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The Modes of the Poisson Distribution of Order 3 and 4

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Abstract: In this article, new properties of the Poisson distribution of order k with parameter λ are found. Based on them, the modes of the Poisson distributions of order $k = 3$ and 4 are derived for λ in $(0, 1)$. They are 0, 3, 5, and 0, 4, 7, 8, respectively, for λ in specified subintervals of (0, 1). In addition, using Mathematica, computational results for the modes of the Poisson distributions of order $k = 2, 3$, and 4 are presented for λ in specified subintervals of $(0, 2)$.

Keywords: poisson distribution of order k; discrete distribution; mode; most probable value; order 4

1. Introduction

Following the papers of Philippou and Muwafi [1], Philippou et al. [2], Philippou [3–5], Aki et al. [6] and Aki [7], there has been an upsurge in the study of distributions of order k (distributions of runs) due to their theoretical importance and great applicability in reliability, start-up demonstration tests, sampling inspection, etc. See, e.g., Ling [8], Mohanty [9], Chang [10], Johnson et al. [11], Shmueli and Kohen [12], Balakrishnan and Koutras [13], Fu and Lou [14], Eryilmaz [15], Rakitzis and Antzoulakos [16], Dafnis et al. [17], Sengar et al. [18], Kwon [19], and references therein. However, the modes of these distributions are not yet known, except for the modes of the geometric distribution of order k and partial results for the mode(s) of the Poisson distribution of order k and the negative binomial distribution of the same order derived by Luo [20], Georghiou et al. [21], Philippou [22], Shao and Fu [23], and Georghiou et al. [24].

The Poisson distribution of order k ($k \geq 1$, integer) with parameter $\lambda(>0)$ say $P_k(\lambda)$, has probability mass function (pmf)

$$f_k(x;\lambda) = e^{-k\lambda} \sum \frac{\lambda^{x_1+x_2+\cdots+x_k}}{x_1!x_2!\cdots x_k!}, \quad x = 0, 1, 2, \ldots, \tag{1}$$

where the summation is taken over all k-tuples of non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$.

It was derived by Philippou et al. [2] as a limit of the negative binomial distribution of order k, and it was named so, since, for $k = 1$, it reduces to the Poisson distribution with parameter λ. It is a special case, for $\lambda_1 = \lambda_2 = \cdots = \lambda_k = \lambda$, of the multiparameter Poisson distribution of order k (Philippou [25]), also known as k stuttering Poisson distribution (Galliher et al. [26], Patel [27]). The latter author discussed the estimation of the parameters of the triple and quadruple stuttering distributions and noted that the cases k = 2, 3, and 4 are more frequently observed in practice.

Let $m_{k,\lambda}$ denote the mode(s) of $f_k(x;\lambda)$, i.e., the value(s) of x for which $f_k(x;\lambda)$ attains its maximum. It is well known that $m_{1,\lambda} = \lambda$ or $\lambda - 1$ if $\lambda \in \mathbb{N}$ and $m_{1,\lambda} = \lfloor \lambda \rfloor$, if λ does not belong to $\mathbb{N}$, where $\lfloor \alpha \rfloor$ denotes the greatest integer part of α. Philippou [3] derived some properties of $f_k(x;\lambda)$ and posed the problem of finding its mode(s) for $k \geq 2$.

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Hirano et al. [28] presented several graphs of $f_k(x;\lambda)$ for $\lambda \in (0,1)$ and $2 \leq k \leq 8$, and Luo [20] derived the following lower bound inequality for $m_{k,\lambda}$,

$$m_{k,\lambda} \geq k\lambda\sqrt[k]{k!} - \frac{1}{2}k(k+1), \quad k \geq 1, \quad \lambda > 0.$$

Georghiou et al. [21] employed the probability generating function of the Poisson distribution of order k to improve the lower bound of Luo [20] and also to give an upper bound for $m_{k,\lambda}$

$$\frac{1}{2}k(k+1)(\lambda-1)+1-\delta_{k,1} \leq m_{k,\lambda} \leq \left\lfloor \frac{1}{2}k(k+1)\lambda \right\rfloor = u_{k,\lambda}, \quad k \geq 1, \quad \lambda > 0, \tag{2}$$

where $\delta_{k,1}$ denotes the Kronecker delta. With the bounds of $m_{k,\lambda}$ in (2), they showed that

$$m_{k,\lambda} = \frac{1}{2}k(k+1)\lambda - \left\lfloor \frac{k}{2} \right\rfloor, \quad 2 \leq k \leq 5, \quad \lambda \in \mathbb{N}. \tag{3}$$

Using the upper bound $u_{k,\lambda}$ of (2) and the definition of $m_{k,\lambda}$, Philippou [22] found that:

(a) For any integer $k \geq 1$ and $0 < \lambda < 2/k(k+1)$, the Poisson distribution of order k has a unique mode $m_{k,\lambda} = 0$.

(b) The Poisson distribution of order 2 has a unique mode $m_{k,\lambda} = 0$ if $0 < \lambda < \sqrt{3}-1$; it has two modes $m_{k,\lambda} = 0$ and 2 if $0 < \lambda = \sqrt{3}-1$, and it has a unique mode $m_{k,\lambda} = 2$ if $\sqrt{3}-1 < \lambda < 1$. (The number $\sqrt{3}-1$ is the positive root (say r_2) of the quadratic equation $\lambda^2 + 2\lambda - 2 = 0$.)

Remark 1. *Since the modes of the Poisson distribution of order k with parameter λ are defined as the values of $x \in \{0,1,2,\ldots\}$, which maximize $f_k(x;\lambda)$, they are its most probable values and they may be obtained numerically for any given positive integer k and positive λ, from*

$$f_k(m_{k,\lambda};\lambda) = \max\Big\{ f_k(x;\lambda) \,\Big|\, x \in \{0,1,2,\ldots,u_k(\lambda)\} \Big\}.$$

In the present short note, we derive some additional properties of $f_k(x;\lambda)$ and find the modes of the Poisson distribution of order $k=3$ and $k=4$ for $0 < \lambda < 1$. Furthermore, Section 3 presents computational results for the modes of the Poisson distributions of order $k = 2$, 3, and 4 for $\lambda \in (0, 2)$. Finally, in Section 4, we briefly discuss our results, give the moment estimator of λ (> 0) for $k \geq 1$, and indicate further research.

2. Main and Preliminary Results

The mode(s) of a discrete probability mass function is (are) its most probable value(s). In this section, we derive the modes of the Poisson distributions of order 3 and 4, respectively, when $0 < \lambda < 1$ (see Propositions 1 and 2). In order to do so, we first state and prove three lemmas, regarding $h_k(x;\lambda) = e^{k\lambda} f_k(x;\lambda)$, which we use, along with relation (2), to prove the propositions.

Because of (1),

$$h_k(x;\lambda) = \sum \frac{\lambda^{x_1+x_2+\cdots+x_k}}{x_1!x_2!\cdots x_k!}, \quad x = 0,1,2,\ldots;\ \lambda > 0, \tag{4}$$

where the summation is taken over all k-tuples of non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$. Note that $h_k(0;\lambda) = 1$ and $h_{k_1}(x;\lambda) = h_{k_2}(x;\lambda)$, for $1 \leq x \leq k_1 \leq k_2$.

Georghiou et al. [21] provided a recursive form of $f_k(x;\lambda)$ as

$$xf_k(x;\lambda) = \sum_{j=1}^{k} j\lambda f_k(x-j;\lambda), \quad x \geq 1. \tag{5}$$

It can be restated, in terms of $h_k(x;\lambda)$, as

$$xh_k(x;\lambda) = \begin{cases} \sum_{j=1}^{x} j\lambda h_k(x-j;\lambda) & 1 \leq x \leq k \\ \\ \sum_{j=1}^{k} j\lambda h_k(x-j;\lambda) & x > k, \end{cases} \tag{6}$$

with $h_k(0;\lambda) = 1$.

Lemma 1. *For $2 \leq x \leq k$ and a fixed $\lambda > 0$,*

$$\lambda \leq h_k(x-1;\lambda) < h_k(x;\lambda).$$

Proof. To avoid the abuse of notation, let $h(x) \equiv h_k(x;\lambda)$. From (4), it is easy to see $h(1) = \lambda$. Using (6), for $2 \leq x \leq k$,

$$\begin{aligned} xh(x) - (x-1)h(x-1) &= \lambda \sum_{j=1}^{x} jh(x-j) - \lambda \sum_{j=1}^{x-1} jh(x-1-j) \\ &= \lambda \sum_{j=1}^{x} jh(x-j) - \lambda \sum_{j=2}^{x} (j-1)h(x-j) \\ &= \lambda \sum_{j=1}^{x} jh(x-j) - \lambda \sum_{j=2}^{x} jh(x-j) + \lambda \sum_{j=2}^{x} h(x-j) \\ &= \lambda h(x-1) + \lambda \sum_{j=2}^{x} jh(x-j) - \lambda \sum_{j=2}^{x} jh(x-j) + \lambda \sum_{j=2}^{x} h(x-j) \\ &= \lambda h(x-1) + \lambda \sum_{j=2}^{x} h(x-j) \\ &= \lambda \sum_{j=1}^{x} h(x-j) \end{aligned}$$

From (6), since $(x-1)h(x-1) = \sum_{j=1}^{x-1} j\lambda h(x-1-j)$, we have

$$\begin{aligned} x\Big[h(x) - h(x-1)\Big] &= xh(x) - (x-1)h(x-1) - h(x-1) \\ &= \lambda \left[\sum_{j=1}^{x} h(x-j) - \frac{1}{x-1} \sum_{j=1}^{x-1} jh(x-1-j) \right] \\ &= \lambda \left[\sum_{j=1}^{x} h(x-j) - \frac{1}{x-1} \sum_{j=2}^{x} (j-1)h(x-j) \right] \\ &= \lambda \left[h(x-1) + \sum_{j=2}^{x} h(x-j) - \frac{1}{x-1} \sum_{j=2}^{x} (j-1)h(x-j) \right] \\ &= \lambda \left[h(x-1) + \sum_{j=2}^{x} \left(1 - \frac{j-1}{x-1}\right) h(x-j) \right] > 0. \end{aligned}$$

Therefore, $h_k(x;\lambda) > h_k(x-1;\lambda)$ for $2 \leq x \leq k$ with a fixed value of $\lambda > 0$. $\square$

Lemma 2. *For $k \geq 2$ and $0 < \lambda < 1$, the equation $h_k(k;\lambda) = h_k(0;\lambda)$ has exactly one root $\lambda = r_k$ $(0 < r_k < 1)$ such that*

$$\begin{cases} h_k(0;\lambda) > h_k(k;\lambda), & \text{if } 0 < \lambda < r_k \\ h_k(0;\lambda) < h_k(k;\lambda), & \text{if } r_k < \lambda < 1. \end{cases}$$

Proof. First, note that $h_k(0;1) = 1$ and $\lim_{\lambda\to 0^+} h_k(0;\lambda) = 1$ because the relation (4) implies $h_k(0;\lambda) = 1$ for $\lambda > 0$. Since, for $k \geq 1$, $h_k(k;\lambda)$ is a polynomial function of λ with positive coefficient only and without constant term, we have $\lim_{\lambda\to 0^+} h_k(k;\lambda) = 0$. Note that $h_k(1;\lambda) = \lambda$ and by Lemma 1, $h_k(1;\lambda) < h_k(2;\lambda) < \cdots < h_k(k;\lambda)$ for $\lambda > 0$. Thus, $h_k(k;1) > h_k(1;1) = 1$ for $k \geq 2$.

Second, let $g_k(\lambda) = h_k(k;\lambda) - h_k(0;\lambda)$. Then, since $\lim_{\lambda\to 0^+} g_k(\lambda) = \lim_{\lambda\to 0^+} h_k(k;\lambda) - \lim_{\lambda\to 0^+} h_k(0;\lambda) = 0 - 1 = -1 < 0$, and $g_k(1) = h_k(k;1) - h_k(0;1) > h_k(1;1) - h_k(0;1) = 1 - 1 = 0$.

Lastly, Since $h_k(x;\lambda)$ is a polynomial function with positive coefficients only, we have $\frac{\partial}{\partial\lambda} g_k(\lambda) > 0$ for $\lambda > 0$, which implies $g_k(\lambda)$ is an increasing function of λ for $0 < \lambda < 1$.

Therefore, $g_k(\lambda)$ has a unique root, say r_k, between 0 and 1 with $h_k(0;\lambda) > h_k(k;\lambda)$, if $0 < \lambda < r_k$ and $h_k(0;\lambda) < h_k(k;\lambda)$, if $r_k < \lambda < 1$. $\square$

Lemma 3. *For $k \geq 2$ and $0 < \lambda \leq r_k < 1$,*

$$h_k(k;\lambda) > h_k(k+1;\lambda),$$

where r_k is defined in Lemma 2.

Proof. For notational simplicity, let $h(x) \equiv h_k(x;\lambda)$. From (6), we have

$$kh(k) = \sum_{j=1}^{k} j\lambda h_k(k-j) \quad \text{and} \quad (k+1)h_k(k+1) = \sum_{j=1}^{k} j\lambda h_k(k+1-j).$$

Thus, with $h(0) = 1$,

$$\begin{aligned} k\Big[h(k) - h(k+1)\Big] &= \sum_{j=1}^{k} j\lambda h(k-j) - \sum_{j=1}^{k} j\lambda h(k+1-j) + h(k+1) \\ &= \sum_{j=1}^{k} j\lambda h(k-j) - \sum_{j=0}^{k-1} (j+1)\lambda h(k-j) + h(k+1) \\ &= \sum_{j=1}^{k} j\lambda h(k-j) - \sum_{j=0}^{k-1} j\lambda h(k-j) - \sum_{j=0}^{k-1} \lambda h(k-j) + h(k+1) \\ &= k\lambda h(0) - \sum_{j=0}^{k-1} \lambda h(k-j) + h(k+1) \\ &= \lambda\left(k - \sum_{j=0}^{k-1} h(k-j)\right) + h(k+1). \end{aligned} \tag{7}$$

Since each $h_k(1;\lambda),\ h_k(2;\lambda), \ldots,\ h_k(k;\lambda)$ are increasing functions of lambda, they all have the maximum value at $\lambda = r_k$ on $\lambda \in (0, r_k]$. Moreover, by Lemma 1 and the definition of

r_k, we have $h_k(1; r_k) < h_k(2; r_k) < \cdots < h_k(k; r_k) = 1$. Therefore, for $0 < \lambda \leq r_k$, the first term of (7) can be written as

$$\begin{aligned} k - \sum_{j=0}^{k-1} h_k(k-j;\lambda) &\geq k - \sum_{j=0}^{k-1} h_k(k-j;r_k) \\ &= k - \Big[h_k(k;r_k) + h_k(k-1;r_k) + \cdots + h_k(1;r_k)\Big] \\ &> k - k\Big[(h_k(k;r_k)\Big] = 0, \end{aligned}$$

and it implies $h_k(k;\lambda) > h_k(k+1;\lambda)$. □

Proposition 1. *Let $m_{3,\lambda}$ denote the mode(s) of the Poisson distribution of order 3 with parameter $\lambda(> 0)$. Let r_3 (=0.6016791318...) and $r_{3,5}$ (=0.9962030611...) be the positive roots of the equations $\lambda^3 + 6\lambda^2 + 6\lambda - 6 = 0$, and $\lambda^4 + 20\lambda^3 + 100\lambda^2 - 120 = 0$, respectively. Then*

$$m_{3,\lambda} = \begin{cases} 0, & \text{if } 0 < \lambda < r_3, \\ 0 \text{ and } 3, & \text{if } \lambda = r_3, \\ 3, & \text{if } r_3 < \lambda < r_{3,5}, \\ 3 \text{ and } 5, & \text{if } \lambda = r_{3,5}, \\ 5, & \text{if } r_{3,5} < \lambda < 1. \end{cases}$$

Proof. The Equation (2) implies

$$0 \leq m_{3,\lambda} \leq \left\lfloor \frac{3}{2}(3+1)\lambda \right\rfloor = \lfloor 6\lambda \rfloor \leq 5, \text{ for } 0 < \lambda < 1.$$

Hence, it is enough we compare the magnitudes of $f_3(x;\lambda)$, or the magnitudes of $h_3(x;\lambda)$ for $0 \leq x \leq 5$. By Lemma 1, we have $h_3(1;\lambda) < h_3(2;\lambda) < h_3(3;\lambda)$ for $0 < \lambda < 1$, and, by Lemma 2 and 3, it is given $1 = h_3(0;\lambda) > h_3(3;\lambda) > h_3(4;\lambda)$ for $0 < \lambda < r_3$. In addition, the maximum of $h_3(5;\lambda)$ on $\lambda \in (0, r_3]$ is $h_3(5;r_k) = r_3^2 + r_3^3 + \frac{1}{6}r_3^4 + \frac{1}{120}r_3^5 < 1$. Hence, $m_{3,\lambda} = 0$ for $0 < \lambda < r_3$.

Furthermore, $h_3(3;\lambda) > h_3(4;\lambda)$ for $0 < \lambda < 1$. To see this, let $d_k(x;\lambda) = h_k(k;\lambda) - h_k(x;\lambda)$. Then

$$\begin{aligned} d_3(4;\lambda) &= h_3(3;\lambda) - h_3(4;\lambda) \\ &= \left(\lambda + \lambda^2 + \frac{1}{6}\lambda^3\right) - \left(\frac{3}{2}\lambda^2 + \frac{1}{2}\lambda^3 + \frac{1}{24}\lambda^4\right) \\ &= \lambda - \frac{1}{2}\lambda^2 - \frac{1}{3}\lambda^3 - \frac{1}{24}\lambda^4, \end{aligned}$$

with $\frac{\partial^2}{\partial\lambda^2} d_3(4;\lambda) = -1 - 2\lambda - \lambda^2/2 < 0$, which implies that $d_3(4;\lambda)$ is concave down for $\lambda > 0$. Since $\lim_{\lambda\to 0^+} d_3(4;\lambda) = 0^+$, and $d_3(4;1) = 1/8 > 0$, we have $h_3(3;\lambda) > h_3(4;\lambda)$ for $0 < \lambda < 1$. Hence, it suffices that we compare the magnitude of $h_3(0;\lambda)$, $h_3(3;\lambda)$ and $h_3(5;\lambda)$ to find the modes for $r_3 < \lambda < 1$.

$$\begin{aligned} d_3(5;\lambda) &= h_3(3;\lambda) - h_3(5;\lambda) \\ &= \left(\lambda + \lambda^2 + \frac{1}{6}\lambda^3\right) - \left(\lambda^2 + \lambda^3 + \frac{1}{6}\lambda^4 + \frac{1}{120}\lambda^5\right) \\ &= \lambda - \frac{5}{6}\lambda^3 - \frac{1}{6}\lambda^4 - \frac{1}{120}\lambda^5. \end{aligned}$$

For $\lambda > 0$, $d_3(5;\lambda)$ is concave down because $\frac{\partial^2}{\partial\lambda^2} d_3(5;\lambda) = -5\lambda - 2\lambda^2 - \lambda^3/6 < 0$. Since $\lim_{\lambda\to 0^+} d_3(5;\lambda) = 0^+$ and $d_3(5;1) = -1/120$, the equation $d_3(5;\lambda) = 0$, equivalently

$\lambda^4 + 20\lambda^3 + 100\lambda^2 - 120 = 0$, has one positive root, say $r_{3,5}$. Thus, $h_3(3;\lambda) > h_3(5;\lambda)$ for $0 < \lambda < r_{3,5}$ and $h_3(3;\lambda) < h_3(5;\lambda)$ for $r_{3,5} < \lambda < 1$. Since $0 < r_3 = 0.601679\ldots < r_{3,5} = 0.996203\ldots < 1$, we have

$$\begin{cases} f_3(0;\lambda) > f_3(3;\lambda) > f_3(5;\lambda), & \text{if } 0 < \lambda < r_3, \\ f_3(3;\lambda) > f_3(0;\lambda) \;\;\&\;\; f_3(3;\lambda) > f_3(5;\lambda), & \text{if } r_3 < \lambda < r_{3,5}, \\ f_3(5;\lambda) > f_3(3;\lambda) > f_3(0;\lambda), & \text{if } r_{3,5} < \lambda < 1. \end{cases}$$

□

Proposition 2. *Let $m_{4,\lambda}$ denote the mode(s) of the Poisson distribution of order 4 with parameter $\lambda(>0)$, and let r_4 (=0.5203510176...), $r_{4,7}$ (=0.7947408725...), and $r_{7,8}$ (=0.8944652714...), respectively, be the positive roots of the equations $\lambda^4 + 12\lambda^3 + 36\lambda^2 + 24\lambda - 24 = 0$, $\lambda^6 + 42\lambda^5 + 630\lambda^4 + 3990\lambda^3 + 7560\lambda^2 - 2520\lambda - 5040 = 0$, and $\lambda^6 + 48\lambda^5 + 840\lambda^4 + 6720\lambda^3 + 18{,}480\lambda^2 - 20{,}160 = 0$. Then*

$$m_{4,\lambda} = \begin{cases} 0, & \text{if } 0 < \lambda < r_4, \\ 0 \text{ and } 4, & \text{if } \lambda = r_4, \\ 4, & \text{if } r_4 < \lambda < r_{4,7}, \\ 4 \text{ and } 7, & \text{if } \lambda = r_{4,7}, \\ 7, & \text{if } r_{4,7} < \lambda < r_{7,8}, \\ 7 \text{ and } 8, & \text{if } \lambda = r_{7,8}, \\ 8, & \text{if } r_{7,8} < \lambda < 1. \end{cases}$$

Proof. The Equation (2) implies

$$0 \le m_{4,\lambda} \le \left\lfloor \frac{4}{2}(4+1)\lambda \right\rfloor = \lfloor 10\lambda \rfloor \le 9, \text{ for } 0 < \lambda < 1.$$

Hence, it is enough we compare the magnitudes of $f_4(x;\lambda)$, or the magnitudes of $h_4(x;\lambda)$ for $0 \le x \le 9$. By Lemma 1, we have $h_4(1;\lambda) < h_4(2;\lambda) < h_4(3;\lambda) < h_4(4;\lambda)$. Thus, we will compare the magnitudes of $h_4(x;\lambda)$ for $4 \le x \le 9$, and $h_4(0;\lambda)$. They are given by

$$\begin{aligned} h_4(4;\lambda) &= \lambda + \frac{3}{2}\lambda^2 + \frac{1}{2}\lambda^3 + \frac{1}{24}\lambda^4 \\ h_4(5;\lambda) &= 2\lambda^2 + \lambda^3 + \frac{1}{6}\lambda^4 + \frac{1}{120}\lambda^5 \\ h_4(6;\lambda) &= \frac{3}{2}\lambda^2 + \frac{5}{3}\lambda^3 + \frac{5}{12}\lambda^4 + \frac{1}{24}\lambda^5 + \frac{1}{720}\lambda^6 \\ h_4(7;\lambda) &= \lambda^2 + 2\lambda^3 + \frac{5}{6}\lambda^4 + \frac{1}{8}\lambda^5 + \frac{1}{120}\lambda^6 + \frac{1}{5040}\lambda^7 \\ h_4(8;\lambda) &= \frac{1}{2}\lambda^2 + 2\lambda^3 + \frac{31}{24}\lambda^4 + \frac{7}{24}\lambda^5 + \frac{7}{240}\lambda^6 + \frac{1}{720}\lambda^7 + \frac{1}{40{,}320}\lambda^8 \\ h_4(9;\lambda) &= \frac{5}{3}\lambda^3 + \frac{5}{3}\lambda^4 + \frac{13}{24}\lambda^5 + \frac{7}{90}\lambda^6 + \frac{1}{180}\lambda^7 + \frac{1}{5040}\lambda^8 + \frac{1}{362{,}880}\lambda^9 \end{aligned}$$

Note that $h_4(x;\lambda)$, for $4 \le x \le 9$, are strictly increasing functions of λ and $h_4(x;0) = 1$. Table 1 displays the function values of $h_4(x;\lambda)$ for $4 \le x \le 9$ with $\lambda = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,$ 1.0, and 1.1. From the function values of the Table 1, we can see $h_4(4;\lambda) = 1$ has a root $r_4 \in (0.5, 0.6)$, $h_4(4;\lambda) = h_4(7;\lambda)$ has a root $r_{4,7} \in (0.7, 0.8)$, and $h_4(7;\lambda) = h_4(8;\lambda)$ has a root $r_{7,8} \in (0.8, 0.9)$.

Table 1. The function values of $h_4(x;\lambda)$ for $x = 0$ and $4 \le x \le 10$ with $\lambda = 0.5, 0.6, 0.7, 0.8, 0.9$, 1.0, and 1.1. The bolded value in each column stands for the maximum of $h_4(x;\lambda)$, which implies $m_{4,\lambda} = x$ with a value of λ given in the corresponding column. Note that $0.5 < r_4 < 0.6 < 0.7 < r_{4,7} < 0.8 < r_{7,8} < 0.9$. The function values are calculated based on substantially tight grid for λ. The table displays only the meaningful λ values.

				λ			
x	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**
0	**1.0000**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0.9401	**1.2534**	**1.6165**	2.0331	2.5068	3.0417	3.6415
5	0.6357	0.9582	1.3644	1.8630	2.4633	3.1750	4.0084
6	0.6107	0.9573	1.4139	1.9980	2.7287	3.6264	4.7129
7	0.5561	0.9101	1.3981	**2.0485**	2.8931	3.9669	5.3085
8	0.4653	0.8035	1.2937	1.9766	**2.8989**	**4.1139**	5.6823
9	0.3307	0.6219	1.0725	1.7351	2.6724	3.9585	5.6799
10	0.2686	0.5273	0.9457	1.5861	2.5258	3.8593	**5.7004**

Therefore,

$$\begin{cases} f_4(0;\lambda) > f_4(x;\lambda) \text{ where } 1 \le x \le 9, & \text{if } 0 < \lambda < r_4, \\ f_4(4;\lambda) > f_4(x;\lambda) \text{ where } 0 \le x \le 3, \text{ or } 5 \le x \le 9 & \text{if } r_4 < \lambda < r_{4,7}, \\ f_4(7;\lambda) > f_4(x;\lambda) \text{ where } 0 \le x \le 6, \text{ or } 8 \le x \le 9 & \text{if } r_{4,7} < \lambda < r_{7,8}, \\ f_4(8;\lambda) > f_4(x;\lambda) \text{ where } 0 \le x \le 7, \text{ or } x = 9 & \text{if } r_{7,8} < \lambda < 1. \end{cases}$$

□

3. More Computational Resutls

This section provides more computational results using the computer algebra system Mathematica. Table 2 shows the modes of Poisson distribution of order $k = 2,\ 3$, and 4 for $0 < \lambda < 2$. For $\lambda > 1$, we observe that the mode values frequently change as λ increases. However, for every value of k, the first two modes are 0 and k for some subintervals of λ between zero and one. We also note that for $k = 2,\ 3,$ and 4, $m_{k,1} = 2,\ 5,$ and 8, (the modes of the Poisson distribution of order k with $\lambda = 1$) as it should, in accordance with (3).

Table 2. The modes of Poisson distribution of order $k = 2,\ 3$, and 4 for $0 < \lambda < 2$. The lower and upper bounds of λ are the approximated values.

$k = 2$ Mode	Interval of λ	$k = 3$ Mode	Interval of λ	$k = 4$ Mode	Interval of λ
0	(0.0000, 0.7321)	0	(0.0000, 0.6017)	0	(0.0000, 0.5204)
2	(0.7321, 1.3412)	3	(0.6017, 0.9962)	4	(0.5204, 0.7947)
4	(1.3412, 1.8851)	5	(0.9962, 1.0612)	7	(0.7947, 0.8945)
5	(1.8851, 2.0000)	6	(1.0612, 1.3881)	8	(0.8945, 1.0950)
		7	(1.3881, 1.4293)	10	(1.0950, 1.2056)
		8	(1.4293, 1.6286)	11	(1.2056, 1.3244)
		9	(1.6286, 1.8197)	12	(1.3244, 1.4332)
		10	(1.8197, 1.9590)	13	(1.4332, 1.5124)
		11	(1.9590, 2.0000)	14	(1.5124, 1.6183)
				15	(1.6183, 1.7215)
				16	(1.7215, 1.8210)
				17	(1.8210, 1.9180)
				18	(1.9180, 2.0000)

4. Moment Estimation of the Parameter λ of $P_k(\lambda)$, Discussion, and Further Research

Despite the upsurge of the study of distributions of order k or runs since the early 1980s, their modes, due to the difficulty of obtaining them, are not known, except for the mode of the geometric distribution of order k and partial results for the modes of the negative binomial and Poisson distributions of order k. Their probability generating functions, however, and moments are well known.

The mean and variance of $P_k(\lambda)$, for example, are (a) $k(k+1)\lambda/2$ and (b) $k(k+1)(2k+1)\lambda/6$ (see, e.g., Philippou [3,25]). By means of them and the method of moments estimation, we now give the moment estimator $\hat{\lambda}$ of λ of $P_k(\lambda)$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $P_k(\lambda)$, and set $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$. Then, the moment estimator of λ is $\hat{\lambda} = 2\bar{X}/[k(k+1)]$. It is unbiased, and has variance $Var(\hat{\lambda}) = 2(2k+1)\lambda/[3k(k+1)n]$. In fact, by the method of moments and (a), $\bar{X} = k(k+1)\hat{\lambda}/2$, which implies $\hat{\lambda} = 2\bar{X}/[k(k+1)]$. It follows by (a) and (b), respectively, that $\hat{\lambda}$ is unbiased for λ, since $E(\hat{\lambda}) = 2E(\bar{X})/[k(k+1)] = \lambda$, and has variance $Var(\hat{\lambda}) = 4Var(\bar{X})/[k^2(k+1)^2] = [4/k^2(k+1)^2] \cdot [k(k+1)(2k+1)\lambda/(6n)] = 2(2k+1)\lambda/[3k(k+1)n]$, which was to be shown.

In the present article, in addition to the above paragraph regarding $P_k(\lambda)$, we derived a few new properties of the Poisson distribution of order k, and using them, along with a result of Georghiou et al. [21], we found the modes or most probable values of the Poisson distributions of order 3 and 4 for λ in the interval $(0,1)$. In addition, using Mathematica and a personal computer, we found the modes of the Poisson distributions of order 2, 3, and 4 for $\lambda \in (0,2)$. We observe that for $k = 2$, 3, and 4, the first two modes are 0 for $0 < \lambda < r_k$, and k for $r_k < \lambda < r_k + l_k$, where l_k stands for the length of the interval on which k is the mode of the Poisson distribution of order k. Further research may include several interesting problems: Is it generally true that $m_{k,\lambda} = 0$ for $k \geq 2$ and $0 < \lambda < r_k$, and $m_{k,\lambda} = k$ for $k \geq 2$ and $r_k < \lambda < r_k + l_k$? Does r_k decrease as k increases? If it does, how fast is r_k decreasing? What positive integers cannot be modes of the Poisson distribution of order k?

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