}} + \frac{N_t + \sum_{i=1}^{k+2} y_{t+c_t}}{(r+1)^{k+2}} + \cdots\right] \\
&= \frac{N_t}{r(1+r)^k} + \frac{d_t[(k+1)r+1]}{r^2(1+r)^k}
\end{aligned}
$$

Thus, the assertions hold.

$\square$