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α-Connections and a Symmetric Cubic Form on a Riemannian Manifold

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Abstract: In this paper, we study the construction of α-conformally equivalent statistical manifolds for a given symmetric cubic form on a Riemannian manifold. In particular, we describe a method to obtain α-conformally equivalent connections from the relation between tensors and the symmetric cubic form.

Keywords: information geometry; statistical manifold; affine geometry; α-connection; conformal equivalence

1. Introduction

Statistical manifolds are important in differential geometry, information geometry, theoretical physics, and so on and the theory of α-connections plays a significant role in these fields [1].

By considering conformal transformations of α-connections, Okamoto, Amari, and Takeuchi obtained an asymptotic theory of sequential estimation [2]. Kurose defined the α-conformal equivalence and α-conformal flatness of statistical manifolds. In addition, he showed that a statistical manifold is (-1)-conformally flat only if the connection is a projectively flat connection with a symmetric Ricci tensor [3]. The projective equivalence of affine hypersurfaces was studied by Nomizu and Sasaki [4]. Relations between statistical manifolds and affine hypersurfaces allow divergences of statistical manifolds to be expressed in terms of geometric divergences.

A Hessian domain is a flat statistical manifold and, conversely, a flat statistical manifold is locally a Hessian domain [5,6]. In our previous paper, we showed that level surfaces of a Hessian potential function are $(+1)$-conformally flat statistical submanifolds of a Hessian domain, and that a $(+1)$-conformally flat statistical manifold with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold [7]. Moreover, we showed that level surfaces of a Hessian domain are $(+1)$-conformally equivalent, and that these α-connections are α-conformally equivalent [8,9].

The divergence of Nagaoka and Amari is defined on a Hessian domain (i.e., a flat statistical manifold). The geometrical divergence of Kurose on a $(+1)$-conformally flat statistical manifold coincides with the restriction of the divergence of Nagaoka and Amari onto a level surface of a Hessian domain. Based on this, we obtained the decomposition of the divergence on a Hessian domain, with respect to the foliation and orthogonal foliation of level surfaces, and not with respect to a flat submanifold and perpendicularly dual geodesic on a flat statistical manifold [7]. Through our decomposition, the infimum of the divergence provides the projection of a point in a Hessian domain onto a level surface along an orthogonal leaf. Gradient systems are important to study the relations between information geometry and integrable dynamical systems [10]. We derived a gradient system different from that of Fujiwara and Amari, using decomposition with respect to the foliation of level surfaces [7].

Recently, relations between α-geometry and many concepts, such as means of positive operators, Bayesian statistics, q-exponential families, and escort distributions, were clarified [11–16].

We conducted a detailed investigation of α-geometry for α values other than $\alpha \in \{1, -1\}$ and, in order to obtain more detailed results, we made comparisons to modern α-geometry, general Riemannian geometry, and classical affine geometry [17–19].

In this paper, we study the construction of α-conformally equivalent statistical manifolds for a given symmetric cubic form on a Riemannian manifold. In particular, we describe a method to obtain α-conformally equivalent connections from a relation between tensors and the symmetric cubic form. In Section 2, we explain the definitions and theorems on α-conformal equivalence of statistical manifolds. In Section 3, we present a method to define α-conformally equivalent statistical manifolds on a Riemannian manifold by a symmetric cubic form.

2. α-Conformal Equivalence of Statistical Manifolds

For a torsion-free affine connection ∇ and a pseudo-Riemannian metric h on a manifold N, the triple (N, ∇, h) is called a statistical manifold if ∇h is symmetric. For a statistical manifold (N, ∇, h), let ∇' be an affine connection on N such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z) \text{ for } X, Y \text{ and } Z \in \Gamma(TN),$$

where $\Gamma(TN)$ is the set of smooth tangent vector fields on N. The affine connection ∇' is torsion free, and $\nabla' h$ is symmetric. Then, ∇' is called the dual connection of ∇, the triple (N, ∇', h) is the dual statistical manifold of (N, ∇, h), and (∇, ∇', h) is the dualistic structure on N [1,5].

For a real number α, statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are said to be α-conformally equivalent if there exists a function ϕ on N such that

$$\bar{h}(X, Y) = e^{\phi} h(X, Y), \tag{1}$$

$$\begin{aligned} h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) h(X, Y) \\ + \frac{1-\alpha}{2} \{d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z)\} \end{aligned} \tag{2}$$

for X, Y, and $Z \in \Gamma(TN)$. Two statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are α-conformally equivalent if and only if the dual statistical manifolds (N, ∇', h) and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. A statistical manifold (N, ∇, h) is called α-conformally flat if (N, ∇, h) is locally α-conformally equivalent to a flat statistical manifold. In addition, a statistical manifold is (-1)-conformally flat if and only if the connection is projectively flat with a symmetric Ricci tensor [3]. For $\alpha = -1$, Equation (2) is written as

$$h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) + d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z). \tag{3}$$

A centro-affine hypersurface is called projectively flat if its affine connection ∇ locally satisfies Equation (3) for a flat affine connection $\bar{\nabla}$. It is known that $\phi = \log \lambda$ for a positive function λ, which is the ratio of coordinates for the projected point in the flat plane to coordinates for a point in the centro-affine hypersurface [4]. This makes a projectively flat centro-affine hypersurface into a (-1)-conformally flat statistical manifold. Conversely, for a given (-1)-conformally flat statistical manifold, we can locally find a projectively flat centro-affine hypersurface with a symmetric Ricci tensor that induces the connection and the Riemannian metric of the (-1)-conformally flat statistical manifold.

Let N be a manifold with a dualistic structure (∇, ∇', h). For $\alpha \in \mathbf{R}$, an affine connection defined by

$$\nabla^{(\alpha)} := \frac{1+\alpha}{2} \nabla + \frac{1-\alpha}{2} \nabla' \tag{4}$$

is called an α-connection of (N, ∇, h). The triple $(N, \nabla^{(\alpha)}, h)$ is also a statistical manifold and $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$. The $(+1)$-connection, (-1)-connection and 0-connection coincide with ∇, ∇' and the Levi–Civita connection of (N, h), respectively.

A Hessian domain is a flat statistical manifold and, conversely, a flat statistical manifold is locally a Hessian domain [6]. In our previous papers, we showed that for $n \geq 2$, an n-dimensional level surface of a Hessian domain of dimension equal to $n+1$ is $(+1)$-conformally flat; that level surfaces of a Hessian domain are $(+1)$-conformally equivalent; and that α-connections of level surfaces are α-conformally equivalent [7–9]. From the proofs of the above stated results, it follows that Theorem 1 below holds also for statistical manifolds not being level surfaces or $(+1)$-conformally flat statistical manifolds.

Theorem 1. *For $\alpha \in \mathbf{R}$, two statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are $(+1)$-conformally equivalent if and only if $(N, \nabla^{(\alpha)}, h)$ and $(N, \bar{\nabla}^{(\alpha)}, \bar{h})$ are α-conformally equivalent, where $\nabla^{(\alpha)}$, $\bar{\nabla}^{(\alpha)}$ are α-connections of (N, ∇, h), $(N, \bar{\nabla}, \bar{h})$, respectively.*

If (N, ∇, h) is a flat statistical manifold, we call $\nabla^{(\alpha)}$ an α-transitively flat connection of (N, ∇, h). It is worth noting, however, that an α-transitively flat connection is not always flat. We call $(N, \tilde{\nabla}, h)$ a statistical manifold with an α-transitively flat connection if there exists a flat statistical manifold (N, ∇, h), such that $\tilde{\nabla}$ coincides with an α-transitively flat connection of (N, ∇, h). In [8], we outlined a procedure to realize a statistical manifold, which is α-conformally equivalent to a statistical manifold with an α-transitively flat connection, in another statistical manifold as a statistical submanifold of codimension one.

Theorem 2 ([8])**.** *A statistical manifold of $\dim n \geq 2$ with a Riemannian metric, which is α-conformally equivalent to a statistical manifold with an α-transitively flat connection for non-zero $\alpha \in \mathbf{R}$, can be locally realized as a submanifold of a statistical manifold of $\dim(n+1)$ with an α-transitively flat connection.*

3. A Method to Define α-Connections on Riemannian Manifolds

In Section 2, results and theorems were discussed under the conditions of a given statistical structure or affine geometric structure. In this section, we show a method to induce α-conformally equivalent statistical manifolds for a given Riemannian manifold, and compare the induced statistical structure with an affine geometric structure.

Theorem 3. *Let (N, h) be a Riemannian manifold with the Levi–Civita connection ∇^{LC}. If $(1,2)$-tensors K, $\bar{K}$, and a function ϕ exist on N and satisfy*

$$h(\bar{K}_X Y, Z) = h(K_X Y, Z) - \{d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z) + d\phi(Z)h(X,Y)\}, \tag{5}$$

$$h(K_X Y, Z) = h(K_Y Z, X) = h(K_Z X, Y) \tag{6}$$

for $X, Y,$ and $Z \in \Gamma(TN)$, then $(N, \nabla^{(\alpha)}, h)$ and $(N, \bar{\nabla}^{(\alpha)}, \bar{h})$ are α-conformally equivalent statistical manifolds, where $\nabla^{(\alpha)} := \nabla^{LC} + (\alpha/2)K$, $\bar{\nabla}^{(\alpha)} := \bar{\nabla}^{LC} + (\alpha/2)\bar{K}$, and where $\bar{\nabla}^{LC}$ is the Levi–Civita connection of $\bar{h} = e^{\phi}h$.

Proof of Theorem 3. Since the Levi–Civita connection preserves a metric h, i.e., $\nabla^{LC}h = 0$, it holds from Equation (6) that

$$h(\nabla^{(\alpha)}_X Y, Z) = h(\nabla^{(\alpha)}_Y Z, X) = h(\nabla^{(\alpha)}_Z X, Y). \tag{7}$$

Thus, $\nabla^{(\alpha)}h$ is symmetric and $(N, \nabla^{(\alpha)}, h)$ is a statistical manifold. By Equations (5) and (6), we have

$$h(\bar{K}_X Y, Z) = h(\bar{K}_Y Z, X) = h(\bar{K}_Z X, Y), \tag{8}$$

$$h(\bar{\nabla}^{(\alpha)}_X Y, Z) = h(\bar{\nabla}^{(\alpha)}_Y Z, X) = h(\bar{\nabla}^{(\alpha)}_Z X, Y). \tag{9}$$

In a similar fashion, $(N, \bar{\nabla}^{(\alpha)}, \bar{h})$ is also a statistical manifold.

By definition, $\nabla^{(\alpha)}$ and $\bar{\nabla}^{(\alpha)}$ satisfy Equation (4) and, moreover, they are α-connections of the statistical manifolds $(N, \nabla^{(1)}, h)$ and $(N, \bar{\nabla}^{(1)}, \bar{h})$, respectively.

Next, we prove the following:

$$h(\bar{\nabla}^{LC}_X Y, Z) = h(\nabla^{LC}_X Y, Z) + \frac{1}{2}\{d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z)\} - \frac{1}{2}d\phi(Z)h(X,Y). \tag{10}$$

It is well known on the Levi–Civita connection that [4]

$$2h(\nabla^{LC}_X Y, Z) = Xh(Y,Z) + Yh(Z,X) - Zh(X,Y) + h([X,Y],Z) + h([Z,X],Y) - h([Y,Z],X). \tag{11}$$

Thus,

$$2\bar{h}(\bar{\nabla}^{LC}_X Y, Z) = X\bar{h}(Y,Z) + Y\bar{h}(Z,X) - Z\bar{h}(X,Y) + \bar{h}([X,Y],Z) + \bar{h}([Z,X],Y) - \bar{h}([Y,Z],X), \tag{12}$$

$$\begin{aligned} 2e^{\phi}h(\bar{\nabla}^{LC}_X Y, Z) &= X(e^{\phi}h(Y,Z)) + Y(e^{\phi}h(Z,X)) - Z(e^{\phi}h(X,Y)) \\ &\quad + e^{\phi}h([X,Y],Z) + e^{\phi}h([Z,X],Y) - e^{\phi}h([Y,Z],X) \\ &= d\phi(X)e^{\phi}h(Y,Z) + d\phi(Y)e^{\phi}h(X,Z) - d\phi(Z)e^{\phi}h(X,Y) + e^{\phi}\{Xh(Y,Z) + Yh(Z,X) - Zh(X,Y)\} \\ &\quad + e^{\phi}\{h([X,Y],Z) + h([Z,X],Y) - h([Y,Z],X)\}. \\ &= e^{\phi}\{d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z) - d\phi(Z)h(X,Y)\} + e^{\phi}\cdot 2h(\nabla^{LC}_X Y, Z). \end{aligned}$$

Therefore, we obtain Equation (10).

Finally, by Equations (5) and (10) and $\nabla^{(\alpha)} := \nabla^{LC} + (\alpha/2)K$, we determine that

$$\begin{aligned} h(\bar{\nabla}^{(\alpha)}_X Y, Z) &= h(\nabla^{(\alpha)}_X Y, Z) - \frac{1+\alpha}{2}d\phi(Z)h(X,Y) \\ &\quad + \frac{1-\alpha}{2}\{d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z)\}. \end{aligned} \tag{13}$$

From the definition by Equations (1) and (2), $(N, \nabla^{(\alpha)}, h)$ and $(N, \bar{\nabla}^{(\alpha)}, \bar{h})$ are α-conformally equivalent statistical manifolds. □

It is described in [20,21] that for a Riemannian manifold (M, h) with a symmetric cubic form C satisfying

$$C(X,Y,Z) = h(K_X Y, Z) = h(K_Y Z, X) = h(K_Z X, Y) \tag{14}$$

the triplet (M, h, C) is a statistical manifold. The $(1,2)$-tensor $K = \nabla^{(1)} - \nabla^{LC}$ is called the difference tensor.

Theorem 3 treats the symmetric cubic form C_0 satisfying

$$C_0(X,Y,Z) = d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z) + d\phi(Z)h(X,Y). \tag{15}$$

By Equation (6), the conformality of the Riemannian metrics h and $\bar{h}$ is not known; however, by adding Equation (5), which provides the difference between K and $\bar{K}$ by the symmetric cubic form C_0, we determined that $\bar{h}$ is a conformal metric of h,

As in Equation (3), the function e^{ϕ} refers to the ratio of coordinates of points in two centro-affine hypersurfaces that realize $(N, h, \nabla^{(-1)})$ and $(N, \bar{h}, \bar{\nabla}^{(-1)})$, respectively. In the literature on affine geometry and statistical information geometry, several symmetric cubic forms were treated to compare with two affine connections. However, there seems to be no description on the comparison of two difference tensors by a cubic form containing the function ϕ in [4], and so on. From Theorem 3, we have thus obtained a new perspective on α-geometry and information geometry.

4. Conclusions

In this paper, we presented a method for the construction of α-conformally equivalent statistical manifolds for a given symmetric cubic form on a Riemannian manifold. For this purpose, we explained the definitions and theorems on Hessian domains and the α-conformal equivalence of statistical manifolds. Finally, we compared the induced statistical structure and affine geometrical structure.

One remaining challenge is to develop a general method for the construction of α-connections on a given Riemannian manifold.

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