

Review

Geometric Entropy of Self-Gravitating Systems

Lorenzo Fatibene^{1,2}, Marco Ferraris¹, Mauro Francaviglia^{1,2,3,*} and Silvio Mercadante^{1,2}

1 Department of Mathematics, University of Torino, Torino 10123, Italy

2 INFN, Sezione di Torino, Iniziativa Specifica Na12, Torino 10125, Italy

3 ESG, University of Calabria, Arcavacata di Rende 87036, Italy

E-Mails: lorenzo.fatibene@unito.it; marco.ferraris@unito.it; mauro.francaviglia@unito.it;
silvio.mercadante@unito.it

* Author to whom correspondence should be addressed.

Received: 30 April 2007 / Accepted: 12 September 2007 / Published: 29 November 2007

Abstract: We shall review different approaches to the entropy of self-gravitating systems in General Relativity. Then we shall discuss in detail the macroscopic approach based on *à la* Clausius point of view. Recent developments will be reviewed discussing the aims as well as the assumptions which the framework is based on.

Keywords: Entropy, Self-Gravitating Systems, Black-Holes.

1. Introduction

The idea that an entropy may be associated to black-hole solutions of field equations of General Relativity (GR) has been first suggested in the '70s by Bekenstein and Hawking (see [1],[2],[3]), relying on earlier works by Christodoulou and Ruffini (see [4],[5]) about the possibility of extracting energy from a black-hole (by describing in detail the physics of particles around a Kerr black-hole solution)*.

Bekenstein's observations were based on a correspondence between the dynamical laws of (non-extreme) black-holes and the classical laws of thermodynamics, which eventually lead him to identify (modulo a universal factor) the entropy with the area A_H of the outer horizon (which Christodoulou had previously found to be proportional to the irreducible mass, i.e. the amount of mass that cannot be reduced further by irreversible processes).

*Christodoulou himself proved (see [4]) that a certain amount of the energy of a black-hole, though not all of it, can be extracted, thus suggesting a thermodynamical interpretation, as the classical entropy is related to the difference between the internal energy and the free energy of a gas.

As Hawking pointed out (see [3]), due to quantum mechanical effects black-holes radiate as black-bodies at a definite temperature T (that is related to the surface gravity κ_H at the horizon H ; such a temperature lies below the cosmic microwave background radiation for stellar black-holes). The temperature T rapidly increases as the mass decreases, so that microscopic black-holes are very hot. Knowing this temperature one can fix the universal factor for Bekenstein's entropy to be $1/4$. Endowing Hawking's temperature with a physical meaning is the key to consider black-hole entropy as a physical quantity.

While both the horizon area A_H of a black-hole and its gravity at surface κ_H are macroscopic geometrical quantities that are characteristic of the solution of Einstein equations under consideration, physically speaking the entropy should be thought in terms of the microscopic states. This duality is well known since the early times of classical gas theory: the entropy is expected to be both a microscopic and a macroscopic quantity depending on the framework.

The microscopic setting based on microstate counting can be also reviewed in terms of information theory [†] and in fact the relation existing between irreducible mass and information was already stated by Christodoulou. The fundamental origin of entropy was recognised by Bekenstein as a measure of our ignorance about the internal state of matter fallen into a black-hole; a black-hole in fact destroys (or permanently hides to an external observer) most of the detailed nature of the matter falling into it, owing to the so called no-hair theorem.

Although Shannon's entropy seems to be different in nature from physical entropies, there is a strict link between them. For example, gas macrostates can be realised by many different microstates and entropy measures our ignorance about the particular microstate which actually realizes the macrostate observed.

One can use Shannon's notion of entropy for a physical system when the following requirements are fulfilled:

- (i) each microstate is *equiprobable*;
- (ii) *ergodicity*: the system fills a dense open set of phase space and the probability of being in a region is proportional to the volume of the region;
- (iii) the phase space accessible to the system is *finite* (so that volumes can be normalised to estimate probabilities).

Under these simple assumptions, in fact, the probability of a macrostate is directly related to the volume (normalised to the accessible volume of phase space) of all microstates determining it. Thence Shannon entropy measures our capacity (or better, our incapacity) to predict the microstate of the system knowing

[†]Following Shannon (see [6]), we may introduce the notion of entropy of a stream of characters (taken from a given alphabet) as a quantification of the predictability of each character's occurrence in terms of those previously transmitted. Shannon entropy is defined as

$$S = - \sum_i p_i \ln p_i \quad (1)$$

where p_i is the probability (or the frequency) of the occurrence of the i^{th} character of the alphabet. The function S is clearly positive definite, vanishing when the stream is constant (just one character repeats on and on, when one) and it is maximal for completely random sequences.

its past history. On the other hand this knowledge is clearly related to the possibility of extracting energy from the system ‡ which is the link with the classical thermodynamical definition of entropy.

Unfortunately this approach is not directly applicable in GR. At a microscopic level, Quantum Gravity is involved, which is still at a very problematic stage as a fully coherent theory. As a result a general definition of entropy in terms of microstate counting is not likely to be available in the very near future.

On the other hand many interesting results have been obtained from a macroscopic viewpoint. The definition of black-holes entropy in terms of the area of its horizon is well established in many general situations (but not in the case of extreme black-holes). Relying on this, it was proposed to universally extend the identification $S = \frac{1}{4}A_H$ (in Planck units). This simplified choice turns out, however, to be invalid for larger classes of solutions.

A great contribution in this direction came from Hawking, Page and Hunter (see [7]), who studied Taub-bolt solutions in full detail. Despite Taub-bolt is a family of singular solutions (see [8],[9],[10]) their singularities are not enclosed within a Killing horizon (due to the so-called *Misner string* running along a z -axis). Although the physical interpretation of Taub-bolt solutions is not universally accepted, it was proven that an entropy should be associated with them. However, the entropy of Taub-bolt solutions deviates from the simple one-quarter of the area law proved for standard black-holes. This consideration led eventually Hawking and his collaborators to propose a new and different interpretation which looks at the entropy as a measure of the obstruction to defining a global foliation of spacetime.

A different approach, deeply related to statistical considerations based on a micro-canonical ensemble formulation, was also pursued by Brown and York (see [11],[12]), who worked on a Hamiltonian formulation for spatially confined gravitational system (see axiom (iii) above). In this framework macroscopical energy and entropy are defined as quasilocal quantities. The main advantage of this approach is a definition of conserved quantities for finite spatially extended systems which improves substantially the standard ADM results (see [13],[14]) for asymptotically flat solutions.

2. The Geometry of Entropy in General Relativity

Microstate counting methods, Taub-bolt example, statistical approaches, altogether seems to indicate a sort of failure of a purely geometrical definition of entropy in GR; or, in other words, they seems to suggest that the geometrically minded identification between the entropy and (the one-quarter of) the horizon area is still to be considered fundamental just for (physically reasonable) stationary black-holes, while in more general situations the geometric character of entropy is more difficult (or even impossible) to be established.

However, starting from 1994 the picture has changed. In that year Wald et al. proposed a prescription (see [15]), which subsequently was generalised by us (see [16],[17],[18],[19]) and shown to be intrinsically and deeply geometrical. In fact, the original prescription of [15] was based on strong and unnecessary hypotheses that partially but radically hide its clever geometrical content; in this formulation Killing horizons a fundamental role which eventually prevents a geometrical interpretation of entropy for the solutions with singularities that are not enclosed by Killing horizons. In our generalization, in-

‡More information we know, more energy we may extract from the system: imagine if we were able to predict when a gas in a box will happen to be wholly contained in the right half of the box!

stead, all the unnecessary hypotheses have been relaxed and we showed both the very geometric nature of our prescription and that all cases mentioned above can be easily encompassed in this new geometrical framework. In other words, the geometric nature of entropy in GR is present also when it is not easily recognisable, and it provides in fact a unifying viewpoint.

The geometric nature of entropy in GR can be understood only after the problem has been split into two parts, one of them being deeply geometrical provided one gives up a control on the other part. This is not a peculiarity of GR. In any macroscopical approach to entropy—e.g. when dealing with Classical Thermodynamics of gases—one has first to predict thermodynamical potentials. It can be easily argued that there is no classical way of computing the temperature in macroscopic thermodynamics unless all other thermodynamical potentials are already known.

Similarly, the notion of temperature in GR is defined in terms of the Hawking radiation that is in turn based on a semiclassical approximation of Quantum Mechanics on a curved background, or some equivalent classical result about Geodesics Geometry (see [20]).

The further identifications between the temperature and other geometrical entities (surface gravity of the horizon [3], the period of time compactification in the Euclidean sector to avoid conical singularities and so on) are necessarily less fundamental than Hawking radiation and they basically represent a coincidence. It is worthwhile to mention that such a coincidence is deeply related to the no-hair theorem: *if the status of a black-hole is described by very few parameters which in turn determine its geometrical features*, then it is trivial that any state function is necessarily a function of these few parameters (or, equivalently, it is a function of the geometrical parameters of the black-hole). If one considers a one-parameter family of black-hole solutions (e.g. Schwarzschild solutions) such that the horizon area can be considered as a parameter for that family, than *any* physical quantity associated with those black-holes can be trivially expressed as a function of the area of the horizon.

From a macroscopic viewpoint, the variational equation for the thermodynamic quantities of a black-hole is (see [4], [7], [14], [21] for details)

$$\delta M = T\delta S + \Omega\delta J + b^a\delta Q_a \quad (2)$$

where: M is the mass of the black-hole (i.e. a measure of the energetic content of the gravitational system), $T = \kappa_H/2\pi$ is its temperature (being κ_H the surface gravity on the horizon H), $S = A_H/4$ is the entropy of the black-hole (being A_H the area of the horizon surface), Ω is the angular velocity of the horizon, J is the angular momentum of the black-hole while the parameters b^a are the Lagrangian multipliers for (possible) gauge charges Q_a (actually $b^a\delta Q_a = b\delta q + \dots$ where b is an electromagnetic thermodynamical potential and q is the electric charge, and the dots represent other terms that can of course appear depending on how liberal we decide to be in allowing other physical quantities, e.g. when more general gauge fields are present).

Equation (2) is known as the *first law of Black-Holes Thermodynamics*. In it the quantities T and Ω are assumed, in this framework, to be external parameters of the macroscopical thermodynamical theory, which have to be obtained by means of physical considerations on the geometry and the dynamics of the black-hole. The deformation operator δ denotes any infinitesimal variation of parameters in the space of solutions, e.g. along a (restricted) family of arbitrarily fixed solutions (as it happens along Kerr–Newman solutions).

The main content of Wald's prescription [15] was to use the first law of thermodynamics to define S whenever all the other quantities are already known. In particular, the quantities $\delta m, \delta J, \delta q, \dots$ can be defined via Nöther theorem as boundary (namely, asymptotic) quantities. Hence the stronger are the tools we have to compute conservation laws the more general are the situations we are able to deal with. In this view variational calculus has a prominent role being the natural setting for studying conservation laws in a global and covariant way and to guide us towards their physical interpretation.

3. The Variational Setting

Here we are going to set the mathematical framework we need to define entropy in GR. In all this Section the main reference for the formalism will be [22]. We shall omit the (sometimes long) details, which can be found there.

The framework is suitable to deal with *natural theories* (namely, generally covariant field theories) as well as with *gauge-natural theories* (namely, field theories with extra gauge covariance). Since we shall be concerned with gravitational theories (with matter interactions) let us assume that fields carry information enough to determine the geometry (more precisely a metric structure) of spacetime.

The exact nature of the dynamical (fundamental) fields, and then how exactly spacetime geometry is encoded by them, actually depends on the context and on the framework adopted (e.g. purely metric, Palatini metric-affine, purely affine, purely tetrad, tetrad-affine, Ashtekar's variables, Chern–Simons formulations, just to mention the most commonly used; see [22]).

Field configurations are parametrized by sections of a natural (or gauge-natural) bundle $\pi : \mathcal{C} \rightarrow M$, called the *configuration bundle*. The base manifold M is spacetime, while fibered coordinates (x^μ, y^i) on \mathcal{C} locally parametrize fundamental fields $y^i(x)$.

The configuration bundle is assumed to be *natural* which means that spacetime diffeomorphisms act on configuration fields. *Gauge natural* refers to a more general situation in which the wider group $\text{Aut}(\mathcal{P})$ of principal automorphisms of a principal structure bundle \mathcal{P} globally acts on fields. Such transformations are called *gauge transformations*. The gauge natural case encompasses the natural case since the group $\text{Aut}(\mathcal{P})$ canonically projects onto the group $\text{Diff}(M)$ owing to the bundle structure of \mathcal{P} .

The *jet prolongations* of the configuration bundle take care of partial derivatives of fields and play the role of Lagrangian phase space. For any bundle $\mathcal{C} \rightarrow M$ there is a canonical infinite chain of bundles, the k -th of which denoted by $J^k\mathcal{C}$:

$$\dots \rightarrow J^{k+1}\mathcal{C} \rightarrow J^k\mathcal{C} \rightarrow \dots \rightarrow J^1\mathcal{C} \rightarrow \mathcal{C} \rightarrow M. \quad (3)$$

A system of local coordinates of $J^k\mathcal{C}$ is $(x^\mu, y^i, y_{\mu}^i, \dots, y_{\mu_1 \dots \mu_k}^i)$ if (x^μ, y^i) denote fibered coordinates on \mathcal{C} . The extra coordinates $(y_{\mu}^i, \dots, y_{\mu_1 \dots \mu_k}^i)$ represent partial derivatives of fields up to order k (hence they are symmetric in the lower indices). Any section, fibered morphism or vector field on \mathcal{C} uniquely and functorially (i.e. preserving compositions) induces a section, a fibered morphism or a vector field on $J^k\mathcal{C}$.

A *Lagrangian* is a bundle morphism $L : J^k\mathcal{C} \rightarrow A_m(M)$, where $m \equiv \dim(M)$ and $A_m(M)$ denotes the bundle of m -forms over M . In terms of the fibered coordinates chosen, we locally have

$$L = \mathcal{L}(x^\mu, y^i, y_{\mu}^i, \dots, y_{\mu_1 \dots \mu_k}^i) ds \quad (4)$$

where $ds = dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$ is the canonical pointwise basis of m -forms induced by the local coordinates x^μ chosen on M .

The Lagrangian describing the dynamics is required to allow $\text{Diff}(M)$ ($\text{Aut}(\mathcal{P})$, respectively) as symmetries in natural (gauge-natural, respectively) theories (see also [16]). Gauge-natural theories are essential, for instance, in tetrads, Ashtekar’s variables or Chern–Simons formulations of GR, where, besides spacetime diffeomorphisms, Lorentz, spin and $\text{SU}(2)$ transformations are assumed to be acting on fields, respectively, and to be Lagrangian symmetries.

Given a vertical vector field X on \mathcal{C} , locally represented as $X = \delta y^i \partial_i$, both the sections of \mathcal{C} and the Lagrangian can be dragged along its flow. One can hence define a (global) bundle morphism $\delta L : J^k \mathcal{C} \rightarrow V^*(J^k \mathcal{C}) \otimes A_m(M)$, related to the *Lagrangian deformation* $\delta_X L$, usually defined in Physics as follows

$$\delta_X L = \langle \delta L \mid j^k X \rangle \tag{5}$$

where $\langle \cdot \mid \cdot \rangle$ denotes the natural duality between $V(J^k \mathcal{C})$ and $V^*(J^k \mathcal{C})$.

Inspired by Spencer cohomology [23],[24], we define a *variational morphism* as a global bundle morphism $\mathbb{A} : J^k \mathcal{C} \rightarrow J^h E^* \otimes A_{m-n}(M)$ where E is some vector bundle. Because of the canonical isomorphisms $V(J^k \mathcal{C}) \simeq J^k V(\mathcal{C})$, the morphism δL turns out to be a variational morphism (with $E = V(\mathcal{C})$, $h = k$ and $n = 0$).

Each variational morphism \mathbb{A} can be globally (and almost canonically) split into two global morphisms

$$\begin{cases} \mathbb{V} : J^{k+h} \mathcal{C} \rightarrow J^{h'} E^* \otimes A_{m-n}(M) \\ \mathbb{S} : J^{k+h-1} \mathcal{C} \rightarrow J^{h-1} E^* \otimes A_{m-n-1}(M) \end{cases} \tag{6}$$

(where $h' = 0$ if $n = 0$ while $h' = h - 1$ when $n > 0$) such that the following splitting property holds for any section X of E

$$\langle \mathbb{A} \mid j^h X \rangle = \langle \mathbb{V} \mid j^{h'} X \rangle + \text{Div} \langle \mathbb{S} \mid j^{h-1} X \rangle . \tag{7}$$

Here Div denotes the *formal divergence operator on forms*, defined by

$$\text{Div}(f) \circ j^{k+1} \sigma = d(f \circ j^k \sigma) , \quad f : J^k Y \rightarrow A(M), \quad \sigma : M \rightarrow \mathcal{C} \tag{8}$$

being $d(\cdot)$ the exterior differential operator on forms on M , σ a section and $A(M) \equiv \oplus_i A_i(M)$ denoting the bundle of all forms over M . In the splitting (7), which is nothing but the global expression for the usual algorithm for computing field equations out of the variation of the Lagrangian, the variational morphism \mathbb{V} is uniquely determined, while \mathbb{S} depends in general on an extra choice (e.g., on a connection used to globalise the integration by parts involved in the splitting algorithm). However, due to the low order of the Lagrangian ($k = 1$ or $k = 2$), in all cases analysed below \mathbb{S} will be uniquely defined (see [25]).

The reason why we use the variational morphisms formulation is that being an algebrization of the classical Variational Calculus, it captures the geometrical essence of it and provide the tools sufficient to deal with global properties of field equations, dependence on boundary conditions and conservation laws (see [16],[22],[26],[27],[28] for more details on this).

In the cases of our interest it is $\mathbb{A} = \delta L$, $E = V(C)$ and we denote the two morphisms \mathbb{V} and \mathbb{S} respectively as $\mathbb{E}(L)$ (called the *Euler-Lagrange morphism*) and $\mathbb{F}(L)$ (called the *Poincaré-Cartan morphism*). Then (7) reads:

$$\langle \delta L | j^k X \rangle = \langle \mathbb{E}(L) | X \rangle + \text{Div} \langle \mathbb{F}(L) | j^{k-1} X \rangle . \tag{9}$$

The morphism $\mathbb{E}(L)$ owes its name to the fact that the Euler-Lagrange field equations are completely encoded into it. Hence, in particular, the Poincaré-Cartan morphism is completely irrelevant for field equations. However, we stress that despite of this, conservation laws obtained via Nöther theorem are strongly dependent on $\mathbb{F}(L)$.

In both natural and gauge-natural theories there exists a vector bundle \mathcal{S} the sections of which are in one-to-one correspondence with infinitesimal symmetries of the Lagrangian. This is why it is called the *symmetry bundle*. It is the tangent bundle TM in the case of a natural theory, while in the gauge-natural case it is the bundle of right invariant vector fields on the structure bundle \mathcal{P} .

A section Ξ of \mathcal{S} is then a vector field acting on (physical) fields and hence it defines Lie derivatives $\mathcal{L}_{\Xi}y$ of fields.

Let us now explore the nature of the symmetry bundle. The condition for a vector field Ξ being an infinitesimal symmetry is the following

$$\langle \delta L | j^k \mathcal{L}_{\Xi}y \rangle = \text{Div}(\alpha(\Xi)) \tag{10}$$

where $\alpha(\Xi)$ is some suitable variational morphism linear with respect to Ξ (e.g., in natural theories Ξ is a spacetime vector field ξ and we have simply $\alpha(\xi) = i_{\xi}L$).

Equation (10) is called the *covariance identity* because it expresses the (gauge or general) covariance of the Lagrangian and contains all computational details of Nöther theorem. In fact, it is easy to use the standard variational techniques introduced above, together with the fact that Lie derivatives $\mathcal{L}_{\Xi}y$ take their values in the space of vertical vector fields, to recast the covariance identity into the form of a conservation law:

$$\text{Div} \mathcal{E}(L, \Xi) = \mathcal{W}(L, \Xi) \quad \begin{cases} \mathcal{E}(L, \Xi) = \langle \mathbb{F}(L) | j^{k-1} \mathcal{L}_{\Xi}y \rangle - \alpha(\Xi) \\ \mathcal{W}(L, \Xi) = - \langle \mathbb{E}(L) | \mathcal{L}_{\Xi}y \rangle \end{cases} \tag{11}$$

Since the *work form* \mathcal{W} clearly vanishes when computed along a solution of field equations $y = \sigma(x)$, the *Nöther current* \mathcal{E} turns out to be conserved on-shell.

Since in both the natural and gauge-natural case infinitesimal symmetries Ξ are (generic) sections of a suitable vector bundle, both \mathcal{E} and \mathcal{W} can be regarded as variational morphisms themselves. They can be then split to define the so-called *Bianchi term* \mathcal{B} , *reduced current* $\tilde{\mathcal{E}}$ and the *superpotential* \mathcal{U} :

$$\begin{cases} \mathcal{E}(L, \Xi) = \tilde{\mathcal{E}}(L, \Xi) + \text{Div} \mathcal{U}(L, \Xi) \\ \mathcal{W}(L, \Xi) = \mathcal{B}(L, \Xi) + \text{Div} \tilde{\mathcal{E}}(L, \Xi) \end{cases} \tag{12}$$

As a consequence of conservation laws (11) one has $\mathcal{B} = 0$ even off-shell (result known as *Bianchi identities*; see [16],[22],[24],[29]), $\tilde{\mathcal{E}} \approx 0$ (i.e., $\tilde{\mathcal{E}} = 0$ on-shell), while \mathcal{U} will enter into the definition of conserved quantities (later appearing in the definition of the entropy) in a Gauss-like fashion.

We explicitly notice that the splitting of \mathcal{E} and \mathcal{W} is variational in nature and then so are conserved quantities, too. Furthermore relying on Spencer cohomology (7) this procedure is algorithmic and also shows clearly (see (11) and (12)) that the Nöther current \mathcal{E} is exact on-shell regardless of the spacetime topology (*weak conservation law*) while the difference $\mathcal{E} - \tilde{\mathcal{E}}$ is exact even off-shell (*strong conservation law*), which is one of the most relevant results that can be obtained from the jet bundle approach to conserved quantities.

Before defining conserved quantities, i.e. physical observables, it is necessary to proceed one step further with the Geometric Variational Calculus machinery. Indeed, it is well known that the definitions of conserved quantities relying on direct integration of the superpotential \mathcal{U} lead, in gravitational theories, to the wrong conserved quantities (e.g., the so-called anomalous factor in asymptotically flat spacetimes [30]) and, even worse, they usually give divergent results when solutions are not asymptotically flat ([16],[31]).

We then proceed as follows. Let X be a vertical vector tangent to the space of solutions and set ([16],[27])

$$[\delta_X Q_B](\Xi) := \int_B \delta_X \mathcal{U}(L, \Xi) - i_\xi \langle \mathbb{F}(L) | j^{k-1} X \rangle \tag{13}$$

where i_ξ denotes the contraction of spacetime forms along the spacetime projection ξ of the infinitesimal symmetry Ξ , B is the (outer) boundary of a (spacelike) $(m - 1)$ -dimensional surface Σ and the evaluation along a solution σ is understood. If (13) is integrable, then $[\delta_X Q_B](\Xi)$ is the *true* variation (along X) of a *true* conserved quantity $Q_B(\Xi)$, i.e. we could write $[\delta_X Q_B](\Xi) = \delta_X [Q_B(\Xi)]$. This definition is supported by the following properties:

- 1) The integrand of (13) does not depend on the variational cohomology class $[L]$ (i.e. the set of Lagrangians that differ from L only by a divergence term) to which the Lagrangian L belongs. This is a rather welcome property and we shall analyse its far reaching consequences later on.
- 2) The quantity $\delta_X Q_B(\Xi)$ defined as in (13) satisfies (see [21],[32],[33]) the symplectic relation:

$$\delta_X Q_{\partial\Sigma}(\Xi) = \int_\Sigma \omega(X, \Xi) \, d^{m-1}x \tag{14}$$

where

$$\omega(X, \xi) := \delta_X \langle \mathbb{F}(L) | j^{k-1} \mathcal{L}_\Xi \sigma \rangle - \mathcal{L}_\xi \langle \mathbb{F}(L) | j^{k-1} X \rangle \tag{15}$$

is the (*pre*)*symplectic current* (see [34]), while $\partial\Sigma$ denotes the boundary of the $(m - 1)$ -dimensional region Σ . The relation (14) shows that the conserved quantity $Q(\Xi)$, if it exists by exact integration of $\delta_X Q(\Xi)$, can be truly considered as the Hamiltonian which generates the evolution of fields along the vector field Ξ .

In any natural theory the information on the energy-momentum content inside a region bounded by a surface B is obtained by integrating the right hand side of (13) on B with respect to generators of diffeomorphisms, i.e. space time vector fields ξ . In particular, when ξ is timelike we refer to $\delta_X Q_B(\xi)$ as the *variation of the energy* inside B and relative to the vector field ξ .

In gauge-natural theories there is a canonical way to build a connection on the structure bundle directly out of fields, so that gauge transformations can be formally split into vertical and horizontal ones

(horizontal and vertical with respect to the dynamical connection which will be known when field equations will have been solved). In this case horizontal symmetries define energy-momentum while vertical symmetries define the so-called *gauge charges*.

We think it is important at this point to make some remarks.

To obtain expression (13) a Lagrangian formulation of the theory is not strictly necessary. Indeed, the Lagrangian formalism is just a—powerful—tool to find the field equations of a field theory; and it is clear that all the Lagrangians belonging to the same equivalence class $[L]$ (see item 1) above) give rise to the same field equations. Furthermore, field equations play undoubtedly an important role in field theories, as they select the admissible configurations and dictate their evolution, but they do not encode the whole of the information contained in a field theory. A relevant part of this information is in fact encoded in the field symmetries. The knowledge on how fields are dragged along infinitesimal generators of symmetries is hence a further fundamental point.

These two sets of complementary information, field equations and symmetries, can be joined together into a unique structure. One can in fact define *the variational Lagrangian* [28] as the contraction of field equations with the Lie derivatives of the fields:

$$L'(j^{2k}\sigma, j^r\Xi) :\equiv - \langle \mathbb{E} | \mathcal{L}_{\Xi} y \rangle \tag{16}$$

where r is a finite integer that depends on the theory. Remembering (11) we set $L'(\Xi) \equiv \mathcal{W}$. Obviously as well as \mathcal{W} , $L'(\Xi)$ vanishes along solutions of L . This is a true Lagrangian and, for the information it encodes, it is a good candidate to represent the fundamental object in terms of which one can derive conserved quantities. In fact, being L' a Lagrangian, it can be handled with the tools of the Calculus of Variations in jet bundles we have introduced so far. In particular we can make use of (9) to obtain the splitting:

$$\begin{aligned} \langle \delta L'(\Xi) | j^a X \rangle &= \langle \mathbb{E}(L'(\Xi)) | X \rangle + \text{Div} \langle \mathbb{F}(L'(\Xi)) | j^{a-1} X \rangle \\ &= \text{Div} \langle \mathbb{F}(L'(\Xi)) | j^{a-1} X \rangle \end{aligned} \tag{17}$$

where a is a finite integer depending on the theory (for simplicity, a can be assumed to be the order of field equations) and $\langle \mathbb{E}(L'(\Xi)) | X \rangle \equiv 0$ since L' is a pure divergence and therefore variationally trivial: $\mathbb{E}(L') = 0$ (see (12) and [28]). Accordingly, since L' is vanishing on-shell we obtain the (weak) conservation law:

$$\text{Div} \langle \mathbb{F}(L'(\Xi)) | j^{a-1} X \rangle \approx 0 . \tag{18}$$

Moreover the morphism $\langle \mathbb{F}(L'(\Xi)) | j^{a-1} X \rangle$ depends linearly on the vector Ξ together with its derivatives up to a finite order. Being Ξ a generic section of a well-defined vector bundle we can again implement Spencer cohomology (7) with respect to the vector Ξ (roughly speaking, we integrate by parts with respect to the the derivatives of vector field Ξ) and we end up with the canonical splitting:

$$\langle \mathbb{F}(L'(\Xi)) | j^{a-1} X \rangle = \delta_X \tilde{\mathcal{E}}(\Xi) + \text{Div} \mathcal{V}(X, \Xi) \tag{19}$$

where $\tilde{\mathcal{E}}$ is the reduced current seen above while $\mathcal{V}(X, \Xi)$ agrees with the right hand side of (13) (see [28] for the technical details).

We feel then ourselves authorised to say that definition (13) captures the fundamental mathematical essence of the theory we are dealing with and endows it with a direct physical interpretation. Namely, *theories that are equivalent as far as their field equations content is concerned produce conserved quantities which are equivalent when computed along equivalent solutions*. Moreover, definition (13), being directly built out from the field equations via the variational Lagrangian $L'(\Xi)$ turns out to be related only with the solutions and not with the specific mathematical rule (i.e. the Lagrangian) we choose to describe the theory. So, for example, the energy enclosed in a region surrounding a black-hole will eventually depend, as physically expected, only on the geometry of spacetime, and not on the dynamical variables (metric, tetrad, connection, ...) nor on the Lagrangians (Hilbert, tetrad-affine, Chern-Simons, ...) we initially chose to describe it.

These techniques for conservation laws have been shown to capture the covariant and global aspects of the physical conserved quantities which enter into the first law of thermodynamics (2). The Kerr-Newman, BTZ and Taub-bolt solutions, as well as Yang-Mills theories, spinor physics, Chern-Simons and Lovelock theories, have been directly and successfully treated covering many different examples with different boundary conditions.

Coming at last back to entropy in GR, it has been shown that in gravitational theories which admit a solution with horizon a symmetry generator Ξ can be found so that its associated $\delta_X Q_B(\Xi)$ is $\delta_X m - \Omega \delta_X J - b \delta_X q$. Hence, using (2), it is clear that $\delta_X Q_B(\Xi)$ defines directly $T \delta_X S$, which can be integrated for S once the temperature T is known. In this way we may recover the expected entropy not only for asymptotically flat black-holes (Kerr-Newman; [16]), but also for asymptotically anti-de Sitter black-holes ([17],[18],[31],[35]), asymptotically de Sitter black-holes (Schwarzschild-de Sitter solution; [21]), asymptotically locally flat and asymptotically locally anti-de Sitter solutions (Taub-bolt, [19],[36]) and, recently, also isolated ([37]) and causal ([21]) horizons.

Let us remark that whenever Ξ is a Killing vector field (a condition which is obviously encountered in stationary spacetimes) from formulae (14) and (15) it directly follows that

$$\delta_X Q_{\partial\Sigma}(\Xi) = 0 . \quad (20)$$

As a consequence entropy is defined as a homological quantity which *can* (but not *need*) be defined on horizons (any surface B homologue to an horizon cross section H , i.e. $B - H = \partial\Sigma$, may play the same role). This property drastically reduces the requirement on existence and type of horizons. We lastly observe that the general case of isolated horizons, where a global Killing vector does not exist, can be handled inside the same framework developed so far (see [37]).

4. Boundary Conditions and Reference Backgrounds in General Relativity

An obvious observation is that by means of definition (13) we are in general able to produce just the *variation* of a conserved quantity, not the conserved quantity *itself*.

This in principle can be done only by fixing a boundary condition. There are many equivalent but different ways (Variational Calculus is essential to prove the equivalence of such different prescriptions) to do so.

One of them is to forgo the covariance and define the so-called *energy-momentum pseudo-tensors* ([38]). A choice which is definitely not much in the spirit of GR.

A more elegant solution would be that of introducing a reference background in the variational principle (of course some sort of flat background is also used in the definition of asymptotically flat solutions, which can be seen as deformations of a Minkowski flat spacetime). It is in fact known ([17],[22],[30],[39],[40]) that denoting

$g_{\mu\nu}$	$\bar{g}_{\mu\nu}$	metrics
$\Gamma_{\beta\nu}^\alpha$	$\bar{\Gamma}_{\beta\nu}^\alpha$	connections
$R_{\beta\mu\nu}^\alpha$	$\bar{R}_{\beta\mu\nu}^\alpha$	Riemann tensors
$R_{\mu\nu}$	$\bar{R}_{\mu\nu}$	Ricci tensors
$R = g^{\mu\nu} R_{\mu\nu}$	$\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu}$	scalar curvatures
$u_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu - \Gamma_{\epsilon(\alpha}^\epsilon \delta_{\beta)}^\mu$	$\bar{u}_{\alpha\beta}^\mu = \bar{\Gamma}_{\alpha\beta}^\mu - \bar{\Gamma}_{\epsilon(\alpha}^\epsilon \delta_{\beta)}^\mu$	

and

$$w_{\alpha\beta}^\mu = u_{\alpha\beta}^\mu - \bar{u}_{\alpha\beta}^\mu \tag{21}$$

one can define a covariant Lagrangian depending on the two fields g and \bar{g} by setting:

$$L_{g\bar{g}} := (R\sqrt{\bar{g}} - d_\mu(\sqrt{\bar{g}} g^{\alpha\beta} w_{\alpha\beta}^\mu) - \bar{R}\sqrt{\bar{g}}) ds . \tag{22}$$

This Lagrangian presents an advantage with respect to the Hilbert’s one (see [16], [22], [30], [39], [41]): the superpotential ensuing from $L_{g\bar{g}}$ provides a conserved quantity which depends on both g and \bar{g} and satisfactorily represents the relative energy–momentum between the two field configurations. In this way \bar{g} can be fixed to be any kind of solution (e.g. Minkowski, pure de Sitter or anti-de Sitter, Taub–NUT) in order to study solutions (e.g. Kerr–Newman, Schwarzschild–de Sitter, BTZ or Taub-bolt) with different asymptotic behaviours (asymptotically flat, asymptotically de Sitter, asymptotically anti-de Sitter or asymptotically locally flat, respectively).

The Lagrangian (22) can be viewed in a general scheme to deal with *relative conserved quantities* (see [42]). We stress that this general scheme is aimed precisely to reproduce the variation of conserved quantities in the form (13).

An important and “philosophical” remark. We are obviously very well aware that there is no universal consensus in current literature, especially in loop quantum gravity (see, e.g., [43]), on the use of backgrounds. While we completely agree that backgrounds should not directly enter the description of Physics at a fundamental level (we trust in the need of a background-free quantization scheme) we also think that more attention should be deserved to *what* exactly a background is. In particular we claim that the reference background entering (22) is completely different in nature from the background assumed in standard String Theory or from the role of Minkowski background in Particle Theory. The differences are both in the physical interpretation and in the behaviour from a variational viewpoint. In Particle Physics, for instance, the background is endowed with a direct physical interpretation (for example, free particles are supposed to move along its geodesics) and some of the fundamental physical structures explicitly depend on it (e.g. the inner product of the Hilbert space of quantum states). This kind of background is considered fixed by both variations and symmetries, in order to kill some terms

in the conservation laws (by means of the condition $\mathcal{L}_\xi \bar{g} = 0$, i.e. restricting to Killing symmetries). These extra terms need to be killed because they do not vanish on-shell, as a consequence of the fact that backgrounds are not varied and one has less field equations than one usually would have in an unconstrained variational principle. Instead, in equation (22) the situation is completely different. In particular it seems to us completely satisfactory from a physical point of view. First of all the background \bar{g} is not endowed with a direct physical interpretation; free particles move along geodesics of g and both g and \bar{g} are unknown until field equations are solved (so that, for example, the Hilbert structure on quantum states cannot depend on them). Secondly g and \bar{g} are completely independent and, as the Lagrangian (22) shows, they do not interact; so the choice of \bar{g} does not influence the dynamics of g . So, what is the role of \bar{g} ? More than a *background* (in the meaning this word suggests) it behaves like a “tare”, accounting for the observer freedom in choosing a vacuum state with respect to which one can compute some physical quantity (e.g. the energy). A completely satisfactory situation indeed, if one consider that just variations of conserved quantities (e.g. energy), are endowed with a direct fundamental role, while the absolute values of conserved quantities are observer-dependent and, as such, non-covariant!

In many “simple” field theories the configuration bundle has a canonical section (this is what happens, e.g., with the zero-section of a vector bundle) which can be selected as a preferred reference background. Hence despite one could certainly look for a reference background-depending variational principle, in all these cases one can escape it. On the contrary, when the configuration bundle has no canonical sections (e.g. when it is $\mathcal{C} = \text{Lor}(M)$, the bundle of Lorentzian metrics over M) such ambiguities have to be dealt with explicitly.

In (22) the fields g and \bar{g} are treated on exactly an equal footing: they play the same role in the variational game. Variations are not required to fix \bar{g} , nor symmetries are. On the contrary, symmetries drag both g and \bar{g} equally. Field equations are doubled (one set for g and one set for \bar{g}) and then also \bar{g} can be in principle chosen obeying Einstein field equations. Furthermore, as already noticed above, the particular form of the Lagrangian (22) ensures that g and \bar{g} can be chosen truly independently because they do not interact: no extra terms will in fact survive in the conservation laws once field equations for both g and \bar{g} are taken into account. Hence there is no need of requiring the symmetry to be a Killing symmetry of the dynamical field g nor of the reference background.

Let us finally notice that the superpotential $\mathcal{U}(L_{g\bar{g}}, \xi)$ obtained from the Lagrangian (22) reproduces the well known Regge–Teitelboim prescription (see [14]) once it is specialised to an asymptotically flat solution and a Minkowski background, so that the variational principle (22) can be regarded as a generalization of this fairly accepted result. Moreover, the variation $\delta_X \int_B \mathcal{U}(L_{g\bar{g}}, \xi)$ reproduces the variational formula (13) when (Dirichlet boundary conditions) the metric g approaches the background \bar{g} on the integration domain B (which can or cannot coincide with spatial infinity; see [41]). In this way we can define the *quasilocal energy* (the energetic content inside a bounded region of spacetime) which is general enough to be adapted to many (if not every) situation, provided the metric and its background are suitably chosen and matched.

As already stated in the Introduction, we have convincing arguments to believe that gravitational systems are truly thermodynamical; and then we should be able to associate to them different kinds of energies, such as, e.g., internal or free energy. To do so, let us look again at (13). Once we fix B in

it, there exist many “energies”, depending both on the symmetry vector field Ξ and on the variational vector field X . In particular Ξ (or better, its projection ξ onto the spacetime manifold M) determines, via its flow parameter, how the observers located on B evolve as the “time” flows. The vector field X fixes instead the boundary conditions (e.g. *à la* Dirichlet or Neumann) on the dynamical fields.

Hence to each choice of the pair (ξ, X) a particular realization of a physical system is associated, endowed then with its own “energy content”.

It was also noticed ([11],[12],[44]) that boundary conditions in GR correspond exactly to boundary conditions of a thermodynamical ensemble and that dynamical fields conjugated to each other in a symplectic sense can be also considered thermodynamically conjugated. Accordingly, different ensembles can be realised by exchanging the fields that are kept fixed as boundary data with their respective boundary conjugated fields. Under this point of view the variational vector field X turns out to be a variation into the ensemble realised with specified boundary conditions (e.g. microcanonical, canonical or grand-canonical ensemble), namely X fixes among the dynamical variables, which ones are the intensive and which ones are the extensive variables.

As already noticed, the Dirichlet boundary conditions (i.e. boundary metric kept fixed) give raise to the internal energy when using the Lagrangian (22). Moreover the Neumann boundary conditions (consisting in keeping fixed the momenta conjugated to the boundary metric) lead ([21]) to another kind of energy, which we call the *gravitational heat*, namely the TS contribution to the internal energy of the thermodynamic system (T = temperature, S = entropy).

It is important to notice at this point that the definition of entropy, as it arises from this prescription, depends on the vector ξ and then it is *observer-dependent*. Hence entropy contributions arise whenever an obstruction exists to globally foliating spacetime into surfaces of constant time (in perfect agreement with the perspective of [7]). In other words, entropy turns out to be closely related to the coordinate singularities of the solution describing spacetime. Let us make some physical considerations in order to explain this important point. For each timelike vector ξ there exists a set of (local) observers evolving with velocity ξ and we can choose an observer-adapted system of coordinates in which the observers themselves are at rest. In such a reference frame the metric coordinate singularities correspond to homological boundaries and, according to the homological properties owned by our definition of entropy - which are basically inherited from the property (20) -, each boundary gives rise to its own contribution to the total entropy. However the metric coordinate singularities correspond physically to regions of inaccessibility for the observers (one-way membranes). Hence entropy turns out to be associated to the impossibility for the observer to reach such regions (see [20]). Extending Beckenstein’s original idea to a much broader context we are then induced to state that *entropy measures the information content beyond the hidden regions*.

We remark that the hidden regions we are referring to not only include the standard black-hole horizons but also a much wider class of singularities, such as causal horizons (e.g. cosmological or Rindler horizons), Taub-bolt and Taub-NUT singularities.

Conclusion and Perspectives

Due to cohomological properties the entropy defined via the first law (2) can be geometrically calculated, e.g. as an integral at space infinity; then the geometrical properties of the horizons are certainly irrelevant. In particular there is no need for the horizon to be Killing (even less to be a bifurcate Killing horizon!). This traditional assumption is related to the wish of reducing the entropy to be an integral at the horizon, with the consequent need to control the limit behavior at the horizon itself, although these features turn out to be unnecessary and produce too stringent requirements which are not justified by phenomenology. For example, in Taub-bolt solutions the Misner string is not dressed by a “good” horizon, although the entropy can be properly defined as a quantity at infinity. The traditional physical quantities at singularities can be recovered by (artificially) splitting the domain of integration into the union of a number of regions surrounding the singularities themselves.

The definition of the entropy notion in GR we described so far covers a wide number of cases, extremely different in spacetime dimension, in asymptotic behaviour or structure of singularities. It successfully deals with Kerr–Newmann, BTZ, Taub-bolt solutions, black-holes in higher dimension. Furthermore, interesting results have also been found for non-stationary solutions (even if specific examples suffer from the lack of exact solutions which are at the same time geometrically well-defined). Hence we consider ourselves satisfied enough by this situation, in particular by the fact that entropy is a geometrical quantity the cohomological properties of which reverberate in the first laws of Thermodynamics. We already compared our approach to Brown-York prescription and to Hamiltonian based formulations, showing that they can be understood within our geometrical framework.

Another issue refers to a general problem with variational prescriptions for conserved quantities. Such prescriptions depend in general on the action principle, not only on the solution. There are cases in which a specific metric field is related to the solution of two different actions.

A simple case is when the two actions differ by boundary terms; in this case we saw that the variation of conserved quantities defined by (13) is independent of boundary action terms. However, there are more general cases in which different actions (i.e. with bulk differences) produce different field equations which share common solutions, or better their solutions induce the same metric fields.

For example, because of universality properties (see [45]), it was shown that the *à la* Palatini actions based on Lagrangian densities of the form $L_f = f(R)\sqrt{g}$ (in the case of a generic analytic function f) produce field equations which are in vacuum equivalent to Einstein vacuum field equations with a suitable cosmological constant.

In such situations conserved quantities depend on the Lagrangian more than on the solutions; there is no *a priori* reason to believe that the two variational principles produce the same conserved quantities. We stress that different conserved quantities would produce different entropies. In the specific case of $f(R)$ theories mentioned above it was shown (see [50]) that, despite the superpotentials actually differ, their difference vanishes on-shell hence producing the same conserved quantities and the same entropy.

Further investigations will be devoted to cope with this problem in general. It would be interesting to prove *a priori* that different equivalent Lagrangians produce the same conserved quantities (or to classify the cases in which this does not happen). These results would enforce the euristic idea that conserved quantities are a characteristic of the solution more than of the variational principle (or at least we would

know when the heuristic expectation fails).

Acknowledgements

This work is partially supported by MIUR: PRIN 2005 on “Leggi di conservazione e termodinamica in meccanica dei continui e teorie di campo”. We also acknowledge the contribution of INFN (Iniziativa Specifica NA12), GNFM of INDAM and the local research funds of Dipartimento di Matematica of Torino University.

References

1. Bekenstein, J.D. Black-Holes and Entropy. *Phys. Rev. D*, **1973**, 7(6), 2333.
2. Bekenstein, J.D. Generalized Second Law of Thermodynamics in Black-Holes Physics. *Phys. Rev. D*, **9**(12), 3292.
3. Hawking, S.W. Particle creation by black holes. *Commun. Math. Phys.* **1975**, 43, 199.
4. Christodoulou, D. Reversible and Irreversible Transformations in Black-Hole Physics. *Phys. Rev. Letters* **1970**, 25, 1596-1597.
5. Christodoulou, D.; Ruffini, R. Reversible Transformations of a Charged Black Hole. *Phys. Rev. D* **1971**, 4, 3552-3555.
6. Shannon, C.E.; Weaver, W. *The Mathematical Theory of Communications*; University of Illinois Press: Urbana, 1949.
7. Hawking, S.W.; Hunter, C.J.; Page, D.N. NUT charge, antide Sitter space, and entropy. *Phys. Rev. D* **1999**, 59, 044033 [hep-th/9809035].
8. Taub, A.H. *Ann. Math.* **1951**, 53, 472.
9. Newman, E.T.; Tamburino, L.; Unti, T.J. Empty-Space Generalization of the Schwarzschild Metric. *Math. Phys.* **1963**, 4, 915.
10. Misner, C.W.; The Flatter Regions of Newman, Unti, and Tamburino’s Generalized Schwarzschild Space. *J. Math. Phys.* **1963**, 4, 924.
11. Brown, J.D.; York, J.W. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D* **1993**, 47(4), 1407.
12. Brown, J.D.; York, J.W. Microcanonical functional integral for the gravitational field. *Phys. Rev. D* **1993**, 47(4), 1420.
13. Arnowitt, R.; Deser, S.; Misner, C.W. In *Gravitation: An Introduction to Current Research*; Witten, L., Ed.; Wiley: New York, 1962; pp. 227.
14. Regge, T.; Teitelboim, C. Role of surface integrals in the Hamiltonian formulation of general relativity. *Annals of Phys.* **1974**, 88, 286.
15. Iyer, V.; Wald, R. Some properties of the Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D* **1994**, 50, 846-864 [gr-qc/9403028].
16. Fatibene, L.; Ferraris, M.; Francaviglia, M.; Raiteri, M. Remarks on Nther Charges and Black Holes Entropy. *Annals of Phys.* **1999**, 275, 27.
17. Fatibene, L.; Ferraris, M.; Francaviglia, M.; Raiteri, M. Remarks on conserved quantities and en-

- tropy of BTZ black hole solutions. I. The general setting. *Phys. Rev. D* **1999**, *60*, 124012.
18. Fatibene, L.; Ferraris, M.; Francaviglia, M.; Raiteri, M. Remarks on conserved quantities and entropy of BTZ black hole solutions.II.BCEA theory. *Phys. Rev. D* **1999**, *60*, 124013.
 19. Fatibene, L.; Ferraris, M.; Francaviglia, M.; Raiteri, M. The Entropy of the TaubBolt Solution. *Ann. Phys.* **2000**, *284*, 197.
 20. (a) Padmanabhan, T. gr-qc/0311036; (b) Padmanabhan, T. gr-qc/0309053; (c) Padmanabhan, T. gr-qc/0308070.
 21. Francaviglia, M.; Raiteri, M. gr-qc/0402080.
 22. Fatibene, L.; Francaviglia, M. *Natural and gauge natural framework for classical field theories*; Kluwer, 2003.
 23. Goldschmidt, H.; Spencer, D.; *J. Diff. Geom.* **1978**, *13*, 455.
 24. (a) Ferraris, M.; Francaviglia, M.; Robutti, O. In *Geometrie et Physique, Proceedings of the Journees Relativistes 1985*, Marseille, 1985; pp.112-125; (b) Choquet-Bruhat, Y.; Coll, B.; Kerner, R.; Lichnerowicz, A., Eds.; Hermann: Paris, 1987.
 25. Ferraris, M. In *Geometrical Methods in Physics, Proceedings of the Conference on Differential Geometry and Its Applications*, Nove Mesto na Morave, Czechoslovakia, Sept 5-9, 1983.
 26. (a) Trautman, A. In *Gravitation: An Introduction to Current Research*; Witten., L., Ed.; Wiley: New York, 1962; pp. 168; (b) Trautman, A. Noether equations and conservation laws. *Commun. Math. Phys.* **1967**, *6*, 248.
 27. (a) Ferraris, M.; Francaviglia, M. *Atti Sem. Mat. Univ. Modena*, **1989**, *37*, 61; (b) Ferraris, M.; Francaviglia, M.; Sinicco, I. *Il Nuovo Cimento*, 107 B, N.11, 1992, p.1303; (c) Ferraris, M.; Francaviglia, M. In *Proceedings of the 7th Italian Conference on General Relativity and Gravitational Physics*, Rapallo, Genoa, Sept 3-6, 1986: World Scientific: Singapore, 1986.
 28. Ferraris, M.; Francaviglia, M.; Raiteri, M. Conserved quantities from the equations of motion: with applications to natural and gauge natural theories of gravitation. *Class. Quant. Grav.* **2003**, *20*, 4043-4066 [gr-qc/0305047].
 29. Ferraris, M.; Francaviglia, M. In *Mechanics, Analysis and Geometry: 200 Years after Lagrange*; Francaviglia, M., Ed.; Elsevier Science Publishers B.V.: Amsterdam, 1991; pp. 451.
 30. (a) Katz, J. A note on Komar's anomalous factor. *Class. Quantum Grav.*, **1985**, *2*, 423-425; (b) Katz, J.; Bicak, J.; Lynden-Bell, D. Relativistic conservation laws and integral constraints for large cosmological perturbations. *Phys. Rev. D* **1997**, *55*(10), 5957-5969.
 31. Allemandi, G.; Francaviglia, M.; Raiteri, M. Charges and energy in ChernSimons theories and Lovelock gravity. *Class. Quant. Grav.* **2003**, *20*, 483-506, gr-qc/0211098.
 32. Iyer, V.; Wald, R.M. Comparison of the Noether charge and Euclidean methods for computing the entropy of stationary black holes. *Phys. Rev. D* **1995**, *52*, 4430-4439 [gr-qc/9503052].
 33. Francaviglia, M.; Raiteri, M. Hamiltonian, energy and entropy in general relativity with non-orthogonal boundaries. *Class. Quantum Grav.* **2002**, *19*, 237-258.
 34. (a) Burnett, G. ; Wald, R.M. A Conserved Current for Perturbations of Einstein-Maxwell Space-Times. *Proc. Royal Soc.: Math. Phys. Sci.* **1990**, *430*, 56; (b) Lee, J.; Wald, R. M.; Local symmetries and constraints. *J. Math. Phys* **1990**, *31*, 725-743.

35. Allemandi, G.; Francaviglia, M.; Raiteri, M. Charges and energy in ChernSimons theories and Lovelock gravity. *Class. Quant. Grav.* **2003**, *20*, 5103-5120 [gr-qc/0308019].
36. Clarkson, R.; Fatibene, L.; Mann, R.B. *Nucl. Phys. B* **2003**, *652*, 348-382 [hep-th/0210280].
37. Allemandi, G.; Francaviglia, M.; Raiteri, M. Energy in EinsteinMaxwell theory and the first law of isolated horizons via the Noether theorem. *Class. Quant. Grav.* **2002**, *19*, 2633-2655 [gr-qc/0110104].
38. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; Freeman: San Francisco, 1973.
39. (a) Ferraris, M.; Francaviglia, M. Covariant first-order Lagrangians, energy-density and superpotentials in general relativity. *Gen. Rel. Grav.* **1990**, *22*(9), 965-985; (b) Ferraris, M.; Francaviglia, M. In *Proceedings of the 8th Italian Conference on General Relativity and Gravitational Physics*, Cavalese, Trento, Aug 30 - Sept 3, 1988; World Scientific: Singapore, 1988; pp. 183.
40. Chen, C.M.; Nester, J.M. A Symplectic Hamiltonian Derivation of Quasilocal Energy-Momentum for GR. *Grav. Cosm.* **2000**, *6*, 257-270 [gr-qc/0001088].
41. Fatibene, L.; Ferraris, M.; Francaviglia, M.; Raiteri, M.J. Nther charges, BrownYork quasilocal energy, and related topics. *Math. Phys.* **2001**, *42*(3), 1173-1195 [gr-qc/0003019].
42. Fatibene, L.; Ferraris, M.; Francaviglia, M. Augmented variational principles and relative conservation laws in classical field theory. *Int. J. Geom. Methods Mod. Phys.* **2005**, *2*(3), 373-392.
43. Rovelli, C. Loop Quantum Gravity. *Living Rev. Relativity*, **1998**, *1*
<http://relativity.livingreviews.org/Articles/lrr-1998-1>
44. Brown, J.D.; Comer, G.L.; Martinez, E.A.; Melmed, J.; Whiting, B.F.; York, J.W. Thermodynamic ensembles and gravitation. *Class. Quant. Grav.* **1990**, *7*, 1433-1444.
45. Borowiec, A.; Ferraris, M.; Francaviglia, M.; Volovich, I. Universality of the Einstein equations for Ricci squared Lagrangians. *Class. Quantum Grav.* **1998**, *15*, 43-55.
46. Wald, R.M. On identically closed forms locally constructed from a field. *J. Math. Phys.* **1993**, *31*, 2378.
47. Racz, I.; Wald, R. Extensions of spacetimes with Killing horizons. *Class. Quantum Grav.* **1992**, *9*, 2643.
48. Bekenstein, J.D. Black holes and the second law. *Nuovo Cim. Lett.* **1972**, *4*, 737.
49. Harada, T.; Iguchi, H.; Nakao, K.-I. [gr-qc/0204008].
50. Fatibene, L.; Ferraris, M.; Francaviglia, M. The Energy of a Solution from Different Lagrangians. *Int. J. Geom. Methods Mod. Phys.* **2006**, *3*(7).
51. Vaz, C.; Witten, L. On the spectrum of the radiation from a naked singularity. *Phys. Lett. B* **1998**, *442*, 90-96 [gr-qc/9804001].