

Conformal Symmetry and Cosmological Entropy Production

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Received: 16 August 2001 / Accepted: 3 March 2002 / Published: 10 March 2002

Abstract: Introducing an effective refraction index of an isotropic cosmic medium, we investigate the cosmological fluid dynamics which is consistent with a conformal, timelike symmetry of a corresponding “optical” metric. We demonstrate that this kind of symmetry is compatible with the existence of a negative viscous pressure and, consequently, with cosmological entropy production. We establish an exactly solvable model according to which the viscous pressure is a consequence of a self-interacting one-particle force which is self-consistently exerted on the microscopic particles of a relativistic gas. Furthermore, we show that a sufficiently high decay rate of the refraction index of an ultrarelativistic cosmic medium results in an inflationary expansion of the universe.

Keywords: General Relativity; Cosmology; Thermohydrodynamics; Kinetic Theory; Inflationary Universe.

1 Introduction

Standard relativistic cosmology relies on the cosmological principle according to which the Universe is spatially homogeneous and isotropic on sufficiently large scales. This symmetry requirement largely fixes the spacetime metric which may be written in the Robertson-Walker form (see, e.g., [1])

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]. \quad (1)$$

The quantity k can take the values $k = 0, \pm 1$ and characterizes the three possible cases of the spatial curvature. Only one unknown function, the scale factor a , has to be determined by the gravitational field equations.

For many purposes the cosmic substratum may be modeled as a fluid. So called fundamental observers are observers which are at rest with respect to the mean motion of the cosmic fluid. The symmetry requirements of the cosmological principle restrict the corresponding effective energy momentum tensor $T_{(eff)}^{ik}$ to be of the structure

$$T_{(eff)}^{ik} = \rho u^i u^k + P h^{ik}. \quad (2)$$

Here, ρ is the energy density measured by an observer comoving with the fluid four-velocity u^i which is normalized by $u^a u_a = -1$. The quantity $h_{ik} \equiv g_{ik} + u_i u_k$ is the spatial projection tensor with $h_{ik} u^k = 0$. The total pressure P is the sum

$$P = p + \Pi \quad (3)$$

of an equilibrium part $p > 0$ and a non-equilibrium part $\Pi \leq 0$ which is connected with entropy production. A perfect fluid is characterized by $\Pi = 0$. A scalar, viscous pressure is the only entropy producing phenomenon which is compatible with the symmetry requirements of the cosmological principle. Neither heat flows nor shear stresses are reconcilable with the assumption of spatial homogeneity and isotropy. This demonstrates that symmetries are generally connected with conditions on the properties and on the dynamics of the matter which generates this spacetime.

Spacetime symmetries are invariantly characterized by symmetry groups. Isometries of a spacetime are described by Killing vectors (KVs). To characterize spacetimes that admit a conformal group one uses conformal Killing vectors (CKVs). A CKV ξ^a of the metric g_{ik} is defined by

$$\mathcal{L}_\xi g_{ik} \equiv \xi_{i;k} + \xi_{k;i} = 2\phi(x) g_{ik}, \quad (4)$$

where \mathcal{L}_ξ is the Lie derivative along the vector field ξ^i . The semicolon denotes the covariant derivative with respect to g_{ik} . For the special case $\phi = 0$ the vector ξ is a KV. Maximally symmetric

spaces have the maximally possible number of KVs. The Minkowski space, e.g., has 10 KVs corresponding to its well-known 10 symmetries and conservation laws. According to the cosmological principle, the four-dimensional spacetime contains maximally symmetric three-dimensional subspaces of constant curvature, corresponding to the spatial symmetries of homogeneity and isotropy, characterized by spacelike KVs. Timelike symmetries are not generally expected to exist in the expanding universe since a timelike KV characterizes a stationary spacetime. However, it is known [2] that under certain circumstances a *conformal*, timelike symmetry is possible in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The existence of timelike or conformal, timelike KVs is closely related to “global” equilibrium properties of perfect fluids. For a gaseous fluid system of particles with mass $m > 0$, the global equilibrium condition can only be satisfied if the quantity $\xi^a \equiv u^a/T$, where T is the fluid temperature, is a (timelike) KV, equivalent to a stationary spacetime. For massless particles on the other hand, the condition for global equilibrium requires u^a/T to be a CKV. A CKV is compatible with the cosmological expansion. Macroscopically, a gas of massless particles obeys the equation of state for radiation, $p = \rho/3$. Only for this perfect fluid equation of state a global equilibrium is possible in the expanding universe. (The fact that there exists an equilibrium also in the non-relativistic limit is not relevant in this paper.) Any deviation from $p = \rho/3$ will destroy the conformal symmetry. In other words, the conformal symmetry singles out perfect fluids with the equation of state for radiation. Apparently, this also implies that a conformal symmetry is incompatible with a non-vanishing entropy production.

The purpose of the present paper is to show that, contrary to this expectation, entropy production due to a scalar viscous pressure may well be consistent with a conformal, timelike symmetry, albeit not a symmetry of the spacetime metric itself but the symmetry of an associated “optical” metric, characterized by a time dependent refraction index. The introduction of an effective refraction index of the medium as a new scalar parameter to characterize the cosmological evolution was shown to be useful in the context of cosmological particle production [3]. Here we explore the role of this parameter in more detail and for a different kind of non-equilibrium states for which the (not necessarily small) deviations from equilibrium are due to an increase in the entropy per particle, whereas the number of particles is preserved. In particular, we demonstrate that a time variation of the refraction index may take into account such kind of entropy producing deviations from the perfect fluid behavior. The standard global equilibrium for $p = \rho/3$ is then recovered as the limiting case for a refraction index unity. In this limit the optical metric coincides with the spacetime metric. Moreover, we show that a time dependent refraction index may be regarded as a consequence of the action of self-interacting forces on the microscopic constituents of an ultra-relativistic cosmic medium. Although entropy is produced, the microscopic particles are governed by an equilibrium distribution function. The deviations from standard equilibrium are mapped onto a non-zero, time varying chemical potential which is related to the refraction index in a

simple way. Furthermore, we investigate the conditions under which the corresponding viscous pressure is sufficiently large to violate the strong energy condition and discuss simple examples for an inflationary evolution of the early universe.

The paper is organized as follows. In section 2 we summarize basic thermodynamic relations for a bulk viscous fluid. Section 3 presents the conformal symmetry concept of an optical metric and clarifies its relation to the production of entropy within the medium. Section 4 is devoted to a microscopic derivation of the fluid dynamics from relativistic gas theory in terms of a self-interacting one-particle force. Cosmological applications, in particular the conditions for power-law inflation, are discussed in section 5, while section 6 sums up our conclusions on the relation between cosmological entropy production and the conformal symmetry of an optical metric. Units have been chosen so that $c = k_B = \hbar = 1$.

2 Basic thermodynamics

We assume the cosmic substratum to be modeled by a homogeneous and isotropic fluid with the energy-momentum tensor (2) and a particle number flow vector N^i , given by

$$N^i = nu^i, \quad (5)$$

where n is the particle number density. The conservation laws for particle number and energy are

$$N^i_{;i} = 0 \quad \Rightarrow \quad \dot{n} + 3Hn = 0 \quad (6)$$

and

$$u_i T_{(eff);k}^{ik} = 0 \quad \Rightarrow \quad \dot{\rho} + 3H(\rho + p + \Pi) = 0, \quad (7)$$

respectively, where $H \equiv \dot{a}/a$ is the Hubble rate and $\dot{n} \equiv n_{,i}u^i$ etc. In the following we recall basic relations for the evolution of the relevant thermodynamical quantities. With the help of the Gibbs equation (see, e.g., [4])

$$Tds = d\frac{\rho}{n} + pd\frac{1}{n}, \quad (8)$$

where s is the entropy per particle and T is the equilibrium temperature, we obtain

$$nT\dot{s} = \dot{\rho} - (\rho + p)\frac{\dot{n}}{n}. \quad (9)$$

Using here the conservation laws (6) and (7) yields

$$nT\dot{s} = -3H\Pi. \quad (10)$$

Since n , T and H are positive, the entropy per particle increases for a negative viscous pressure Π . From the Gibbs-Duhem relation (see, e.g., [4])

$$dp = (\rho + p) \frac{dT}{T} + nT d\left(\frac{\mu}{T}\right), \quad (11)$$

where μ is the chemical potential, it follows that

$$\left(\frac{\mu}{T}\right) \dot{} = \frac{\dot{p}}{nT} - \frac{\rho + p}{nT} \frac{\dot{T}}{T}. \quad (12)$$

We assume equations of state in the general form

$$p = p(n, T), \quad \rho = \rho(n, T), \quad (13)$$

i.e., particle number density and temperature are taken as the independent thermodynamical variables. Differentiating the latter relation and using the balances (6) and (7) provides us with an evolution law for the temperature,

$$\frac{\dot{T}}{T} = -3H\Sigma \frac{\partial p}{\partial \rho}, \quad (14)$$

where

$$\Sigma \equiv 1 - \frac{\dot{s}}{3H} \frac{n}{\partial p / \partial T} \quad (15)$$

and

$$\frac{\partial p}{\partial \rho} \equiv \frac{(\partial p / \partial T)_n}{(\partial \rho / \partial T)_n}, \quad \frac{\partial \rho}{\partial T} \equiv \left(\frac{\partial \rho}{\partial T}\right)_n.$$

Moreover, we have used the general relation

$$\frac{\partial \rho}{\partial n} = \frac{\rho + p}{n} - \frac{T}{n} \frac{\partial p}{\partial T},$$

which follows from the fact that the entropy is a state function, i.e., $\partial^2 s / \partial n \partial T = \partial^2 s / \partial T \partial n$. For the special case of a perfect fluid, i.e., $\dot{s} = 0$, and for ultrarelativistic matter, i.e., $p = \rho/3$, the temperature law (14) specifies to

$$\frac{\dot{T}}{T} = -\frac{\dot{a}}{a} \quad \Rightarrow \quad aT = \text{const} \quad \left(\dot{s} = 0, p = \frac{\rho}{3}\right), \quad (16)$$

which is the well-known behavior $T \propto a^{-1}$ for the radiation temperature in an expanding universe.

With the help of Eqs. (13), (6), (9) and (14), the time evolution of the pressure may generally be written as

$$\dot{p} = c_s^2 \dot{\rho} - 3H\Pi \left[\frac{\partial p}{\partial \rho} - c_s^2 \right], \tag{17}$$

where

$$c_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_{ad} = \frac{n}{\rho + p} \frac{\partial p}{\partial n} + \frac{T}{\rho + p} \frac{(\partial p / \partial T)^2}{\partial \rho / \partial T} \tag{18}$$

is the square of the adiabatic sound velocity c_s [see, e.g. [5]]. Using Eq. (17) in Eq. (12) yields

$$\left(\frac{\mu}{T} \right)' = 3H \left\{ \frac{\rho + p}{nT} \left[\frac{\partial p}{\partial \rho} - c_s^2 \right] - \frac{\Pi}{nT} \left[\frac{\partial p}{\partial \rho} - \frac{\rho + p}{T \partial \rho / \partial T} \right] \right\}. \tag{19}$$

The relations so far are completely general. Below we shall use them in connection with a relativistic Maxwell-Boltzmann gas with internal self-interactions. It is expedient to emphasize that a viscous pressure Π according to (19) influences the time behavior of the quantity μ/T . Later on we shall be interested in a special case in which this non-equilibrium term is the only “source” for $(\mu/T)'$.

3 Conformal symmetry of optical metrics

Given a spacetime metric g_{ik} and an isotropic medium with a four-velocity u^a and a refraction index n_r , one introduces the “optical” metric \bar{g}_{ik} by [6]

$$\bar{g}_{ik} \equiv g_{ik} + \left(1 - \frac{1}{n_r^2} \right) u_i u_k = h_{ik} - \frac{1}{n_r^2} u_i u_k. \tag{20}$$

Optical metrics are known to be helpful in simplifying the equations of light propagation in isotropic, refractive media. With respect to a metric \bar{g}_{ik} light propagates as in vacuum. Here we are interested in the role of optical metrics in relativistic gas dynamics. Guided by the connection between spacetime symmetries and equilibrium properties, outlined in the introduction, our starting point is to assume $\xi^a \equiv u^a/T$ to be a conformal Killing vector of the optical metric, i.e.,

$$\mathcal{L}_\xi \bar{g}_{ik} = 2\psi \bar{g}_{ik}. \tag{21}$$

The main objective of this paper is to explore the fluid dynamics which is compatible with the latter requirement. In particular, we are interested in the question to what extent the CKV

condition (21) admits the production of entropy in a cosmological context. In a first step, using the relations

$$\mathcal{L}_\xi \frac{u^i}{T} = 0, \quad \mathcal{L}_\xi \frac{u_i}{T} = \frac{u^a}{T} \mathcal{L}_\xi g_{ia}, \quad \mathcal{L}_\xi S = \frac{\dot{S}}{T},$$

the last one being valid for any scalar S , the left-hand side of Eq. (21) is written as

$$\mathcal{L}_\xi \bar{g}_{ik} = \mathcal{L}_\xi g_{ik} + 2 \frac{u_i u_k}{T} \left[\left(1 - \frac{1}{n_r^2} \right) \frac{\dot{T}}{T} + \frac{\dot{n}_r}{n_r^3} \right] + 2 \left[1 - \frac{1}{n_r^2} \right] u_k u^a \mathcal{L}_\xi g_{ia}. \quad (22)$$

In a second step we find an explicit expression for the first term on the right-hand side of Eq. (22). As any symmetric tensor, the Lie derivative of g_{ik} may be decomposed into contributions parallel and perpendicular to the four-velocity. For homogeneous and isotropic media such a split amounts to

$$\mathcal{L}_\xi g_{ik} \equiv \left(\frac{u_i}{T} \right)_{;k} + \left(\frac{u_k}{T} \right)_{;i} = \frac{2}{T} \frac{\dot{a}}{a} g_{ik} + \frac{2}{T} \left(\frac{\dot{T}}{T} + \frac{\dot{a}}{a} \right) u_i u_k, \quad (23)$$

where we have used that in a FLRW universe $u_{;k}^i = H h_k^i \Rightarrow u_{;i}^i = 3\dot{a}/a$. Combining Eqs. (22) and (23) we obtain

$$\mathcal{L}_\xi \bar{g}_{ik} = \frac{2}{T} \frac{\dot{a}}{a} h_{ik} + 2 \frac{u_i u_k}{n_r^2 T} \left(\frac{\dot{n}_r}{n_r} + \frac{\dot{T}}{T} \right) \quad (24)$$

for the left-hand side of Eq. (21). Together with the definition (20) one finds

$$\psi = \frac{1}{T} \frac{\dot{a}}{a}$$

for the conformal factor in (21) and

$$\frac{\dot{n}_r}{n_r} = - \left[\frac{\dot{a}}{a} + \frac{\dot{T}}{T} \right] \quad (25)$$

for the change rate of the refraction index. The last relation implies

$$n_r \propto \frac{1}{aT},$$

i.e., all deviations from $aT = \text{const}$ (pure radiation, $p = \rho/3$, cf. Eq. (16)) have been mapped onto the time-varying refraction index n_r . This demonstrates explicitly, that n_r is indeed a useful

quantity to characterize a fluid dynamics with a temperature evolution that is different from that for $\dot{s} = 0$ and $p = \rho/3$.

Inserting now the general temperature law (14) into Eq. (25), we obtain

$$\frac{\dot{n}_r}{n_r} = - \left[1 - 3 \frac{\partial p}{\partial \rho} \Sigma \right] \frac{\dot{a}}{a} . \quad (26)$$

The variable Σ depends on \dot{s} according to Eq. (15). Consequently, Eq. (26) establishes a relation between the rate of change of the refraction index and the expansion rate for a viscous cosmological fluid under the condition of a conformal symmetry of the optical metric \bar{g}_{ik} . This implies the statement that a conformal symmetry (21) does not contradict a production of entropy. To obtain a better understanding of this feature, we present a derivation for such kind of fluid dynamics for a relativistic gas in the following section.

4 Kinetic theory

The particles of a relativistic gas are assumed to move under the influence of a four-force F^i in between elastic binary collisions, described by Boltzmann's collision integral $C[f, f]$. The equations of motion of the gas particles are

$$\frac{dx^i}{d\gamma} = p^i , \quad \frac{Dp^i}{d\gamma} = F^i , \quad (27)$$

where γ is a parameter along the particle worldline which for massive particles may be related to the proper time τ by $\gamma = \tau/m$. Since the particle four-momenta are normalized according to $p^i p_i = -m^2$, the force F^i has to satisfy the relation $p_i F^i = 0$. The corresponding equation for the invariant one-particle distribution function $f = f(x(\tau), p(\tau))$ may be written as (cf. [4, 7, 8, 9])

$$L[f] + \frac{\partial(F^i f)}{\partial p^i} = C[f, f] , \quad (28)$$

where

$$L[f] \equiv p^i \frac{\partial f}{\partial x^i} - \Gamma_{ii}^k p^i p^l \frac{\partial f}{\partial p^k} \quad (29)$$

is the Liouville operator. It will turn out that a suitably chosen effective one-particle force F^i may give rise to a time-dependent refraction index on the macroscopic level. The particle number flow 4-vector N^i and the energy momentum tensor T^{ik} are defined in a standard way (see, e.g., [7, 8]) as

$$N^i = \int dP p^i f(x, p) , \quad T^{ik} = \int dP p^i p^k f(x, p) . \quad (30)$$

Since we shall identify the particle number flow in (30) with the corresponding quantity in Eq. (5), we have used here the same symbol, although these expressions may not coincide in the general case. The integrals in the definitions (30) and in the following are integrals over the entire mass shell $p^i p_i = -m^2$. The entropy flow vector S^a is given by [7, 8]

$$S^a = - \int p^a [f \ln f - f] dP , \tag{31}$$

where we have restricted ourselves to the case of classical Maxwell-Boltzmann particles. Using the general relationship [9]

$$\left[\int p^{a_1} \dots p^{a_n} p^b f dP \right]_{;b} = \int p^{a_1} \dots p^{a_n} L[f] dP , \tag{32}$$

we find for the balance equations

$$N_{;a}^a = \int dP \left\{ C[f, f] - \frac{\partial(f F^i)}{\partial p^i} \right\} = 0 \tag{33}$$

and

$$T_{;k}^{ak} = \int dP p^a \left\{ C[f, f] - \frac{\partial(f F^i)}{\partial p^i} \right\} = \int dP f F^a . \tag{34}$$

In particular, the energy balance becomes

$$u_a T_{;k}^{ak} = \int dP f u_a F^a . \tag{35}$$

The entropy production density is a sum of two terms:

$$S_{;i}^i = \sigma_c + \sigma_F . \tag{36}$$

Here,

$$\sigma_c \equiv - \int dP C[f, f] \ln f \tag{37}$$

is the familiar contribution of Boltzmann's collision integral, while

$$\sigma_F \equiv - \int dP F^i \frac{\partial f}{\partial p^i} = \int dP f \frac{\partial F^i}{\partial p^i} \tag{38}$$

takes into account an entropy production due to the action of the force F^i . Since Boltzmann's H theorem guarantees $\sigma_c \geq 0$, we have

$$S_{;i}^i - \sigma_F \geq 0 . \tag{39}$$

We restrict ourselves to the class of forces which admit solutions of Eq. (28) that are of the type of Jüttner's distribution function

$$f^0(x, p) = \exp[\alpha + \beta_a p^a], \quad (40)$$

where $\alpha = \alpha(x)$ and $\beta_a(x)$ is timelike. For $f \rightarrow f^0$ the collision integral vanishes, i.e., $C[f^0, f^0] = 0$. With f replaced by f^0 in the definitions (30) and (31), the quantities N^a , T^{ab} and S^a may be split with respect to the unique four-velocity u^a according to

$$N^a = nu^a, \quad T^{ab} = \rho u^a u^b + ph^{ab}, \quad S^a = nsu^a. \quad (41)$$

Here we have identified the general fluid quantities of the previous section with those following from the dynamics of a Maxwell-Boltzmann gas. Note that the energy-momentum tensor T^{ik} in Eqs. (30) and (41) does *not* coincide with the tensor $T_{(eff)}^{ik}$ introduced in Eq. (2). The exact integral expressions for n , ρ , p are given, e.g., in [4]). The entropy per particle s is

$$s = \frac{\rho + p}{nT} - \frac{\mu}{T}. \quad (42)$$

For $f \rightarrow f^0$ the energy balance (35) takes the form

$$\dot{\rho} + 3H(\rho + p) = - \int dP f^0 u_a F^a. \quad (43)$$

The entropy production density becomes

$$S_{;a}^a = \sigma_F = n\dot{s} = - \int dP \beta_i F^i. \quad (44)$$

With the identification $\beta_i = u_i/T$ we may write

$$n\dot{s} = -3H \frac{\Pi}{T}, \quad (45)$$

where we have introduced the quantity Π by

$$3H\Pi \equiv \int dP u_i F^i. \quad (46)$$

Notice that we have identified this kinetic theory based viscous pressure with the corresponding phenomenological quantity of the previous sections. Consequently, the energy balance (43) takes the form (7). This implies the conclusion that *a non-vanishing viscous pressure may generally be compatible with an equilibrium-type distribution function, provided, the gas particles move in a suitable force field*. Conventionally, a bulk viscous pressure arises as a consequence of deviations

from an equilibrium distribution. Whereas these deviations have to be small in standard non-equilibrium theories, a corresponding restriction is absent in the present case.

Substituting the distribution (40) into Eq. (28) we obtain

$$p^a \alpha_{,a} + \beta_{(a;b)} p^a p^b = -\beta_i F^i - \frac{\partial F^i}{\partial p^i} . \tag{47}$$

To make further progress, we have to choose specific force types. The most general force compatible with the cosmological principle is [10]

$$F^i = -F (E p^i + p^a p_a u^i) , \tag{48}$$

where $E \equiv -u_i p^i$ is the particle energy measured by a comoving observer. Generally, F may still depend on E . With

$$\begin{aligned} \frac{\partial F^i}{\partial p^i} &= -\frac{\partial F}{\partial p^i} (E p^i + p^a p_a u^i) - F (-u_i p^i + 4E + 2p_i u^i) \\ &= \frac{\partial F}{\partial E} (m^2 - E^2) - 3FE , \end{aligned} \tag{49}$$

the condition (47) then specifies to

$$p^a \alpha_{,a} + \beta_{(a;b)} p^a p^b = -3F u_a p^a + \left(\frac{\partial F}{\partial E} - \frac{F}{T} \right) h_{ab} p^a p^b . \tag{50}$$

Now we restrict ourselves to the case $\partial F/\partial E = 0$ and consider massless particles $p^i p_i = -m^2 = 0$, which macroscopically corresponds to an equation of state $p = \rho/3$. As a consequence we find

$$\alpha_{,a} = -3F u_a \quad \Rightarrow \quad \dot{\alpha} = 3F \tag{51}$$

and

$$\beta_{(a;b)} \equiv \frac{1}{2} \left[\left(\frac{u_a}{T} \right)_{;b} + \left(\frac{u_b}{T} \right)_{;a} \right] = \frac{H}{T} g_{ab} - \frac{F}{T} u_a u_b . \tag{52}$$

Combining the relations (51) and (52), we obtain

$$\beta_{(a;b)} = \frac{1}{2} \mathcal{L}_\xi g_{ab} = \frac{H}{T} g_{ab} - \frac{\dot{\alpha}}{3T} u_a u_b . \tag{53}$$

Using the equation of state $p = nT = \rho/3$ in Eq. (12) and identifying $\alpha = \mu/T$, we have

$$\left(\frac{\mu}{T} \right)' = \dot{\alpha} = -3 \left(\frac{\dot{a}}{a} + \frac{\dot{T}}{T} \right) . \tag{54}$$

In the standard case (without a force contribution) one has $\alpha = \text{const}(= 0)$ and recovers the familiar cooling rate $\dot{T}/T = -\dot{a}/a$ for a relativistic fluid. We conclude that *the force field manifests itself in the existence of a non-vanishing, time dependent chemical potential*. From (54) and (53) we recover

$$T u^a u^b \beta_{(a;b)} = \frac{T}{2} u^a u^b \mathcal{L}_\xi g_{ab} = \frac{\dot{T}}{T} , \tag{55}$$

which shows the consistency of our approach. Comparing (53) with (23) we obtain

$$\frac{\dot{T}}{T} + \frac{\dot{a}}{a} = -\frac{\dot{\alpha}}{3} \quad \Rightarrow \quad \frac{\dot{n}_r}{n_r} = \frac{\dot{\alpha}}{3} = F , \tag{56}$$

i.e., with the identification $\dot{n}_r/n_r = \dot{\alpha}/3$ we recover the conformal symmetry relation (25). Moreover, we have traced back the time variation of the refraction index to a self-consistent interaction within the system. From the moment equation (34) it follows that

$$T_{;k}^{ak} = F u_k T^{ak} \quad \Rightarrow \quad u_a T_{;k}^{ak} = -\dot{\rho} - 3H(\rho + p) = F\rho . \tag{57}$$

Since $\rho = 3nT$ we reproduce Eq. (7) with

$$3H\Pi = 3nTF \quad \Rightarrow \quad F = \frac{\Pi}{nT} H . \tag{58}$$

This is consistent with (46) and (48). The quantity F depends on macroscopic fluid quantities. According to (27) and (48), these macroscopic quantities determine the motion of the individual microscopic particles which themselves are the constituents of the macroscopic medium. Consequently, the force F^i describes a self-interaction of the fluid. This self-interaction is accompanied by a time variation of the chemical potential, equivalent to a time dependent refraction index.

From the entropy production density (44) we find

$$S_{;a}^a = -\frac{F}{T} u_a u_b T^{ab} = -\frac{3H\Pi}{T} = n\dot{s} . \tag{59}$$

This means

$$\dot{s} = -\frac{3H\Pi}{nT} , \tag{60}$$

consistent with (45) and the phenomenological relation (10). The ratio $\dot{s}/(3H) = -\Pi/(nT)$ which characterizes the quantity Σ is just the ratio between non-equilibrium pressure Π and equilibrium pressure $p = nT$. Via Eqs. (51) and (58), relation (60) also implies

$$\left(\frac{\mu}{T}\right)' = \frac{3H\Pi}{nT} = -\dot{s} = 3\frac{\dot{n}_r}{n_r} = 3F . \tag{61}$$

With the first equation in (61) we have recovered the special case of Eq. (19) for $p = nT$, $\rho = 3nT$, $c_s^2 = \partial p / \partial \rho = 1/3$. Furthermore, we have precised, how a decaying refraction index generates a negative contribution to the pressure, equivalent to positive entropy production. Integrating the part $(\mu/T)^\cdot = 3\dot{n}_r/n_r$ of Eq. (61) by assuming $\mu/T \rightarrow 0$ for $n_r \rightarrow 1$ (standard radiation with $\mu = 0$ and $n_r = 1$), we find

$$\exp\left[\frac{\mu}{T}\right] = n_r^3. \quad (62)$$

Consequently, the distribution function (40) can be written as

$$f^0 = \exp\left[\frac{\mu}{T}\right] \exp\left[-\frac{E}{T}\right] = n_r^3 \exp\left[-\frac{E}{T}\right], \quad (63)$$

i.e., the dissipative force gives rise to a time varying factor n_r^3 in the equilibrium-type distribution function.

In order to check the consistency of the entire framework, we may now use the force (48) with (58) in the microscopic equations of motion in Eq. (27). Contracting the latter with the macroscopic four-velocity results in

$$\frac{D(u_i p^i)}{d\gamma} = u_i F^i + u_{i;k} p^i p^k, \quad (64)$$

where we have used that

$$\frac{D u^i}{d\gamma} = u^i{}_{;n} p^n.$$

Since $u_{i;k} = H h_{ik}$ for the homogeneous, isotropic case under consideration here and with Eq. (48) for $m = 0$ and (58) this reduces to

$$-\frac{dE}{d\gamma} = \left(1 + \frac{\Pi}{nT}\right) H E^2. \quad (65)$$

Since $dE/d\gamma = E_{,m} p^m = E \dot{E}$ [cf. Eq. (27)] we obtain finally

$$-\frac{\dot{E}}{E} = \left(1 + \frac{\Pi}{nT}\right) \frac{\dot{a}}{a} \Rightarrow \frac{\dot{E}}{E} = \frac{\dot{T}}{T}. \quad (66)$$

It follows that the distribution function (63) is indeed maintained for particles under the action of the force F^i , which proves the consistency of our approach. While the second factor in the expression (63) for f^0 remains constant, the first one changes according to Eq. (61).

5 Conformal symmetric universe

Now we combine the non-equilibrium fluid dynamics of the previous section with the gravitational field equations. In a homogeneous and isotropic, spatially flat universe Einstein's equations reduce to

$$8\pi G\rho = 3H^2, \quad \dot{H} = -4\pi G(\rho + p + \Pi). \quad (67)$$

The relation between Π and \dot{n}_r in Eq. (61) may be solved with respect to the non-equilibrium pressure,

$$\Pi = \frac{1}{3} \frac{\dot{n}_r/n_r}{\dot{a}/a} \rho. \quad (68)$$

For $\rho + 3P$ we obtain

$$\rho + 3P = \frac{1}{3} \left(\frac{\dot{n}_r/n_r}{\dot{a}/a} + 2 \right). \quad (69)$$

The case $\rho + 3P = 0$ is equivalent to

$$-\frac{\dot{n}_r}{n_r} = 2\frac{\dot{a}}{a}. \quad (70)$$

Consequently, for a decaying refraction index with $|\dot{n}_r/n_r| \geq 2\dot{a}/a$ the strong energy condition $\rho + 3P > 0$ is violated and we have accelerated expansion. This may be explicitly demonstrated for a simple example. With Π from (68) the energy balance (7) may be written

$$\dot{\rho} + \left(4\frac{\dot{a}}{a} + \frac{\dot{n}_r}{n_r} \right) \rho = 0 \quad \Rightarrow \quad \rho n_r a^4 = \text{const}. \quad (71)$$

The simplest case is to assume a decay rate which is proportional to the Hubble rate:

$$\frac{\dot{n}_r}{n_r} = -4\beta\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho a^{4(1-\beta)} = \text{const}. \quad (72)$$

The constant factor β in the range $0 \leq \beta < 1$ has been chosen for convenience. Of course, β can be assumed constant at most piecewise. Integration of Friedmann's equation yields

$$a \propto t^{\frac{1}{2(1-\beta)}}. \quad (73)$$

Obviously, there is accelerated expansion $\ddot{a} \geq 0$ for $\beta \geq 1/2$, which is consistent with (70). The limit $\beta = 0$ corresponds to the standard radiation dominated universe. If β approaches $\beta \approx 1$, we have $\rho \approx \text{const}$ and, according to Eq. (67), an approximately constant Hubble rate H , i.e.,

exponential expansion. Any β in the range $0 \leq \beta < 1$ is connected with entropy production. Combining (68) and (72) we find

$$\Pi = -\frac{4}{3}\beta\rho \quad (74)$$

and, with (60),

$$\dot{s} = 12\beta\frac{\dot{a}}{a} = 6\frac{\beta}{(1-\beta)t} . \quad (75)$$

The entropy per particle grows logarithmically with the cosmic time. According to (58), (74), and (48), the corresponding force is

$$F^i = 4\beta H E p^i . \quad (76)$$

The action of this force on the microscopic constituents of the medium realizes a power-law evolution (73) of the universe, which is a manifestation of a conformal symmetry (21) of the optical metric (20) and, moreover, implies a positive entropy production according to (75). This represents an exactly solvable model, both macroscopically and microscopically, of a specific non-equilibrium configuration of a self-interacting cosmic gas.

6 Conclusions

We have shown that the introduction of a refraction index n_r of the cosmic medium as a new scalar parameter allows us to understand early phases of the cosmological evolution as a specific non-equilibrium configuration of a self-interacting relativistic gas. The latter is characterized by an equilibrium-type distribution function of the microscopic particles. Deviations from equilibrium are mapped onto a time-varying chemical potential which is related to n_r in a simple way. The refraction index defines an optical metric $\bar{g}_{ik} = g_{ik} - (1 - n_r^{-2})u_i u_k$ which coincides with the spacetime metric g_{ik} for $n_r = 1$. A decaying refraction index gives rise to a negative contribution to the scalar pressure of the cosmic substratum. We have established an exactly solvable, self-consistent model for this type of dynamics which is connected with entropy production, but at the same time realizes a conformal, timelike symmetry of the optical metric. In other words, for any deviation from the behavior of an ultrarelativistic perfect fluid which is due to a scalar, viscous pressure, one may find a refraction index such that the evolution of the universe appears as the manifestation of a conformal, timelike symmetry of the corresponding optical metric. For a decay rate of the refraction index which is of the order of the Hubble expansion rate, the associated negative pressures may lead to a violation of the strong energy condition $\rho + 3P > 0$. Consequently, the resulting dynamics is inflationary.

7 Acknowledgment

This paper was supported by the Deutsche Forschungsgemeinschaft and by NATO.

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