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Atangana–Baleanu and Caputo Fabrizio Analysis of Fractional Derivatives for Heat and Mass Transfer of Second Grade Fluids over a Vertical Plate: A Comparative Study

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Abstract: This communication addresses a comparison of newly presented non-integer order derivatives with and without singular kernel, namely Michele Caputo–Mauro Fabrizio (CF) ${}^{CF}\left(\partial^{\beta}/\partial t^{\beta}\right)$ and Atangana–Baleanu (AB) ${}^{AB}\left(\partial^{\alpha}/\partial t^{\alpha}\right)$ fractional derivatives. For this purpose, second grade fluids flow with combined gradients of mass concentration and temperature distribution over a vertical flat plate is considered. The problem is first written in non-dimensional form and then based on AB and CF fractional derivatives, it is developed in fractional form, and then using the Laplace transform technique, exact solutions are established for both cases of AB and CF derivatives. They are then expressed in terms of newly defined M-function $\mathbf{M}_q^p(z)$ and generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$. The obtained exact solutions are plotted graphically for several pertinent parameters and an interesting comparison is made between AB and CF derivatives results with various similarities and differences.

Keywords: AB and CF derivatives; M-function $\mathbf{M}_q^p(z)$ and generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$; convection flow; heat and mass transfer; exact solutions

1. Introduction

Generally, there is no refusing the fact that the non-Newtonian liquids are more conventional in comparison with Newtonian liquids due to their industrial and technological applications. The major non-Newtonian liquids include several materials, for instance lubricants, clay coatings, paints, drilling mud, certain oils, clay coatings, greases, shampoos, polymer solutions, yoghurt, paints, blood, ketchup and several others. These liquids exhibit the non-linear relationship between rate of strain and stress of flow. Due to reliance on rate of strain and stress, the non-Newtonian liquids flow becomes subtle and very complicated. In the literature, several models of non-Newtonian liquids have been launched by scientists and researchers for identifying the rheological properties and typical characteristics. Among them, the most popular model for non-Newtonian fluids is the second grade liquids model which enables prediction of differences in normal stresses [1–10]. Heat generation impacts are applicable abundantly; this is due to the thermal performance of working liquids. These impacts can be exhibited during the manufacturing process with the of disposal of radioactive waste material and rubber and plastic sheets, the dislocating of fluids in packed bed reactors, and the storage of food stuffs, to name just a few. Furthermore, the characteristics of fluid flow for heat transfer are frequently applicable in industrial processes, for instance, extraction of polymers, hot rolling and crystal growing, wire drawing, glassblowing, cooling of metallic plates, and many others. In addition, the heat and mass transfer of non-Newtonian fluid flows has attained a significant role due to their thermal conductivities.

Furthermore, modeling of many phenomena mostly rely on fractional calculus, and it has become a valuable tool in engineering applications, technological development, and industrial sciences for the description of the complex dynamics. Nowadays, fractional calculus has become a burning topic in research due to two reasons/weaknesses: problem of the singular kernel with locality and problem of the non-singular kernel with non-locality. In order to avoid the problem of the singular kernel, Michele Caputo and Mauro Fabrizio proposed a fractional derivative by employing an exponential function [11]. Indeed, the claim of a singular kernel for the fractional derivative operator is not based on their observations; even they suggested their fractional derivative operator is appropriate for various physical problems. Itrat et al. [12] employed the time-fractional Caputo–Fabrizio derivative on the advection-diffusion equation for tracing out the fundamental solutions using the Laplace transform. They investigated numerical solutions for the fractional diffusion phenomenon and normal advection-diffusion process. Atangana et al. [13] analyzed the Keller–Segel Model by applying a fractional derivative without a singular kernel (Caputo–Fabrizio fractional derivative). They employed fixed point theorem to investigate the existence of the coupled-solutions with numerical simulations. Atangana et al. [14] traced out RLC (resistor (R), an inductor (L), and a capacitor (C)) electrical circuit with an extension by implementing the time-fractional Caputo–Fabrizio derivative. Nehad et al. [2] obtained analytical solutions for heat transfer of second grade fluids by applying fractional Caputo–Fabrizio derivatives over vertical oscillating plates. They also introduced newly published special functions for the heat transfer phenomenon of second grade fluids under the influence of Laplace transforms. Briefly, a few recent studies using fractional Caputo–Fabrizio derivatives are referenced in [15–19]. On the other hand, Atangana–Baleanu suggested recently that there are two general definitions of fractional order derivatives in the Riemann–Liouville and Caputo sense. They claimed that their fractional derivative has a fractional integral as the anti-derivative of their operators. Atangana–Baleanu fractional order derivative has non-locality as well as non-singularity of the kernel based on the generalized Mittag–Leffler function [20]. The generalized Casson fluid model has been analyzed for comparative study using the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives with chemical reaction and heat generation by Nadeem et al. [20]. They investigated exact solutions via Atangana–Baleanu and Caputo–Fabrizio fractional derivatives and compared their results graphically. Motivated by the above research work, our aim is to compare newly presented non-integer order derivatives with and without singular kernel, namely Michele Caputo–Mauro Fabrizio (CF) ${}^{CF}\left(\partial^{\beta}/\partial t^{\beta}\right)$ and Atangana–Baleanu (AB) ${}^{AB}(\partial^{\alpha}/\partial t^{\alpha})$ fractional derivatives, respectively. This article proposes to employ AB and CF fractional derivatives on second grade fluid flow free convection due to the combined gradients of mass concentration and temperature distribution. The problem is solved via Laplace transform technique and the results for velocity, temperature, and concentration are obtained. The solutions are expressed in terms of the newly defined M-function $\mathbf{M}_q^p(z)$ and the generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$. Results are then plotted, compared, and discussed. This work is on heat transfer together with mass transfer, as in most physical phenomena the heat transfer is accompanied with mass transfer. Therefore, this work will be of great significance in thermal systems at both the macro-and micro levels. Moreover, this work will be useful in fundamental flow visualization studies on a micro-scale for the two-phase phenomena required for the development of fundamentally-based flow pattern maps and models.

2. Formulation of Problem and Governing Equations

Here, we consider an unsteady second grade fluid for free convection flow of an incompressibility that occupies the space above an infinitely extended plate in the xy plane, and the plate is normal in the y-axis. In the beginning, the temperature is at T_∞, and the concentration level on the plate is C_∞ while plate and fluid are at rest. At $t = 0^+$, the heat and mass transfer from the plate to the fluid is raised to the temperature T_w, and the concentration level near the plate is C_w. We assume the temperature distribution, mass concentration, and velocity field are functions of (y, t). The constraint of incompressibility is identically satisfied when such types of flow occur. Taking the usual Boussinesq approximation, the governing boundary layer equations are [1–3]:

$$\frac{\partial w(y,t)}{\partial t} - \frac{\alpha_1}{\rho}\frac{\partial^3 w(y,t)}{\partial y^2\,\partial t} - \nu\frac{\partial^2 w(y,t)}{\partial y^2} = g\beta_C(C(y,t) - C_\infty) + g\beta_T(T(y,t) - T_\infty),\ y,t > 0, \tag{1}$$

$$\frac{C_p\,\rho}{k}\frac{\partial T(y,t)}{\partial t} - \frac{\partial^2 T(y,t)}{\partial y^2} = 0, \quad y,t > 0, \tag{2}$$

$$\frac{1}{D}\frac{\partial C(y,t)}{\partial t} - \frac{\partial^2 C(y,t)}{\partial y^2} = 0, \quad y,t > 0, \tag{3}$$

where $w(y,t)$, $T(y,t)$, $C(y,t)$, α_1, ρ, ν, g, β_C, β_T, C_p, k, and D are velocity field, temperature distribution and mass concentration, second grade fluid parameter, constant density, the kinematic viscosity of the fluid, gravitational acceleration, volumetric coefficient of expansion for concentration, volumetric coefficient of thermal expansion, heat capacity at constant pressure, thermal conductivity, and mass diffusivity, respectively. Subject to the initial and boundary conditions, with the assumption of no slip between fluid and plate are

$$w(0,t) = A_0 H(t)\,t^p, \quad T(0,t) = T_w, \quad C(0,t) = C_w, \quad t \geq t_0, \qquad t > 0, \tag{4}$$

$$w(y,0) = 0, \quad T(y,0) = T_\infty, \quad C(y,0) = C_\infty, \qquad y > 0, \tag{5}$$

$$w(\infty,t) = 0, \quad T(\infty,t) = T_\infty, \quad C(\infty,t) = C_\infty \qquad t > 0, \tag{6}$$

employing the dimensionless variables into Equations (1)–(6), we have

$$G_r = \frac{\nu\,\beta\,g\,(T_w - T_\infty)}{A_0^3}, \quad P_r = \frac{c_p\,\mu}{k}, \quad S_c = \frac{v}{D} \quad ,\alpha_2 = \frac{A_0^2\,\alpha_1}{\nu\,\mu}, \quad t^* = \frac{A_0^2 t}{\nu}, \quad y^* = \frac{A_0 y}{\nu}, \quad w^* = \frac{w}{A_0}, \\ T = \frac{T - T_\infty}{T_w - T_\infty}, \quad C = \frac{C - C_\infty}{C_w - C_\infty}. \tag{7}$$

we obtain the dimensionless problem by dropping the star notation [20]

$$\frac{\partial w(y,t)}{\partial t} - \left(1 + \alpha_2\frac{\partial}{\partial t}\right)\frac{\partial^2 w(y,t)}{\partial y^2} = G_r T(y,t) + G_m C(y,t), \tag{8}$$

$$\frac{\partial T(y,t)}{\partial t} - \frac{1}{P_r}\frac{\partial^2 T(y,t)}{\partial y^2} = 0, \tag{9}$$

$$\frac{\partial C(y,t)}{\partial t} - \frac{1}{S_c}\frac{\partial^2 C(y,t)}{\partial y^2} = 0, \tag{10}$$

The initial and boundary conditions are

$$w(0,t) = A_0 H(t)\,t^p, \quad T(0,t) = t, \quad C(0,t) = t, \quad t \geq 0, \qquad t > 0, \tag{11}$$

$$w(y,0) = 0, \quad T(y,0) = 0, \quad C(y,0) = 0, \qquad y > 0, \tag{12}$$

$$w(\infty,t) = 0, \quad T(\infty,t) = 0, \quad C(\infty,t) = 0 \qquad t > 0. \tag{13}$$

3. Problem Calculation

3.1. Analytic Solutions with Atangana–Baleanu Fractional Derivative

In order to generate the Atangana–Baleanu fractional model for second grade fluid, we replace governing partial differential equations with respect to time by the Atangana–Baleanu fractional operator of the order $0 < \alpha < 1$; Equations (8)–(10) become

$$^{AB}\left(\frac{\partial^\alpha T(y,t)}{\partial t^\alpha}\right) = \frac{1}{P_r}\frac{\partial^2 T(y,t)}{\partial y^2}, \tag{14}$$

$$^{AB}\left(\frac{\partial^\alpha C(y,t)}{\partial t^\alpha}\right) = \frac{1}{S_c}\frac{\partial^2 C(y,t)}{\partial y^2}, \tag{15}$$

$$^{AB}\left(\frac{\partial^\alpha w(y,t)}{\partial t^\alpha}\right) - \left\{1 + \alpha_2{}^{AB}\left(\frac{\partial^\alpha}{\partial t^\alpha}\right)\right\}\frac{\partial^2 w(y,t)}{\partial y^2} = G_r T(y,t) + G_m C(y,t), \tag{16}$$

where, $\frac{\partial^\alpha w(y,t)}{\partial t^\alpha}$ is the Atangana–Baleanu fractional operator of order α defined as [20]

$$^{AB}\left(\frac{\partial^\alpha w(y,t)}{\partial t^\alpha}\right) = \frac{1}{1-\alpha}\int_0^t w'(y,t)\mathbf{E}_\alpha\left(\frac{-\alpha(z-t)^\alpha}{1-\alpha}\right)dt. \tag{17}$$

For Equation (17), $\mathbf{E}_\alpha(-t^\alpha) = \sum\limits_{m=0}^{\infty} \frac{(-t)^{\alpha m}}{\Gamma(1+\alpha m)}$ is the Mittage–Leffler function.

Employing discrete Laplace transform to Equations (14)–(16) and taking $\eta = \frac{1}{1-\alpha}$, we arrive at

$$\frac{\eta q^\alpha \,\overline{T}(y,q)}{q^\alpha + \eta\alpha} = \frac{1}{P_r}\frac{\partial^2 \overline{T}(y,q)}{\partial y^2}, \tag{18}$$

$$\frac{\eta q^\alpha \,\overline{C}(y,q)}{q^\alpha + \eta\alpha} = \frac{1}{S_c}\frac{\partial^2 \overline{C}(y,q)}{\partial y^2}, \tag{19}$$

$$\frac{\eta q^\alpha \,\overline{w}(y,q)}{q^\alpha + \eta\alpha} - \left\{1 + \frac{\alpha_2 \eta q^\alpha}{q^\alpha + \eta\alpha}\right\}\frac{\partial^2 \overline{w}(y,q)}{\partial y^2} = G_r\overline{T}(y,q) + G_m\overline{C}(y,q). \tag{20}$$

Applying initial and boundary conditions (11)–(13) to Equations (18)–(20), we obtain,

$$\overline{T}(y,q) = \frac{1}{q^2}e^{-y\sqrt{\frac{P_r\,\eta\, q^\alpha}{q^\alpha+\eta\alpha}}}, \tag{21}$$

$$\overline{C}(y,q) = \frac{1}{q^2}e^{-y\sqrt{\frac{S_c\,\eta\, q^\alpha}{q^\alpha+\eta\alpha}}}, \tag{22}$$

$$\overline{w}(y,q) = \frac{A_0 P!}{q^{P+1}}e^{-y\sqrt{\frac{\eta\, q^\alpha}{q^\alpha(1+\eta\alpha_2)+\eta\alpha}}} - \frac{G_m(q^\alpha+\eta\alpha)^2\, e^{-y\sqrt{\frac{S_c\,\eta\, q^\alpha}{q^\alpha+\eta\alpha}}}}{q^2(q_1 q^{2\alpha} + q_2 q^\alpha)} - \frac{G_r\,(q^\alpha+\eta\alpha)^2\, e^{-\sqrt{\frac{P_r\eta\, q^\alpha}{q^\alpha+\eta\alpha}}}}{q^2(q_3 q^{2\alpha} + q_4 q^\alpha)}, \tag{23}$$

where, $q_1 = S_c\eta^2\alpha_2 - \eta$, $q_2 = S_c\eta + S_c\alpha\eta^2 - \eta^2\alpha$, $q_3 = P_r\eta^2\alpha_2 - \eta$, and $q_4 = P_r\eta + P_r\eta^2\alpha - \eta^2\alpha$.

Writing Equations (21)–(23) into series form, we traced an equivalent form as

$$\overline{T}(y,q) = \frac{1}{q^2} + \sum_{\Lambda_1=1}^{\infty}\frac{\left(y\sqrt{p_r\eta}\right)^{\Lambda_1}}{\Lambda_1!}\sum_{\Lambda_2=0}^{\infty}\frac{(-\eta\alpha)^{\Lambda_2}\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)}{\Lambda_2!\Gamma\left(\frac{\Lambda_1}{2}\right)}\frac{1}{q^{\Lambda_2+2}}, \tag{24}$$

$$\overline{C}(y,q)=\frac{1}{q^2}+\sum_{\Lambda_1=1}^{\infty}\frac{\left(y\sqrt{s_c}\eta\right)^{\Lambda_1}}{\Lambda_1!}\sum_{\Lambda_2=0}^{\infty}\frac{(-\eta\alpha)^{\Lambda_2}\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)}{\Lambda_2!\Gamma\left(\frac{\Lambda_1}{2}\right)}\frac{1}{q^{\Lambda_2+2}}, \tag{25}$$

$$\begin{aligned}\overline{w}(y,q)={}&\frac{A_0 p!}{q^{p+1}}+A_0 p!\sum_{\Lambda_1=1}^{\infty}\frac{1}{\Lambda_1!}\left(\frac{-y\sqrt{\eta}}{\sqrt{1+\eta\alpha_2}}\right)^{\Lambda_1}\sum_{\Lambda_2=0}^{\infty}\frac{1}{\Lambda_2!}\left(\frac{-\eta\alpha}{1+\eta\alpha_2}\right)^{\Lambda_2}\frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)}{\Gamma\left(\frac{\Lambda_1}{2}\right)}\frac{1}{q^{\frac{\Lambda_1}{2}-\frac{\eta\Lambda_1}{2}+\Lambda_2+p+1}}\\&-\frac{G_m}{q_2}\sum_{\Lambda_1=0}^{\infty}\frac{\left(-y\sqrt{\eta s_c}\right)^{\Lambda_1}}{\Lambda_4!}\sum_{\Lambda_2=0}^{\infty}\frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\alpha}\right)^{\Lambda_2}\sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_1}{q_2}\right)^{L}\sum_{\Lambda_3=0}^{\infty}\frac{(-\eta\alpha)^{\Lambda_3}}{\Lambda_3!}\frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)\Gamma(3)}{\Gamma\left(\frac{\Lambda_1}{2}\right)\Gamma(3-\Lambda_3)}\frac{1}{q^{\alpha+\Lambda_2\alpha+\Lambda_3\alpha-2\alpha-\Lambda_4\alpha+2}}\\&-\frac{G_r}{q_4}\sum_{\Lambda_1=0}^{\infty}\frac{\left(-y\sqrt{\eta\, p_r}\right)^{\Lambda_1}}{\Lambda_1!}\sum_{\Lambda_2=0}^{\infty}\frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\alpha}\right)^{\Lambda_2}\sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_3}{q_4}\right)^{L}\sum_{\Lambda_3=0}^{\infty}\frac{(-\eta\alpha)^{\Lambda_3}}{\Lambda_3!}\frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)\Gamma(3)}{\Gamma\left(\frac{\Lambda_1}{2}\right)\Gamma(3-\Lambda_3)}\frac{1}{q^{\alpha+\Lambda_2\alpha+\Lambda_3\alpha-2\alpha-\Lambda_4\alpha+2}}.\end{aligned} \tag{26}$$

Inverting Equations (24)–(26) by Laplace transform and expressing Equations (24)–(26) in terms of generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q$ and newly published generalized M-function $\mathbf{M}_q^p(z)$

$$T(y,t)=t+\sum_{\Lambda_1=1}^{\infty}\frac{\left(y\sqrt{p_r\eta}\right)^{\Lambda_1}}{\Lambda_1!}{}_1\mathbf{\Psi}_2\left[-\eta\alpha t\left|\begin{matrix}\left(\frac{\Lambda_1}{2},1\right)\\\left(\frac{\Lambda_1}{2},0\right),(2,1)\end{matrix}\right.\right], \tag{27}$$

$$C(y,t)=t+\sum_{\Lambda_1=1}^{\infty}\frac{\left(y\sqrt{s_c\eta}\right)^{\Lambda_1}}{\Lambda_1!}{}_1\mathbf{\Psi}_2\left[-\eta\alpha t\left|\begin{matrix}\left(\frac{\Lambda_1}{2},1\right)\\\left(\frac{\Lambda_1}{2},0\right),(2,1)\end{matrix}\right.\right], \tag{28}$$

$$\begin{aligned}w(y,t)={}&A_0H(t)t^p+A_0p!\sum_{\Lambda_1=1}^{\infty}\frac{1}{\Lambda_1!}\left(\frac{-y\sqrt{\eta}}{\sqrt{1+\eta\alpha_2}}\right)^{\Lambda_1}\mathbf{M}_2^1\left[\frac{-\eta\alpha t}{1+\eta\alpha_2}\left|\begin{matrix}\left(\frac{\Lambda_1}{2},1\right)\\\left(\frac{\Lambda_1}{2},1\right),\left(\frac{\Lambda_1}{2}-\frac{\Lambda_1\alpha}{2}+p+1,1\right)\end{matrix}\right.\right]\\&-\frac{G_m}{q_2}\sum_{\Lambda_1=0}^{\infty}\frac{\left(-y\sqrt{\eta s_c}\right)^{\Lambda_1}}{\Lambda_4!}\sum_{\Lambda_2=0}^{\infty}\frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\alpha}\right)^{\Lambda_2}\sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_1}{q_2}\right)^{L}\mathbf{M}_3^2\left[-\eta\alpha t\left|\begin{matrix}\left(\frac{\Lambda_1}{2},1\right),(3,0)\\\left(\frac{\Lambda_1}{2},0\right),(3-\Lambda_3,0),(\alpha+\alpha\Lambda_3-2\alpha-\alpha\Lambda_4+2,\alpha)\end{matrix}\right.\right]\\&-\frac{G_r}{q_4}\sum_{\Lambda_1=0}^{\infty}\frac{\left(-y\sqrt{\eta\, p_r}\right)^{\Lambda_1}}{\Lambda_1!}\sum_{\Lambda_2=0}^{\infty}\frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\alpha}\right)^{\Lambda_2}\sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_3}{q_4}\right)^{L}\mathbf{M}_3^2\left[-\eta\alpha t\left|\begin{matrix}\left(\frac{\Lambda_1}{2},1\right),(3,0)\\\left(\frac{\Lambda_1}{2},0\right),(3-\Lambda_3,0),(\alpha+\alpha\Lambda_3-2\alpha-\alpha\Lambda_4+2,\alpha)\end{matrix}\right.\right].\end{aligned} \tag{29}$$

with, the properties of generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$

$$\sum_{\chi=0}^{\infty}\frac{(-L)^{\chi}\prod_{h=1}^{f}\Gamma(m_h+M_h\chi)}{\chi!\prod_{h=1}^{g}\Gamma(n_h+N_h\chi)}={}_f\mathbf{\Psi}_g\left[L\left|\begin{matrix}(m_1,M_1),(m_2,M_2),\ldots,(m_f,M_f)\\(n_1,N_1),(n_2,N_2),\ldots,(n_f,N_f)\end{matrix}\right.\right], \tag{30}$$

and the newly defined generalized M-function $\mathbf{M}_q^p(z)$ is:

$$t^{n_g-1}\sum_{\chi=0}^{\infty}\frac{(-L)^{\chi}\prod_{h=1}^{f}\Gamma(m_h+M_h\chi)}{\chi!\prod_{h=1}^{g}\Gamma(n_h+N_h\chi)}=\mathbf{M}_g^f\left[L\left|\begin{matrix}(m_1,M_1),(m_2,M_2),\ldots,(m_f,M_f)\\(n_1,N_1),(n_2,N_2),\ldots,(n_f,N_f)\end{matrix}\right.\right]. \tag{31}$$

3.2. Analytic Solutions with Caputo–Fabrizio Fractional Derivative

In order to generate the Caputo–Fabrizio fractional model for second grade fluid, we replace the governing partial differential equations with respect to time by the Caputo–Fabrizio fractional operator of order $0<\beta<1$, Equations (8)–(10) become:

$$^{CF}\left(\frac{\partial^{\beta}T(y,t)}{\partial t^{\beta}}\right)=\frac{1}{P_r}\frac{\partial^2 T(y,t)}{\partial y^2}, \tag{32}$$

$$^{CF}\left(\frac{\partial^{\beta}C(y,t)}{\partial t^{\beta}}\right)=\frac{1}{S_c}\frac{\partial^2 C(y,t)}{\partial y^2}, \tag{33}$$

$$ {}^{CF}\left(\frac{\partial^\beta w(y,t)}{\partial t^\beta}\right) - \left\{1 + \alpha_2 {}^{CF}\left(\frac{\partial^\beta}{\partial t^\beta}\right)\right\}\frac{\partial^2 w(y,t)}{\partial y^2} = G_r T(y,t) + G_m C(y,t), \tag{34} $$

where, $\frac{\partial^\beta w(y,t)}{\partial t^\beta}$ is the Caputo–Fabrizio fractional operator of order β defined as [11]:

$$ {}^{CF}\left(\frac{\partial^\beta w(y,t)}{\partial t^\beta}\right) = \frac{1}{1-\beta}\int_0^t w'(y,t) Exp\left(\frac{-\beta(z-t)}{1-\beta}\right) dt. \tag{35} $$

Employing discrete Laplace transform to Equations (32)–(34) and taking $\eta = \frac{1}{1-\beta}$, we arrive at

$$ \frac{\eta q\, \overline{T}(y,q)}{q+\eta\beta} = \frac{1}{P_r}\frac{\partial^2 \overline{T}(y,q)}{\partial y^2}, \tag{36} $$

$$ \frac{\eta q\, \overline{C}(y,q)}{q+\eta\beta} = \frac{1}{S_c}\frac{\partial^2 \overline{C}(y,q)}{\partial y^2}, \tag{37} $$

$$ \frac{\eta q\, \overline{w}(y,q)}{q+\eta\beta} - \left\{1 + \frac{\alpha_2 \eta q}{q+\eta\beta}\right\}\frac{\partial^2 \overline{w}(y,q)}{\partial y^2} = G_r \overline{T}(y,q) + G_m \overline{C}(y,q). \tag{38} $$

Applying initial and boundary conditions (11)–(13) to Equations (36)–(38), we obtain,

$$ \overline{T}(y,q) = \frac{1}{q^2} e^{-y\sqrt{\frac{P_r \eta q}{q+\eta\beta}}}, \tag{39} $$

$$ \overline{C}(y,q) = \frac{1}{q^2} e^{-y\sqrt{\frac{S_c \eta q}{q+\eta\beta}}}, \tag{40} $$

$$ \overline{w}(y,q) = \frac{A_0 P!}{q^{P+1}} e^{-y\sqrt{\frac{\eta q}{q(1+\eta\alpha_2)+\eta\beta}}} - \frac{G_m (q+\eta\beta)^2 e^{-y\sqrt{\frac{S_c \eta q}{q+\eta\beta}}}}{q^2(q_1 q^2 + q_2 q)} - \frac{G_r (q+\eta\beta)^2 e^{-\sqrt{\frac{P_r \eta q}{q+\eta\beta}}}}{q^2(q_3 q^2 + q_4 q)}, \tag{41} $$

where, $q_1 = S_c \eta^2 \alpha_2 - \eta$, $q_2 = S_c \eta + S_c \eta^2 - \eta^2$, $q_3 = P_r \eta^2 \alpha_2 - \eta$, and $q_4 = P_r \eta + P_r \eta^2 - \eta^2$.

Writing Equations (39)–(41) into series form, we find an equivalent form as

$$ \overline{T}(y,q) = \frac{1}{q^2} + \sum_{\Lambda_1=1}^{\infty} \frac{(y\sqrt{p_r \eta})^{\Lambda_1}}{\Lambda_1!} \sum_{\Lambda_2=0}^{\infty} \frac{(-\eta\beta)^{\Lambda_2}\Gamma\left(\Lambda_2 + \frac{\Lambda_1}{2}\right)}{\Lambda_2!\Gamma\left(\frac{\Lambda_1}{2}\right)} \frac{1}{q^{\Lambda_2+2}}, \tag{42} $$

$$ \overline{C}(y,q) = \frac{1}{q^2} + \sum_{\Lambda_1=1}^{\infty} \frac{(y\sqrt{s_c}\eta)^{\Lambda_1}}{\Lambda_1!} \sum_{\Lambda_2=0}^{\infty} \frac{(-\eta\beta)^{\Lambda_2}\Gamma\left(\Lambda_2 + \frac{\Lambda_1}{2}\right)}{\Lambda_2!\Gamma\left(\frac{\Lambda_1}{2}\right)} \frac{1}{q^{\Lambda_2+2}}, \tag{43} $$

$$ \begin{aligned} \overline{w}(y,q) = {} & \frac{A_0 p!}{q^{p+1}} + A_0 p! \sum_{\Lambda_1=1}^{\infty} \frac{1}{\Lambda_1!}\left(\frac{-y\sqrt{\eta}}{\sqrt{1+\eta\alpha_2}}\right)^{\Lambda_1} \sum_{\Lambda_2=0}^{\infty} \frac{1}{\Lambda_2!}\left(\frac{-\eta\beta}{1+\eta\alpha_2}\right)^{\Lambda_2} \frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)}{\Gamma\left(\frac{\Lambda_1}{2}\right)} \frac{1}{q^{\Lambda_2+p+1}} \\ & - \frac{G_m}{q_2} \sum_{\Lambda_1=0}^{\infty} \frac{(-y\sqrt{\eta s_c})^{\Lambda_1}}{\Lambda_4!} \sum_{\Lambda_2=0}^{\infty} \frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\beta}\right)^{\Lambda_2} \sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_1}{q_2}\right)^{L} \sum_{\Lambda_3=0}^{\infty} \frac{(-\eta\beta)^{\Lambda_3}}{\Lambda_3!} \frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)\Gamma(3)}{\Gamma\left(\frac{\Lambda_1}{2}\right)\Gamma(3-\Lambda_3)} \frac{1}{q^{\Lambda_2+\Lambda_3-\Lambda_4+1}} \\ & - \frac{G_r}{q_4} \sum_{\Lambda_1=0}^{\infty} \frac{(-y\sqrt{\eta\, p_r})^{\Lambda_1}}{\Lambda_1!} \sum_{\Lambda_2=0}^{\infty} \frac{1}{\Lambda_2!}\left(\frac{-1}{\eta\beta}\right)^{\Lambda_2} \sum_{\Lambda_4=0}^{\infty}\left(\frac{-q_3}{q_4}\right)^{L} \sum_{\Lambda_3=0}^{\infty} \frac{(-\eta\beta)^{\Lambda_3}}{\Lambda_3!} \frac{\Gamma\left(\Lambda_2+\frac{\Lambda_1}{2}\right)\Gamma(3)}{\Gamma\left(\frac{\Lambda_1}{2}\right)\Gamma(3-\Lambda_3)} \frac{1}{q^{\Lambda_2+\Lambda_3-\Lambda_4+1}}. \end{aligned} \tag{44} $$

Inverting Equations (24)–(26) by Laplace transform and expressing Equations (24)–(26) in terms of generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$ and newly defined generalized M-function $\mathbf{M}_q^p(z)$:

$$ T(y,t) = t + \sum_{\Lambda_1=1}^{\infty} \frac{(y\sqrt{p_r \eta})^{\Lambda_1}}{\Lambda_1!}\, {}_1\mathbf{\Psi}_2\left[-\eta\beta t \left| \begin{matrix} \left(\frac{\Lambda_1}{2}, 1\right) \\ \left(\frac{\Lambda_1}{2}, 0\right), (2,1) \end{matrix} \right. \right], \tag{45} $$

$$C(y,t) = t + \sum_{\Lambda_1=1}^{\infty} \frac{\left(y\sqrt{s_c\eta}\right)^{\Lambda_1}}{\Lambda_1!}\, {}_1\mathbf{\Psi}_2 \left[-\eta\beta t \left| \begin{matrix} \left(\frac{\Lambda_1}{2}, 1\right) \\ \left(\frac{\Lambda_1}{2}, 0\right), (2,1) \end{matrix} \right. \right], \tag{46}$$

$$\begin{aligned} w(y,t) = {} & \mathrm{A}_0 H(t) t^p + \mathrm{A}_0 p! \sum_{\Lambda_1=1}^{\infty} \frac{1}{\Lambda_1!} \left(\frac{-y\sqrt{\eta}}{\sqrt{1+\eta\alpha_2}} \right)^{\Lambda_1} \mathbf{M}_2^1 \left[\frac{-\eta\beta t}{1+\eta\alpha_2} \left| \begin{matrix} \left(\frac{\Lambda_1}{2}, 1\right) \\ \left(\frac{\Lambda_1}{2}, 1\right), (p+1,1) \end{matrix} \right. \right] \\ & - \frac{G_m}{q_2} \sum_{\Lambda_1=0}^{\infty} \frac{\left(-y\sqrt{\eta s_c}\right)^{\Lambda_1}}{\Lambda_4!} \sum_{\Lambda_2=0}^{\infty} \frac{1}{\Lambda_2!} \left(\frac{-1}{\eta\beta} \right)^{\Lambda_2} \sum_{\Lambda_4=0}^{\infty} \left(\frac{-q_1}{q_2} \right)^{L} \mathbf{M}_3^2 \left[-\eta\beta t \left| \begin{matrix} \left(\frac{\Lambda_1}{2}, 1\right), (3,0) \\ \left(\frac{\Lambda_1}{2}, 0\right), (3,-1), (\Lambda_2 - \Lambda_4 + 1, 1) \end{matrix} \right. \right] \\ & - \frac{G_r}{q_4} \sum_{\Lambda_1=0}^{\infty} \frac{\left(-y\sqrt{\eta\, p_r}\right)^{\Lambda_1}}{\Lambda_1!} \sum_{\Lambda_2=0}^{\infty} \frac{1}{\Lambda_2!} \left(\frac{-1}{\eta\beta} \right)^{\Lambda_2} \sum_{\Lambda_4=0}^{\infty} \left(\frac{-q_3}{q_4} \right)^{L} \mathbf{M}_3^2 \left[-\eta\beta t \left| \begin{matrix} \left(\frac{\Lambda_1}{2}, 1\right), (3,0) \\ \left(\frac{\Lambda_1}{2}, 0\right), (3,-1), (\Lambda_2 - \Lambda_4 + 1, 1) \end{matrix} \right. \right]. \end{aligned} \tag{47}$$

4. Results and Discussion

A comparative study of a second grade fluid problem with the combined gradients of mass concentration and temperature distribution was studied via newly presented non-integer order derivatives, namely Caputo–Mauro Fabrizio (CF) and Atangana–Baleanu (AB) fractional derivatives, respectively. Analytical solutions have been established in both cases of CF and AB fractional derivatives via Laplace transform and expressed in terms of newly defined M- function $\mathbf{M}_q^p(z)$ and generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$. In order to justify the validity of comparison, the rheology of several pertinent parameters was compared graphically for CF and AB fractional derivatives with various similarities and differences and some consequential points. The analytical general solutions of temperature distribution, mass concentration, and velocity field have been obtained. They are expressed in the form of the generalized Hyper-geometric function ${}_p\mathbf{\Psi}_q(z)$ and newly defined M-function $\mathbf{M}_q^p(z)$.

Figure 1 is plotted for temperature distribution to show the effects of the Prandlt number in which the thermal boundary layer is scattering in both cases of fractional derivatives. It can be noted that temperature distribution has reciprocal behavior for heat transfer over the whole domain of the plate. Physically, Prandlt number defines the ratio of momentum diffusivity to thermal diffusivity. In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers.

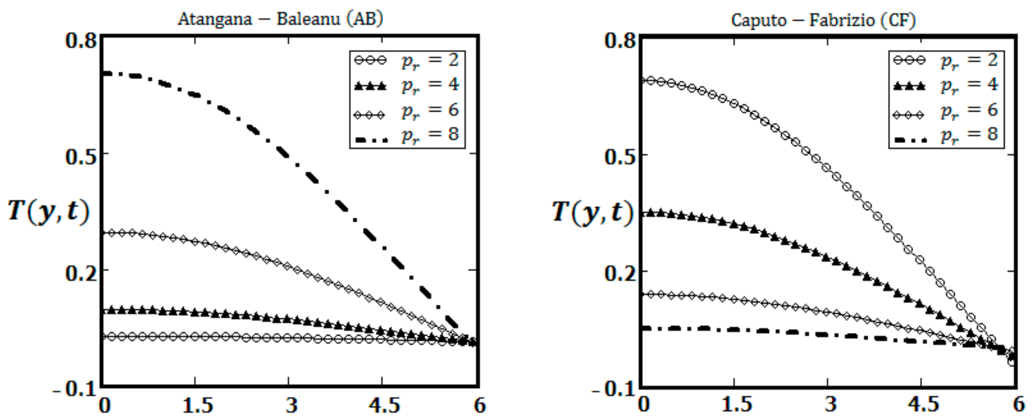

Figure 1. Profile of the temperature distribution for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $\alpha = \beta = 0.3,\ \mu = 12.7,\ t = 2\,\mathrm{s}$, and with different values of P_r.

The mass transfer analog of the Prandtl number is the Schmidt number. This is a dimensionless number defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity, and it is used to characterize fluid flows in which there are simultaneous momentum and mass diffusion convection processes. Figure 2 elucidates the influences on the Schmidt number on mass concentration. It is observed that the Schmidt number behavior is identical to that of the Prandtl number.

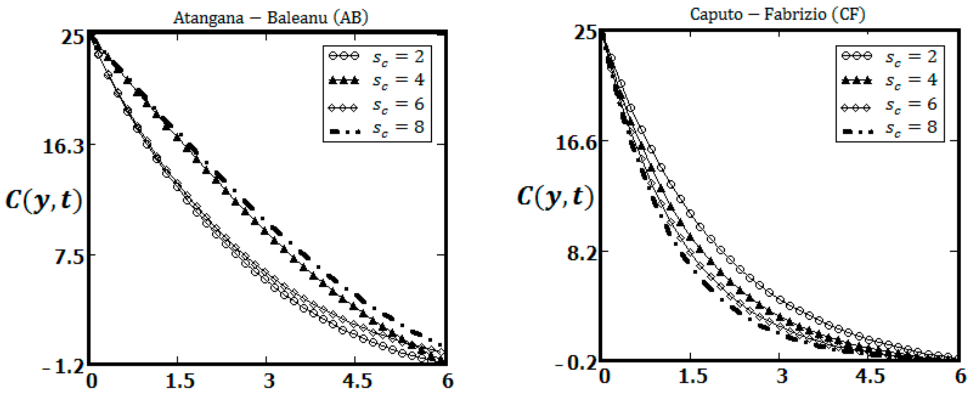

Figure 2. Profile of the mass concentration for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $\alpha = \beta = 0.3, \nu = 6.1, t = 2\,\mathrm{s}$, and with different values of s_c.

Figures 3 and 4 are prepared to characterize the fluid flow for buoyancy and viscous forces due to natural convection. It can be seen in the velocity field that an increase in the Grashof number or the modified Grashof number have similar effects on velocity. In a physical sense, as expected, when the Grashof number and the modified Grashof number are increased, then fluid flow rises due to the thermal buoyancy effects.

Figure 5 demonstrates the effects of second grade fluid on the velocity field which results in opposing fluid flow. It is further noted from Figure 5 that the velocity field via the AB fractional derivative is an increasing function and a decreasing function via the CF fractional derivative. This reversal flow of fluid may be due to the effects of non-locality as well as non-singularity of the kernels. The same is also examined in Figure 6 by taking different values of fractional parameters of AB and CF fractional derivatives.

Figure 7 reveals the influential conclusion that four different values of time are taken for the velocity field. It is worth noting that for shorter time $t = 0.2$, the velocity field investigated by the CF approach moves faster in comparison with that of the velocity field investigated by the AB approach. It is also clear that when time $t = 0.4$, the velocity fields investigated by both approaches have identical behavior. On the other hand, in the case of increasing time, the velocity field traced out by the AB approach is greater in comparison to that of the CF approach.

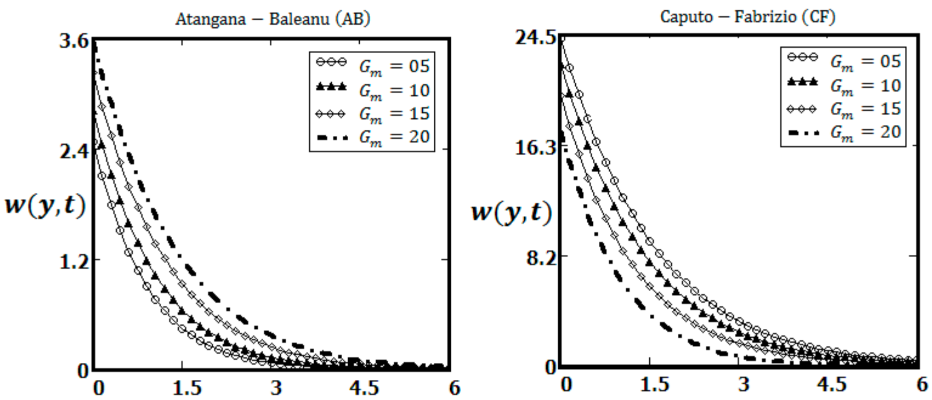

Figure 3. Profile of the velocity field for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $A_0 = 2.5, \alpha_2 = 6, P_r = 2, S_c = 1.7, G_r 4.6, \alpha = \beta = 0.3, \omega = 0.5, p = 2, t = 2\,\mathrm{s}$, and with different values of G_m.

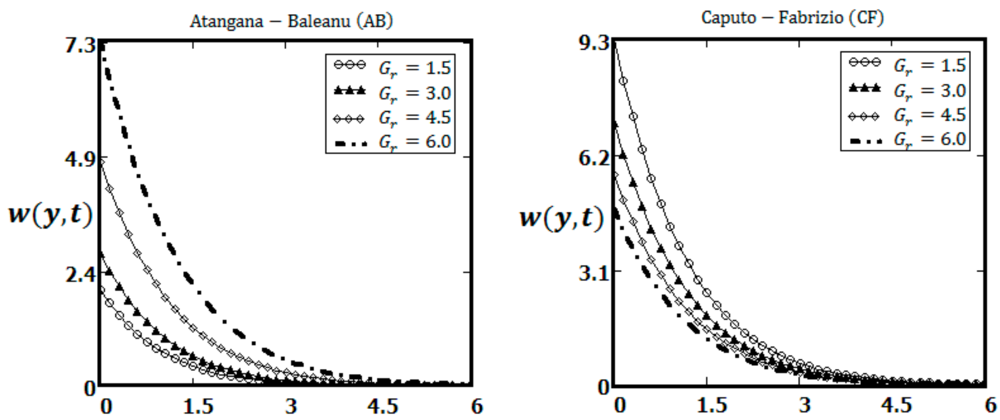

Figure 4. Profile of the velocity field for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $A_0 = 10, \alpha_2 = 2.1, P_r = 3, S_c = 1.7, G_m = 3, \alpha = \beta = 0.3, \omega = 0.5, p = 2, t = 2\,$s, and with different values of G_m.

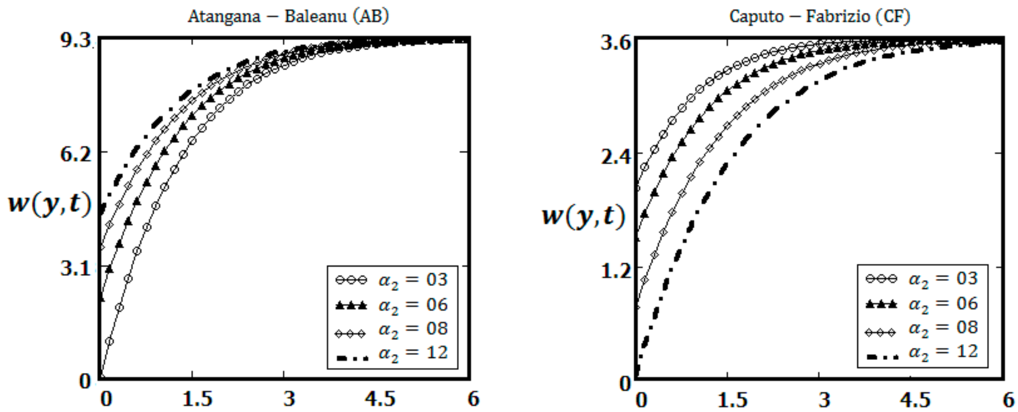

Figure 5. Profile of the velocity field for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $A_0 = 2, P_r = 8, S_c = 0.8, G_m = 14.1, G_r = 2.9, \alpha = \beta = 0.3, \omega = 0.5, p = 2, t = 2\,$s, and with different values of α_2.

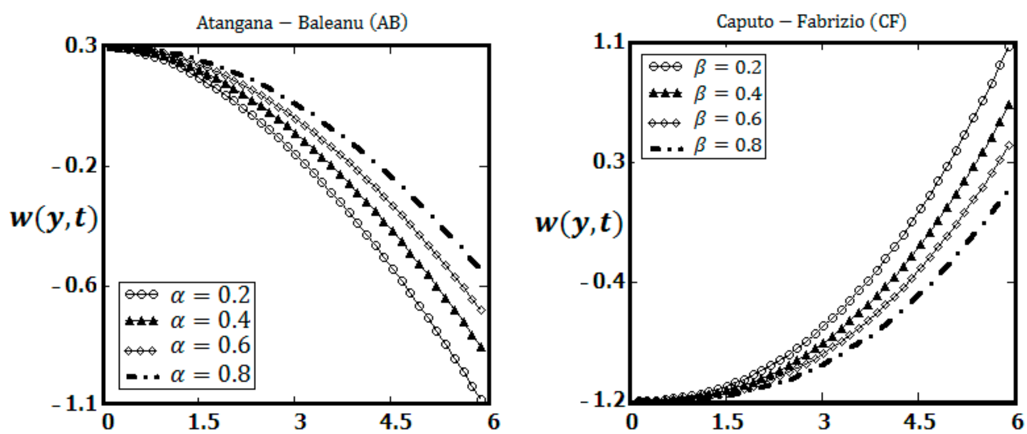

Figure 6. Profile of the velocity field for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $A_0 = 9, \alpha_2 = 3, P_r = 2.3, S_c = 4.1, G_m = 2.43, G_r = 0.2, \omega = 0.5, p = 2, t = 2\,$s, and with different values of α and β.

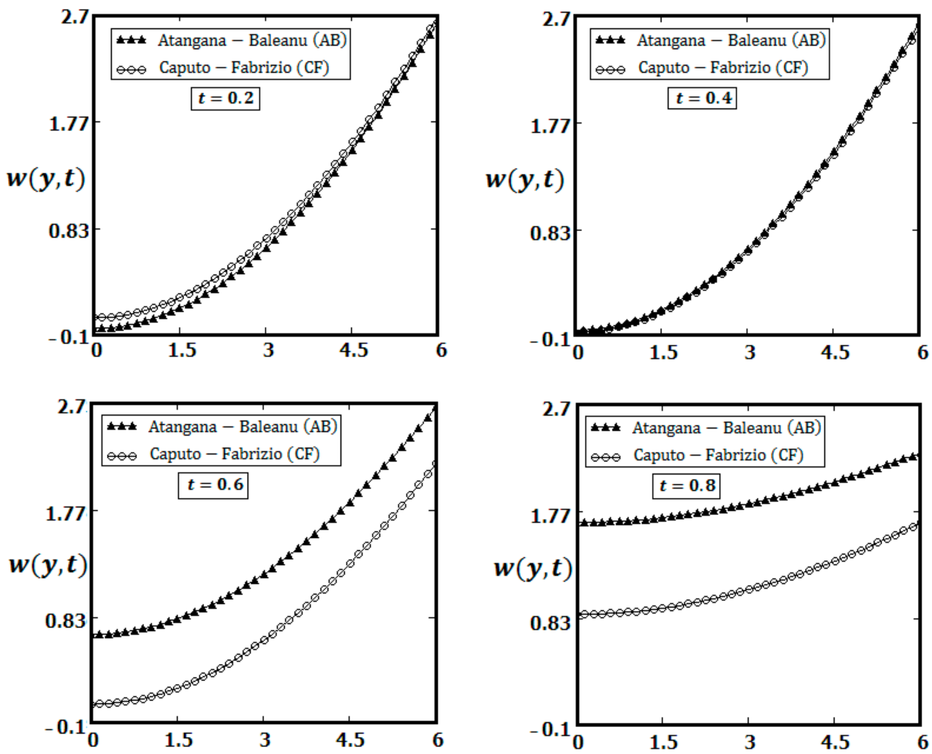

Figure 7. Comparison of the velocity field for Atangana–Baleanu verses Caputo–Fabrizio fractional derivatives when $A_0 = 7, \alpha_2 = 2, P_r = 12, S_c = 4, G_r = 0.6, G_m = 0.2, \alpha = \beta = 0.3, \omega = 0.5, p = 2$, and with different values of t.

It is worth mentioning that limiting cases for this problem can also be considered in order to retrieve a few solutions from the published literature. Firstly, the analytical solutions of both cases of fractional derivatives can be reduced to ordinary derivatives by taking fractional parameters equal to 1. The corresponding solutions for viscous fluid can also be obtained as a special case by taking a second grade parameter equal to zero. The general analytical solution of the first problem of Stokes' can be recovered by taking oscillating frequency equal to zero. The present solutions sudden plate motion become identical to the solution obtained by Shah and Khan ([2], see Equations (22) and (26)) when $Gm = 0$ and the plate is suddenly moved. This comparison is shown in Figure 8. Clearly the solutions obtained by Shah and Khan [2] are in in excellent agreement with the present limiting solutions. This also confirms the accuracy of the present work.

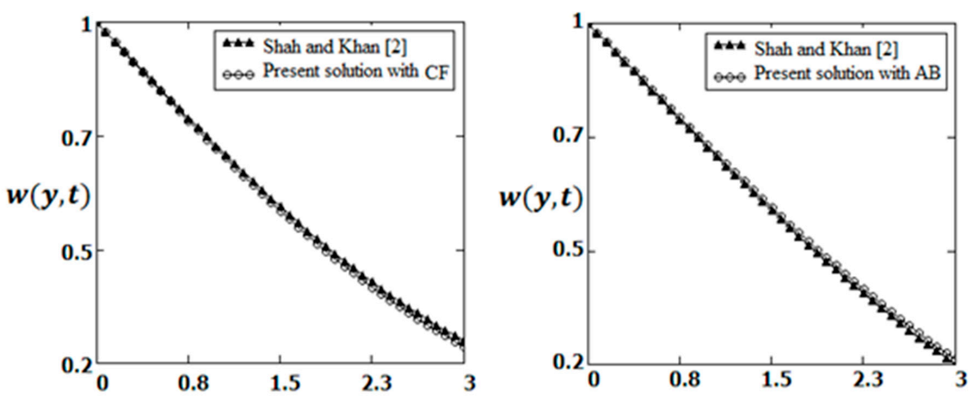

Figure 8. Comparison of the present velocity field when $Gm = 0$, $p = 0$, and Shah and Khan [2] when $w = 0$ via Atangana–Baleanu (AB) and Caputo–Fabrizio (CF) fractional derivatives.

5. Concluding Remarks

This study investigated the comparative analysis of the Atangana–Baleanu fractional and Caputo–Fabrizio fractional approaches for heat and mass transfer of a second grade fluid. Graphs were plotted for several rheological parameters via two different fractional approaches and discussed in detail. Results from analytical solutions showed that Atangana–Baleanu fractional derivatives have reciprocal behavior to Caputo–Fabrizio fractional derivatives. The results also indicate that in a comparison of the two fractional derivatives, the Atangana–Baleanu fractional model moves faster than the Caputo–Fabrizio fractional model. Moreover, the present solutions were compared with published results and were found to be in excellent agreement.

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Author Contributions: Arshad Khan and Kashif Ali Abro did the mathematical formulation; Ilyas Khan and Asifa Tassaddiq performed the solutions; Kashif Ali Abro and Ilyas Khan performed the numerical simulations and plotted the results; Arshad Khan and Asifa Tassaddiq analyzed and discussed the results, Arshad Khan, Kashif Ali Abro, Asifa Tassaddiq, and Ilyas Khan wrote the paper. All the authors contributed equally. All authors have read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

T_∞	Ambient fluid temperature
C_∞	Species concentration away from plate
T_w	Wall temperature
C_w	Concentration level near plate
$w(y,t)$	Velocity field
$T(y,t)$	Temperature distribution
$C(y,t)$	Mass concentration
α_1	Second grade fluid parameter
ρ	Constant density of fluid
ν	Kinematic viscosity of fluid
g	Gravitational acceleration
C_p	Heat capacity at constant pressure
β_c	Volumetric coefficient of expansion for mass Concentration
β_T	Volumetric coefficient of thermal expansion
k	Thermal conductivity
D	Mass diffusivity
A_0	Non zero parameter
$H(t)$	Unit step function
α_2	Material parameter
p_r	Prandtl number
s_c	Schmidt number
G_m	Thermal Grashof number
G_r	Modified Grashof number
α	Fractional parameter of Atangana–Baleanu fractional model
β	Fractional parameter of Caputo–Fabrizio fractional model
${}^{AB}\left(\frac{\partial^\alpha}{\partial t^\alpha}\right)$	Atangana–Baleanu fractional operator
${}^{CF}\left(\frac{\partial^\beta}{\partial t^\beta}\right)$	Caputo–Fabrizio fractional operator
$\mathbf{E}_\alpha(-t^\alpha)$	Mittage-Leffler function

η, q_1, q_2, q_3, q_4	Letting variables
$_p\mathbf{\Psi}_q$	Generalized hyper-geometric function
$\mathbf{M}_q^p(z)$	Generalized M-function
q	Laplace transforms parameter
t	Time

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