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Some Iterative Properties of $(\mathcal{F}_1, \mathcal{F}_2)$-Chaos in Non-Autonomous Discrete Systems

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Abstract: This paper is concerned with invariance $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled sets under iterations. The main results are an extension of the compound invariance of Li–Yorke chaos and distributional chaos. New definitions of $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled sets in non-autonomous discrete systems are given. For a positive integer k, the properties $P(k)$ and $Q(k)$ of Furstenberg families are introduced. It is shown that, for any positive integer k, for any $s \in [0, 1]$, Furstenberg family $\overline{M}(s)$ has properties $P(k)$ and $Q(k)$, where $\overline{M}(s)$ denotes the family of all infinite subsets of $\mathbb{Z}^+$ whose upper density is not less than s. Then, the following conclusion is obtained. D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set of $(X, f_{1,\infty})$ if and only if D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set of $(X, f_{1,\infty}^{[m]})$.

Keywords: nonautonomous discrete system; Furstenberg family; scrambled sets; chaos

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1. Introduction

Chaotic properties of a dynamical system have been extensively discussed since the introduction of the term chaos by Li and Yorke in 1975 [1] and Devaney in 1989 [2]. To describe some kind of unpredictability in the evolution of a dynamical system, other definitions of chaos have also been proposed, such as generic chaos [3], dense chaos [4], Li–Yorke sensitivity [5], and so on. An important generalization of Li–Yorke chaos is distributional chaos, which is given in 1994 by B. Schweizer and J. Smítal [6]. Then, theories related to scrambled sets are discussed extensively (see [7–12] and others). In 1997, the Furstenberg family was introduced by E. Akin [13]. J. Xiong, F. Tan described chaos with a couple of Furstenberg Families. $(\mathcal{F}_1, \mathcal{F}_2)$-chaos has also been defined [14]. Moreover, $\mathcal{F}$-sensitivity was given in [15] and shadowing properties were discussed in [16]. Most existing papers studied the chaoticity in autonomous discrete systems (X, f). However, if a sequence of perturbations to a system are described by different functions, then there are a sequence of maps to describe them, giving rise to non-autonomous systems. Non-autonomous discrete systems were precisely introduced in [17], in connection with non-autonomous difference equations (see [18,19] and some references therein).

Let (X, ρ) (briefly, X) be a compact metric space and consider a sequence of continuous maps $f_n : X \to X, n \in \mathbb{N}$, denoted by $f_{1,\infty} = (f_1, f_2, \cdots)$. This sequence defines a non-autonomous discrete system $(X, f_{1,\infty})$. The orbit of any point $x \in X$ is given by the sequence $(f_1^n(x)) = Orb(x, f_{1,\infty})$, where $f_1^n = f_n \circ \cdots \circ f_1$ for $n \geq 1$, and f_1^0 is the identity map.

For $m \in \mathbb{N}$, define

$$g_1 = f_m \circ \cdots \circ f_1, g_2 = f_{2m} \circ \cdots \circ f_{m+1}, \ldots, g_p = f_{pm} \circ \cdots \circ f_{(p-1)m+1}, \cdots.$$

Call $(X, g_{1,\infty})$ a compound system of $(X, f_{1,\infty})$.

Also, denote $g_{1,\infty}$ by $f_{1,\infty}^{[m]}$ and denote $f_n^k = f_{n+k-1} \circ \cdots \circ f_n$ for $n \geq 1$. By [5], if $(f_n)_{n=1}^{\infty}$ converges uniformly to a map f. Then, for any $m \geq 2 (m \in \mathbb{N})$, the sequence $(f_n^{n+m-1})_{n=1}^{\infty}$ converges uniformly to f^m.

In the present work, some notions relating to Furstenberg families and properties $P(k)$, $Q(k)$ are recalled in Sections 2 and 3. Section 4 states some definitions about $(\mathcal{F}_1, \mathcal{F}_2)$-chaos. In Section 5, it is proved that, under the conditions of property $P(k)$ and positive shift-invariant, $f_{1,\infty}$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, strong $\mathcal{F}$-chaos) implies $f_{1,\infty}^{[k]} (k \in \mathbb{Z}^+)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, strong $\mathcal{F}$-chaos). If the conditions property $Q(k)$ and negative shift-invariant both hold, the above conclusion can be inversed. As a conclusion, for arbitrary s and t in $[0,1]$, for every $k \in \mathbb{Z}^+$, $f_{1,\infty}$ and $f_{1,\infty}^{[k]}$ can share the same $(\overline{M}(s), \overline{M}(t))$-scrambled set (Theorem 3).

In this paper, it is always assumed that all the maps f_n, $n \in \mathbb{N}$, are surjective. It should be noted that this condition is needed by most papers dealing with this kind of system (for example, [20–23]). It is assumed that sequence $(f_n)_{n=1}^{\infty}$ converges uniformly. The aim of this paper is to investigate the $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled sets of $f_{1,\infty}$.

2. Furstenberg Families

Let $\mathcal{P}$ be the collection of all subsets of the positive integers set $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. A collection $\mathcal{F} \subset \mathcal{P}$ is called a Furstenberg family if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Obviously, the collection of all infinite subsets of $\mathbb{Z}^+$ is a Furstenberg family, denoted by $\mathcal{B}$.

Define the dual family $k\mathcal{F}$ of a Furstenberg family $\mathcal{F}$ by

$$k\mathcal{F} = \{F \in \mathcal{P} : \mathbb{Z}^+ - F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \phi \text{ for any } F' \in \mathcal{F}\}.$$

It is clear that $k\mathcal{F}$ is a Furstenberg family and $k(k\mathcal{F}) = \mathcal{F}$ (see [13]).

For $F \in \mathcal{P}$, $i \in \mathbb{Z}^+$, let $F - i = \{j - i \geq 0 : j \in F\}$ and $F + i = \{j + i \geq 0 : j \in F\}$. Furstenberg family $\mathcal{F}$ is positive shift-invariant if $F + i \in \mathcal{F}$ for every $F \in \mathcal{F}$ and any $i \in \mathbb{Z}^+$. Furstenberg family $\mathcal{F}$ is negative shift-invariant if $F - i \in \mathcal{F}$ for every $F \in \mathcal{F}$ and any $i \in \mathbb{Z}^+$. Furstenberg family $\mathcal{F}$ is shift-invariant if it is positive shift-invariant and negative shift-invariant.

The following shows a class of Furstenberg families which is related to upper density.

Let $F \subset \mathcal{P}$. The upper density and the lower density of F are defined as follows:

$$\overline{\mu}(F) = \limsup_{n\to\infty} \frac{\#(F \cap \{0, 1, \ldots, n-1\})}{n}, \quad \underline{\mu}(F) = \liminf_{n\to\infty} \frac{\#(F \cap \{0, 1, \ldots, n-1\})}{n},$$

where $\#(A)$ denotes the cardinality of the set A.

For any s in $[0,1]$, set $\overline{M}(s) = \{F \in \mathcal{B} : \overline{\mu}(F) \geq s\}$.

Proposition 1. *For any s in $[0,1]$, $\overline{M}(s)$ is shift-invariant Furstenberg family. And $\overline{M}(0) = \mathcal{B}$.*

Proof.

(i) Let $F_1, F_2 \in \overline{M}(s), F_1 \subset F_2$, then, $\forall n \in \mathbb{N}$ (where $\mathbb{N} = \{1, 2, 3, \ldots\}$),

$$\overline{\mu}(F_1) = \limsup_{n\to\infty} \frac{\#(F_1 \cap \{0, 1, \ldots, n-1\})}{n} \leq \limsup_{n\to\infty} \frac{\#(F_2 \cap \{0, 1, \ldots, n-1\})}{n} = \overline{\mu}(F_2)$$

Thus, $F_1 \in \overline{M}(s)$ (i.e., $\overline{\mu}(F_1) \geq s$) implies $F_2 \in \overline{M}(s)$ (i.e., $\overline{\mu}(F_1) \geq s$). So, $\overline{M}(s) (\forall s \in [0,1])$ are Furstenberg families.

(ii) Let $F \in \overline{M}(s)$, that is, $\overline{\mu}(F) = \limsup_{n\to\infty} \frac{\#(F\cap\{0,1,\ldots,n-1\})}{n} \geq s$. Denote $F = \{t_1, t_2, \cdots\}$ (where $t_k \in \mathbb{Z}^+$, $t_{k_1} < t_{k_2}(k_1 < k_2)$), then $F + i = \{t_1 + i, t_2 + i, \cdots\}$ and $F - i = \{t_{k_1} - i, t_{k_2} - i, \cdots\}(t_{k_j} - i \geq 0)$ for any $i \in \mathbb{Z}^+$.

$$\limsup_{n\to\infty} \frac{\#((F+i)\cap\{0,1,\ldots,n-1\})}{n} = \limsup_{n\to\infty} \frac{\#(\{t_1+i, t_2+i, \cdots\}\cap\{0,1,\ldots,n-1\})}{n}$$
$$= \limsup_{n\to\infty} \frac{\#(\{t_1, t_2, \cdots\}\cap\{0,1,\ldots,n-1\})}{n} = \overline{\mu}(F) \geq s$$

and

$$\limsup_{n\to\infty} \frac{\#((F-i)\cap\{0,1,\ldots,n-1\})}{n} \geq \limsup_{n\to\infty} \frac{\#(F\cap\{0,1,\ldots,n-1\})-i}{n} = \overline{\mu}(F) \geq s$$

So, $\overline{M}(s)$ is shift-invariant.

(iii) Obviously,

$$\overline{M}(0) = \{F \in \mathcal{B} : \overline{\mu}(F) \geq 0\} = \{F \in \mathcal{B} : \limsup_{n\to\infty} \frac{\#(F\cap\{0,1,\ldots,n-1\})}{n} \geq 0\} = \mathcal{B}.$$

This completes the proof.

□

3. Properties $P(k)$, $Q(k)$ of Furstenberg Families

Definition 1. *Let k be a positive integer and $\mathcal{F}$ be a Furstenberg family.*

(1) For any $F \in \mathcal{F}$, if there exists an integer $j \in \{0, 1, \cdots, k-1\}$ such that $F_{k,j} = \{i \in \mathbb{Z}^+ : ki + j \in F\} \in \mathcal{F}$, we say $\mathcal{F}$ have property $P(k)$;

(2) If $F_k = \{ki + j \in \mathbb{Z}^+ : j \in \{0, 1, \cdots, k-1\}, i \in F\} \in \mathcal{F}$, we say $\mathcal{F}$ have property $Q(k)$.

The following proposition is given by [24]. For completeness, we give the proofs.

Proposition 2. *For any $s \in [0, 1]$ and any $k \in \mathbb{Z}^+$, $\overline{M}(s)$ have properties $P(k)$ and $Q(k)$.*

Proof.

(1) If $k = 1$, $\forall F \in \overline{M}(s)$, $F_{1,0} = \{i \in \mathbb{Z}^+ : i \in F\} = F$, i.e., there exists an integer $j = 0$ such that $F_{k,j} \in \overline{M}(s)$. The following will discuss the case $k > 1$.

If $s = 0$, $\overline{M}(0) = \mathcal{B}$. $\forall F \in \mathcal{B}$, $\forall k \in \mathbb{Z}^+$, obviously, there exist $j \in \{0, 1, \ldots, k-1\}$ such that $F_{k,j} \in \mathcal{B}$.

If $0 < s \leq 1$, suppose properties $P(k)$ does not hold. Then there exists a $F \in \overline{M}(s)$ such that $\overline{\mu}(F_{k,j}) < s$ for every $j \in \{0, 1, \ldots, k-1\}$.

For any $j \in \{0, 1, \ldots, k-1\}$, put $\varepsilon_j > 0$ which satisfied $\overline{\mu}(F_{k,j}) < s - \varepsilon_j$. One can find a sufficiently large number N such that, $n \geq N$, $\#_n(F_{k,j}) < n(s - \varepsilon_j)$ (where $\#_n(F_{k,j})$ denotes the cardinality of the set $F_{k,j} \cap \{0, 1, \ldots, n-1\}$). Then $\#_n(F^c_{k,j}) > n - n(s - \varepsilon_j)$, where $F^c_{k,j}$ denotes the complementary set of $F_{k,j}$.

Give an integer $m = kn + l_m > kN$, $l_m \in \{0, 1, \ldots, k-1\}$. By the definition of $F_{k,j}$, $ki + j \notin F$ if $i \notin F_{k,j}$. And $ki_1 + j_1 \neq ki_2 + j_2$ if $i_1, i_2 \in \{0, 1, \ldots, n-1\}$, $j_1, j_2 \in \{0, 1, \ldots, k-1\}$ and $j_1 \neq j_2$. Then

$$\#_m(F^c) \geq \sum_{j=0}^{k-1} \#_n(F^c_{k,j}) > \sum_{j=0}^{k-1} (n - n(s - \varepsilon_j)).$$

So,

$$\#_m(F) < m - \sum_{j=0}^{k-1}(n - n(s-\varepsilon_j)).$$

Put $\varepsilon = min\{\varepsilon_j : j = 0, 1, \ldots, k-1\}$, then

$$\overline{\mu}(F) = \limsup_{n\to\infty} \frac{\#_m(F)}{m} \leq \lim_{n\to\infty} \frac{m - \sum_{j=0}^{k-1}(n - n(s-\varepsilon_j))}{m} \leq \lim_{n\to\infty} \frac{m - k(n - n(s-\varepsilon))}{m}$$

$$= \lim_{n\to\infty} \frac{kn + l_m - kn + kn(s-\varepsilon)}{kn + l_m} = s - \varepsilon < s$$

This contradicts to $\overline{\mu}(F) \geq s$.

(2) Similarly, just consider the case $k > 1, 0 < s \leq 1$.

Suppose properties $Q(k)$ does not hold. Then there exists an integer $F \in \overline{M}(s)$ such that $\overline{\mu}(F_k) < s$. Put $\varepsilon > 0$ which satisfied $\overline{\mu}(F_k) < s - \varepsilon$. One can find a sufficiently large number N such that, $m \geq N$, $\#_m(F_k) < m(s-\varepsilon)$. Give a $m = kn + l_m > kN(m \geq N)$, $l_m \in \{0, 1, \ldots, k-1\}$. By the definition of F_k, $ki + j \in F_k(j \in \{0, 1, \ldots, k-1\})$ if $i \in F$. And $ki_1 + j_1 \neq ki_2 + j_2$ if $i_1 \neq i_2$ and $j_1, j_2 \in \{0, 1, \ldots, k-1\}$. Then

$$k(\#_n(F)) \leq \#_m(F_k) < m(s-\varepsilon).$$

So,

$$\overline{\mu}(F) \leq \lim_{n\to\infty} \frac{m(s-\varepsilon)}{kn} = \lim_{n\to\infty} \frac{(kn + l_m)(s-\varepsilon)}{kn} = s - \varepsilon \leq s.$$

This contradicts to $\overline{\mu}(F) \geq s$.

This completes the proof.

□

4. $(\mathcal{F}_1, \mathcal{F}_2)$-Chaos in Non-Autonomous Systems

Now, we state the definition of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos in nonautonomous systems.

Definition 2. *Let (X, ρ) be a compact metric space, $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families. $\mathcal{D} \subset X$ is called a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $(X, f_{1,\infty})$ (briefly, $f_{1,\infty}$), if $\forall\, x \neq y \in \mathcal{D}$, the following two conditions are satisfied:*

(i) $\forall t > 0$, $\left\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\right\} \in \mathcal{F}_1$;
(ii) $\exists \delta > 0$, $\left\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\right\} \in \mathcal{F}_2$.

The pair (x, y) which satisfies the above two conditions is called an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled pair of $f_{1,\infty}$.

$f_{1,\infty}$ is said to be $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic if there exists an uncountable $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$. If $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, $f_{1,\infty}$ is said to be $\mathcal{F}$-chaotic and (x, y) is an $\mathcal{F}$-scrambled pair. $f_{1,\infty}$ is said to be strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic if there are some $\delta > 0$ and an uncountable subset $\mathcal{D} \subset X$ such that for any $x, y \in \mathcal{D}$ with $x \neq y$, the following two conditions holds:

(i) $\left\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\right\} \in \mathcal{F}_1$ *for all* $t > 0$;
(ii) $\left\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\right\} \in \mathcal{F}_2$.

$f_{1,\infty}$ is said to be strong $\mathcal{F}$-chaos if it is strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$.

Let us recall the definitions of Li-Yorke chaos and distributional chaos in non-autonomous systems (see [25,26]).

Definition 3. *Assume that $(X, f_{1,\infty})$ is a non-autonomous discrete system. If $x, y \in X$ with $x \neq y$, (x, y) is called a Li–Yorke pair if*

$$\limsup_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) > 0 \quad \textit{and} \quad \liminf_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) = 0.$$

The set $\mathcal{D} \subset X$ is called a Li–Yorke scrambled set if all points $x, y \in \mathcal{D}$ with $x \neq y$, (x, y) is a Li–Yorke pair. $f_{1,\infty}$ is Li–Yorke chaotic if X contains an uncountable Li–Yorke scrambled set.

Assume that $(X, f_{1,\infty})$ is a non-autonomous discrete system. For any pair of points $x, y \in X$, define the upper and lower (distance) distributional functions generated by $f_{1,\infty}$ as

$$F_{xy}^*(t, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,t)}(\rho(f_1^i(x), f_1^i(y)))$$

and

$$F_{xy}(t, f_{1,\infty}) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)}(\rho(f_1^i(x), f_1^i(y)))$$

respectively. Where $\chi_{[0,t)}$ is the characteristic function of the set $[0, t)$, i.e., $\chi_{[0,t)}(a) = 1$ when $a \in [0, t)$ or $\chi_{[0,t)}(a) = 0$ when $a \notin [0, t)$.

Definition 4. *$f_{1,\infty}$ is distributionally chaotic if exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$, we have that $F_{xy}^*(t, f_{1,\infty}) = 1$ for all $t > 0$ and $F_{xy}(t, f_{1,\infty}) = 0$ for some $\delta > 0$.*

The set D is a distributionally scrambled set and the pair (x, y) a distributionally chaotic pair.

It is not difficult to obtain that the pair (x, y) is a $(\overline{M}(0), \overline{M}(0))$-scrambled pair if and only if (x, y) is a Li–Yorke scrambled pair, and the pair (x, y) is a $(\overline{M}(1), \overline{M}(1))$-scrambled pair if and only if (x, y) is a distributionally scrambled pair. In fact,

$$\overline{M}(0) = \mathcal{B}, \overline{M}(1) = \{F \in \mathcal{B} : \limsup_{n\to\infty} \frac{\#(F \cap \{1, 2, \ldots, n\})}{n} = 1\}.$$

Then, $\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\} \in \overline{M}(0)$ for any $t > 0$ and $\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\} \in \overline{M}(0)$ for some $\delta > 0$ is equivalent to that $\limsup_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) > 0$ and $\liminf_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) = 0$. $\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\} \in \overline{M}(1)$ for any $t > 0$ and $\{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\} \in \overline{M}(1)$ for some $\delta > 0$ is equivalent to that $F_{xy}^*(t, f_{1,\infty}) = 1$ and $F_{xy}(\delta, f_{1,\infty}) = 0$.

Hence, $(\overline{M}(0), \overline{M}(0))$-chaos is Li–Yorke chaos and $(\overline{M}(1), \overline{M}(1))$-chaos is distributional Chaos.

5. Main Results

Theorem 1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families with property $P(k)$, where k is a positive integer. $\mathcal{F}_1$ is positive shift-invariant. If the system $(X, f_{1,\infty})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, then the system $(X, f_{1,\infty}^{[k]})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos too.*

Proof. If D is an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$, the following proves that D is an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}^{[k]}$.

(i) Since X is compact and $f_i (i \in \mathbb{N})$ are continuous, then, for any $j \in \{1, 2, \ldots, k-1\}$, $f_{s_1}, \ldots, f_{s_{k-j}}$ are uniformly continuous (where $f_{s_1}, \ldots, f_{s_{k-j}}$ are freely chosen from the sequence $f_i (i \in \mathbb{N})$). That is, for any $\delta > 0$, there exists a $\delta^* > 0$, $\forall a, b \in X$, $\rho(a, b) < \delta^*$ implies $\rho(f_{s_{k-j}} \circ \cdots \circ f_{s_1}(a), f_{s_{k-j}} \circ \cdots \circ f_{s_1}(b)) < \delta$ $(j = 1, 2, \ldots, k-1)$.

Since D is an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$, then, $\forall x \neq y \in D$, for the above δ^*, we have

$$F = \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < \delta^*\} \in \mathcal{F}_1.$$

And because $\mathcal{F}_1$ have property $P(k)$, there exists some $j \in \{1, 2, \ldots, k-1\}$ such that

$$F_{k,j} = \{i \in \mathbb{Z}^+ : ki + j \in F\} = \{i \in \mathbb{Z}^+ : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) < \delta^*\} \in \mathcal{F}_1.$$

By the selection of δ^*, we put $s_r = ki + j + r(r = 1, 2, \ldots, k-j)$, then

$$F_{k,j} \subset \{i \in \mathbb{Z}^+ : \rho(f_1^{ki+j+k-j}(x), f_1^{ki+j+k-j}(y)) < \delta\} = \{i \in \mathbb{Z}^+ : \rho(f_1^{k(i+1)}(x), f_1^{k(i+1)}(y)) < \delta\}.$$

Write $F_{k,j} + 1 = \{i + 1 : i \in \mathbb{Z}^+, ki + j \in \mathcal{F}_1\}(\forall j = 1, 2, \ldots, k-1)$, then $F_{k,j} + 1 \subset \{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) < \delta\}$.

By the positive shift-invariant of $\mathcal{F}_1$ and $F_{k,j} \in \mathcal{F}_1$, we have $F_{k,j} + 1 \in \mathcal{F}_1$. And with the hereditary upwards of $\mathcal{F}_1$, for any $x, y \in D : x \neq y$, $\forall \delta > 0$, $\{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) < \delta\} \in \mathcal{F}_1$.

(ii) Since D is a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$, then, for the above $x, y \in D(x \neq y)$, $\exists \varepsilon^* > 0$, such that $E = \{n \in \mathbb{Z}^+ : \rho(f_1^n(x), f_1^n(y)) > \varepsilon^*\} \in \mathcal{F}_2$. And because $\mathcal{F}_2$ have property $P(k)$, then, there exists some $j \in \{1, 2, \ldots, k-1\}$ such that

$$E_{k,j} = \{i \in \mathbb{Z}^+ : ki + j \in E\} = \{i \in \mathbb{Z}^+ : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \varepsilon^*\} \in \mathcal{F}_2.$$

X is compact and $f_i(i \in \mathbb{N})$ are continuous, then, for any $j \in \{1, 2, \ldots, k-1\}$, $f_{s_1}, \ldots, f_{s_j}$ are uniformly continuous (where $f_{s_1}, \ldots, f_{s_j}$ are freely chosen from the sequence $f_i(i \in \mathbb{N})$). For the above $\varepsilon^* > 0$, $\exists \varepsilon > 0$, $\forall p, q \in X$ satisfied $\rho(p, q) \leq \varepsilon$, inequality $\rho(f_{s_j} \circ \cdots \circ f_{s_1}(p), f_{s_j} \circ \cdots \circ f_{s_1}(q)) \leq \varepsilon^*$ holds.

The following will prove that $\{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \varepsilon\} \in \mathcal{F}_2$.

Suppose $\{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \varepsilon\} \notin \mathcal{F}_2$, then

$$\mathbb{Z}^+ - \{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \varepsilon\} = \{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) \leq \varepsilon\} \in k\mathcal{F}_2.$$

By the selection of ε^*, we put $s_r = ki + r(r = 1, 2, \ldots, j)$, then

$$\{i \in \mathbb{Z}^+ : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) \leq \varepsilon^*\} \in k\mathcal{F}_2.$$

So,

$$\{i \in \mathbb{Z}^+ : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \varepsilon^*\} \notin k\mathcal{F}_2,$$

This contradicts $E_{k,j} \in \mathcal{F}_2$.

Hence, for $x \neq y \in D$ in (i), there exists a $\varepsilon > 0$ such that $\{i \in \mathbb{Z}^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \varepsilon\} \in \mathcal{F}_2$.

Combining with (i) and (ii), $f_{1,\infty}^{[k]}$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos.

This completes the proof.

□

Theorem 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families with property $Q(k)$, where k is a positive integer. $\mathcal{F}_2$ is negative shift-invariant. If the system $(X, f_{1,\infty}^{[k]})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, then the system $(X, f_{1,\infty})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos too.*

Proof. If D is a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}^{[k]}$, the following prove that D is a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$.

(i) Similar to Theorem 1, for any $j \in \{1, 2, \ldots, k-1\}$, $f_{s_1}, \ldots, f_{s_j}$ are uniformly continuous (where $f_{s_1}, \ldots, f_{s_j}$ are freely chosen from the sequence $f_i (i \in \mathbb{N})$). That is, for any $\delta > 0$, there exists a $\delta^* > 0$, $\forall a, b \in X$, $\rho(a, b) < \delta^*$ implies $\rho(f_{s_j} \circ \cdots \circ f_{s_1}(a), f_{s_j} \circ \cdots \circ f_{s_1}(b)) < \delta \, (j = 1, 2, \ldots, k-1)$.

For any pair of distinct points $x, y \in D$, for the above δ^*, one has

$$F = \{n \in \mathbb{Z}^+ : \rho(f_1^{kn}(x), f_1^{kn}(y)) < \delta^*\} \in \mathcal{F}_1.$$

By the selection of δ^*, for $\forall n \in F$, $\forall j \in \{1, 2, \cdots, k-1\}$, put $s_r = ki + j + r (r = 1, 2, \ldots, j)$, then $\rho(f_1^{kn+j}(x), f_1^{kn+j}(y)) < \delta$. And because $\mathcal{F}_1$ have property $Q(k)$, then

$$F_k = \{kn + j \in \mathbb{Z}^+ : j = 1, 2, \ldots, k-1, n \in F\} \in \mathcal{F}_1.$$

Notice that $F_k \subset \{m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) < \delta\}$, then $\{m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) < \delta\} \in \mathcal{F}_1$.

(ii) Since D is an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}^{[k]}$, then, for the above $x, y \in D (x \neq y)$, there exist $\varepsilon^* > 0$, such that $E = \{n \in \mathbb{Z}^+ : \rho(f_1^{kn}(x), f_1^{kn}(y)) > \varepsilon^*\} \in \mathcal{F}_2$.

For any $j \in \{1, 2, \ldots, k-1\}$, $f_{s_1}, \ldots, f_{s_j}$ are uniformly continuous (where $f_{s_1}, \ldots, f_{s_j}$ are freely chosen from the sequence $f_i (i \in \mathbb{N})$), then, for the above $\varepsilon^* > 0$, there exist $\varepsilon > 0$ such that $\rho(p, q) < \varepsilon (p, q \in X)$ implies $\rho(f_{s_j} \circ \cdots \circ f_{s_1}(p), f_{s_j} \circ \cdots \circ f_{s_1}(q)) \leq \varepsilon^* (j = 1, 2, \ldots, k-1)$. That is, $\rho(f_1^k(p), f_1^k(q)) > \varepsilon^* (p, q \in X)$ implies $\rho(f_1^j(p), f_1^j(q)) > \varepsilon (j = 1, 2, \ldots, k-1)$.

$\forall n \in E$, $\forall j = 1, 2, \ldots, k-1$, put $s_r = k(n-1) + r (r = 1, 2, \ldots, j)$, then

$$\rho(f_1^{k(n-1)+j}(x), f_1^{k(n-1)+j}(y)) > \varepsilon.$$

Since $\mathcal{F}_2$ is negative shift-invariant, then $E - 1 \in \mathcal{F}_2$. And because $\mathcal{F}_2$ have property $Q(k)$, then $(E-1)_k \in \mathcal{F}_2$, i.e., $\{k(n-1) + j \in \mathbb{Z}^+ : n - 1 \in E - 1, j = 1, 2, \ldots, k-1\} \in \mathcal{F}_2$. Combining $(E-1)_k \subset \{m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) > \varepsilon\}$ with the hereditary upwards of $\mathcal{F}_2$, we have $\{m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) > \varepsilon\} \in \mathcal{F}_2$.

By (i) and (ii), D is an $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_{1,\infty}$.

This completes the proof.

□

Similarly, the following corollaries hold.

Corollary 1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families with property $P(k)$, where k is a positive integer. $\mathcal{F}_1$ is positive shift-invariant. If the system $(X, f_{1,\infty})$ is $\mathcal{F}$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, or strong $\mathcal{F}$-chaos), then the system $(X, f_{1,\infty}^{[k]})$ is $\mathcal{F}$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, or strong $\mathcal{F}$-chaos).*

Corollary 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families with property $Q(k)$, where k is a positive integer. $\mathcal{F}_2$ is negative shift-invariant. If the system $(X, f_{1,\infty}^{[k]})$ is $\mathcal{F}$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, or strong $\mathcal{F}$-chaos), then the system $(X, f_{1,\infty})$ is $\mathcal{F}$-chaos (strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, or strong $\mathcal{F}$-chaos).*

Combining with Propositions 1 and 2, Theorems 1 and 2, and Corollarys 1 and 2, the following conclusions are obtained.

Theorem 3. *Let s and t are arbitrary two numbers in $[0, 1]$, then*

(1) If D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set (or strong $(\overline{M}(s), \overline{M}(t))$-scrambled set) of $f_{1,\infty}$, then, for every $k \in \mathbb{Z}^+$, D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set(or strong $(\overline{M}(s), \overline{M}(t))$-scrambled set) of $f_{1,\infty}^{[k]}$.

(2) For some positive integer k, if D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set (or strong $(\overline{M}(s), \overline{M}(t))$-scrambled set) of $f_{1,\infty}^{[k]}$, then D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set (or strong $(\overline{M}(s), \overline{M}(t))$-scrambled set) of $f_{1,\infty}$.

Proof.

(1) By Proposition 1, $\overline{M}(s)$ is shift-invariant (obviously positive shift-invariant). And because $\overline{M}(s), \overline{M}(t)$ are two Furstenberg families with property $P(k)$ (Proposition 2). Then, according to the proof of Theorem 1, if D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set of $f_{1,\infty}$, then, for every $k \in \mathbb{Z}^+$, D is an $(\overline{M}(s), \overline{M}(t))$-scrambled set of $f_{1,\infty}^{[k]}$.

(2) In the same way, (2) holds.

This completes the proof.

□

With the preparations in Section 4, we have

Corollary 3.

(1) If D is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}$, then, for every $k \in \mathbb{Z}^+$, D is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}^{[k]}$.

(2) For some positive integer k, if D is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}^{[k]}$, then, D is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}$.

Remark 1. *In the non-autonomous systems, the iterative properties of Li–Yorke chaos and distributional chaos are discussed in [25,26] before. The conclusions in Corollary 3 remains consistent with them.*

This paper has presented several properties of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, and strong $\mathcal{F}$-chaos. There are some other problems, such as generically $\mathcal{F}$-chaos and $\mathcal{F}$-sensitivity, to discuss. Moreover, property $P(k)$ is closely related to congruence theory. Follow this line, one can consider other Furstenberg families which consist of number sets with some special characteristics.

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