Entropy for the Quantized Field in the Atom-Field Interaction: Initial Thermal Distribution

Luis Amilca Andrade-Morales †, Braulio M. Villegas-Martínez † and Hector M. Moya-Cessa *

Instituto Nacional de Astrofísica, Óptica y Electrónica, Calle Luis Enrique Erro No. 1,
Tonantzintla 72840, Mexico; amilka_789@hotmail.com (L.A.A.-M.); misalo15@hotmail.com (B.M.V.-M.)

* Correspondence: hmmc@inaoep.mx; Tel.: +52-222-266-3100

† These authors contributed equally to this work.

Abstract: We study the entropy of a quantized field in interaction with a two-level atom (in a pure state) when the field is initially in a mixture of two number states. We then generalise the result for a thermal state; i.e., an (infinite) statistical mixture of number states. We show that for some specific interaction times, the atom passes its purity to the field and therefore the field entropy decreases from its initial value.

Keywords: von Neumann entropy; mixed states; purity

1. Introduction

The quantum state of a system is given by the von Neumann entropy [1], which may be calculated by the expectation value of the entropy operator [2]

$$S = Tr\{\rho \hat{S}\} \tag{1}$$

with $\hat{S} = -\ln \rho$, where ρ is the density matrix describing the quantum mechanical system.

In quantum information processing, it is important to have ways to measure entanglement, entropy being one of the most important; however, negativity and concurrence [3] may also be used to look for the different degrees of entanglement of two given subsystems.

It is relatively simple to calculate entropies associated with interacting systems—for instance, a two-level atom and a quantised field—when both systems initially are prepared in pure states. However, if one of the systems is initially considered in a statistical mixture, the difficulty of calculating such entropies is increased to the point that it may become impossible to obtain analytical results. This is because when two systems, A (for atom) and F (for field) interact, the entropies obey a triangle inequality [4]

$$|S_A - S_F| \leq S_{AF} \leq S_A + S_F, \tag{2}$$

where S_{AF} is the entropy of the total system, and S_A and S_F are the entropies of the atom and field, respectively. An important result from this equation is that, if the total entropy is zero, the entropies for the atom and field will be equal. Therefore, if in a lossless atom–field interaction, one can generate pure initial states for atom (excited, ground, or any pure superposition), and for the field (one can consider coherent states, superposition of coherent states [5], or squeezed states [6], to name some examples), their entropies will be equal after evolution. Therefore, in the atom–field interaction, one is able to find the entropy for the field given by the (simpler to calculate) entropy of the atom.

A question arises: How to compute the field's entropy in the case where one of the subsystems (let us say the quantised field) is not in a pure state? In this case, the above triangle inequality is of no use, and there is not a general answer.

2. Initial Mixed State

The Hamiltonian for a two-level atom interacting with a quantized field is given by [7,8] (we set $\hbar = 1$)

$$H = \omega a^\dagger a + \omega_0 \frac{\sigma_z}{2} + g(a^\dagger \sigma_- + \sigma_+ a) \tag{3}$$

where a and $a^\dagger$ are the annihilation and creation operators, respectively, ω is the frequency of the quantized field, σ_z is the atomic inversion operator, σ_+ and σ_- are the raising and lowering Pauli matrices, respectively, g is the interaction constant between the atom and the quantized field, and ω_0 is the atomic transition frequency. Although we are considering here a two-level atom interacting with a quantised field, these kinds of Hamiltonians may also be realized in ion–laser interactions [9,10].

To be more specific, the Pauli spin operators in the 2×2 matrix representation read

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

They obey the commutation relations $[\sigma_+, \sigma_-] = \sigma_z$ and $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$.

We now consider the atom and field to be in resonance ($\omega = \omega_0$) and get rid off the (free) Hamiltonians for the field and the atom. We need to solve the Schrödinger equation (see [11] for techniques for its solution)

$$i\frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle \tag{5}$$

and by transforming $|\psi\rangle = T|\phi\rangle$, with $T = \exp[-i(\omega a^\dagger a + \omega \frac{\sigma_z}{2})t]$, a Schrödinger equation in the interaction picture is obtained for $|\phi\rangle$ with the interaction Hamiltonian given by

$$H_I = g(a^\dagger \sigma_- + \sigma_+ a). \tag{6}$$

The evolution operator, $\exp(-iH_I t)$, may then be easily obtained as

$$U_I(t) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \tag{7}$$

with the matrix elements given by

$$\begin{aligned} U_{11}(t) &= \cos gt\sqrt{\hat{n}+1}, \\ U_{12}(t) &= -i\sin gt\sqrt{\hat{n}+1}V \\ U_{21}(t) &= -iV^\dagger \sin gt\sqrt{\hat{n}+1} \\ U_{22}(t) &= \cos gt\sqrt{\hat{n}}, \end{aligned} \tag{8}$$

and where $\hat{n} = a^\dagger a$ is the so-called number operator and V is the London (phase) operator [12,13]

$$V = \sum_{m=0}^{\infty} |m\rangle\langle m+1| = \frac{1}{\sqrt{\hat{n}+1}}a, \tag{9}$$

with $|m\rangle$ number or Fock states; i.e., eigenstates of the harmonic oscillator.

In a sense, V and $V^\dagger$ are the true annihilation and creation operators, as they do actually annihilate or create a single excitation of the harmonic oscillator (in our case, a single photon of the quantized electromagnetic field).

Extreme care has to be taken when doing calculations with the London phase operator, as—contrary to the usual annihilation and creation operators that obey the commutation relation $[a, a^\dagger] = 1$—the former operators obey the more complex commutation relation $[V, V^\dagger] = |0\rangle\langle 0|$, where $|0\rangle\langle 0|$ is the so-called vacuum projection operator.

If we consider an initial excited state for the atom, in the 2×2 matrix representation given by

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{10}$$

and a statistical mixture of number states for the field, namely,

$$\rho_F(0) = P_k |k\rangle\langle k| + P_{k+1}|k+1\rangle\langle k+1|, \tag{11}$$

with $P_k + P_{k+1} = 1$, the initial atom–field density matrix is given by

$$\rho(0) = P_k \begin{pmatrix} |k\rangle\langle k| & 0 \\ 0 & 0 \end{pmatrix} + P_{k+1} \begin{pmatrix} |k+1\rangle\langle k+1| & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

the evolved density matrix is given by the solution to the von Neumann equation (as we are dealing now with mixed states, i.e., density matrices)

$$\rho_F(t) = U(t)\rho_F(0)U^\dagger(t), \tag{13}$$

that is explicitly written as

$$\rho(t) = \begin{pmatrix} P_k|c_k\rangle\langle c_k| + P_{k+1}|c_{k+1}\rangle\langle c_{k+1}| & P_k|c_k\rangle\langle s_{k+1}| + P_{k+1}|c_{k+1}\rangle\langle s_{k+2}| \\ P_k|s_{k+1}\rangle\langle c_k| + P_{k+1}|s_{k+2}\rangle\langle c_{k+1}| & P_k|s_{k+1}\rangle\langle s_{k+1}| + P_{k+1}|s_{k+2}\rangle\langle s_{k+2}| \end{pmatrix}, \tag{14}$$

where

$$|c_k\rangle = \cos gt\sqrt{k+1}|k\rangle, \qquad |s_k\rangle = \sin gt\sqrt{k}|k\rangle. \tag{15}$$

By tracing over the atomic basis, we can obtain the field density matrix

$$\rho_F(t) = P_k|c_k\rangle\langle c_k| + P_{k+1}|c_{k+1}\rangle\langle c_{k+1}| + P_k|s_{k+1}\rangle\langle s_{k+1}| + P_{k+1}|s_{k+2}\rangle\langle s_{k+2}|, \tag{16}$$

from where we can calculate the entropy as

$$
\begin{aligned}
S(t) = &-P_k \cos^2 gt\sqrt{k+1}\ln P_k \cos^2 gt\sqrt{k+1} - P_{k+1}\sin^2 gt\sqrt{k+2}\ln P_{k+1}\sin^2 gt\sqrt{k+2} \\
&- \left(P_{k+1}\cos^2 gt\sqrt{k+2} + P_k \sin^2 gt\sqrt{k+1}\right)\ln\left(P_{k+1}\sin^2 gt\sqrt{k+2} + P_k \sin^2 gt\sqrt{k+1}\right).
\end{aligned} \tag{17}
$$

We plot the entropy in Figure 1 where it is shown that for such an initial mixed state for the field, the entropy can take values close to zero about $gt \approx 8.2$, and in this way, the purity of the atom is transferred completely to the field. In Figure 2, we plot $\cos^2 gt\sqrt{k+2}$ and $\sin^2 gt\sqrt{k+1}$ for $k = 5$. When these two terms are equal, it may be seen from Equation (16) that only the third term survives, and the purity of the atom is completely transferred to the field.
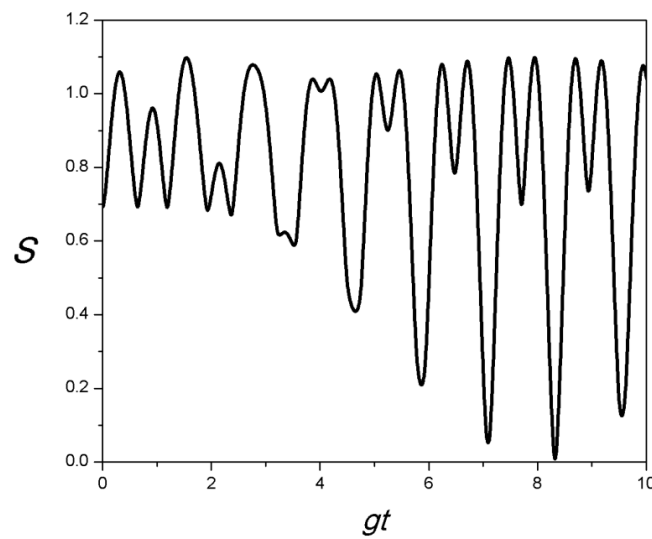
Figure 1. Field entropy for an atom initial excited state and the field in an equally weighted statistical mixture of number sates, with $k = 5$; i.e., $\rho_F(0) = \frac{1}{2}(|5\rangle\langle 5| + |6\rangle\langle 6|)$.
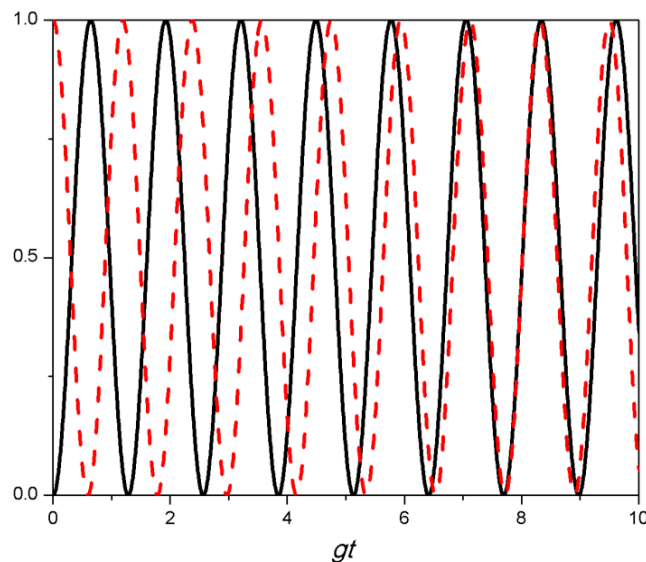

Figure 2. We plot $\cos^2 gt\sqrt{k+2}$ and $\sin^2 gt\sqrt{k+1}$ for $k = 5$. It can be seen that about $gt \approx 8.2$, both terms are close to one, which means that the terms that survive in Equation (15) correspond to the *ket* $|k = 6\rangle$.

3. Thermal Distribution as Initial Field State

From the treatment done in the former Section, we now consider a thermal distribution as initial field,

$$\rho_F(0) = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|, \tag{18}$$

with $P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}$ and $\bar{n}$ the average thermal number of photons. In this case, the evolved field density matrix reads

$$\rho_F(t) = P_0 \cos^2 gt|0\rangle\langle 0| + \sum_{n=0}^{\infty}\left(P_n \sin^2 gt\sqrt{n+1} + P_{n+1}\cos^2 gt\sqrt{n+2}\right)|n+1\rangle\langle n+1|, \tag{19}$$

from where we write down the entropy for the field

$$
\begin{aligned}
S_F(t) \;=\;& -P_0 \cos^2 gt \ln \cos^2 gt \\
& - \sum_{n=0}^{\infty} \left(P_n \sin^2 gt\sqrt{n+1} + P_{n+1} \cos^2 gt\sqrt{n+2} \right) \ln \left(P_n \sin^2 gt\sqrt{n+1} + P_{n+1} \cos^2 gt\sqrt{n+2} \right).
\end{aligned}
\tag{20}
$$

We plot the field entropy in Figure 3, where we can note that a strong decrease of entropy can happen; i.e., a field purification can be realized by interacting a thermal field with a (pure) excited atom.
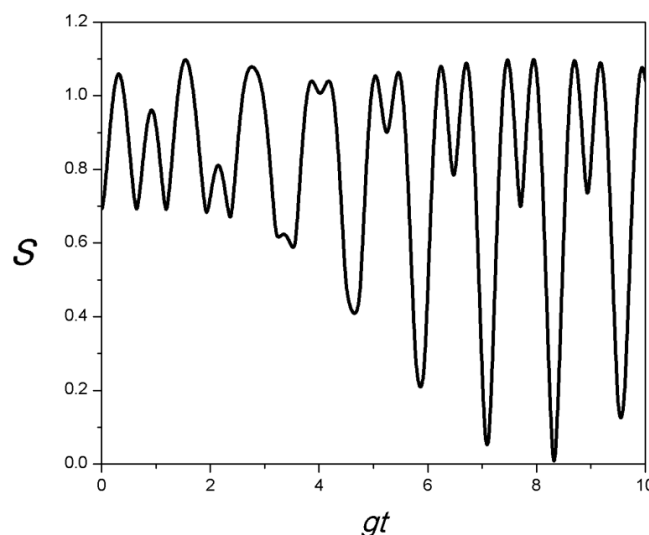

Figure 3. We plot the field entropy as a function of *gt* for the atom initially in its excited state and the field in a thermal distribution with an average number of photons, $\bar{n} = 0.5$.

4. Conclusions

We have shown that it is possible to decrease the entropy of a thermal field by interacting it with a two-level atom initially in its excited state. With the help of an initial field in a simpler mixture (i.e., just two number states), we constructed the density matrix for the extended mixture of number states given by the thermal distribution. In the case of the simpler mixture, it was shown that a complete transfer of purity may take place. A simpler case (because of its periodicity) —that of a two-photon interaction [14]—has already been studied. We studied the ideal case of light–matter interaction (i.e., where no interaction of the total system with an environment takes place; see for instance [15]); however, because we are considering initial states with only diagonal terms in the field density matrix, it is expected that for sufficiently small cavity decay, our study describes the interaction well, as the environment severely affects off-diagonal matrix elements, quickly destroying coherent effects. Finally, effects such as the decrease of entropy of a subsystem by swapping purity could also be realized in ion–laser interaction [9,10] and semiconductor microcavities [16], to name some other possibilities.

Author Contributions: Luis Amilca Andrade-Morales and Braulio M. Villegas-Martínez conceived the idea and developed it under Hector M. Moya-Cessa supervision. The manuscript was written by all authors, who have read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Von Neumann, J. *Mathematical Foundations of Quantum Mechanics*; Princeton University Press: Princeton, NJ, USA, 1955.
2. Moya-Cessa, H. Entropy operator and associated Wigner function. *Int. J. Quant. Inf.* **2007**, *5*, 149–155.
3. Berrada, K.; Ei Bas, M.; Eleuch, H.; Hassouni, Y. A comparative study of negativity and concurrence based on spin coherent states. *Int. J. Mod. Phys. C* **2010**, *21*, 291–305.
4. Araki, H.; Lieb, E. Entropy inequalities. *Commun. Math. Phys.* **1970**, *18*, 160–170.
5. Vidiella-Barranco, A.; Moya-Cessa, H.; Buzek, V. Interaction of superpositions of coherent states of light with two-level atoms. *J. Mod. Opt.* **1992**, *39*, 1441–1459.
6. Moya-Cessa, H.; Vidiella-Barranco, A. Interaction of squeezed states of light with two-level atoms. *J. Mod. Opt.* **1992**, *39*, 2481–2499.
7. Jaynes, E.T.; Cummings, F.W. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proc. IEEE* **1963**, *51*, 89–109.
8. Shore, B.W.; Knight, P.L. The Jaynes-Cummings model. *J. Mod. Opt.* **1993**, *40*, 1195–1238.
9. Moya-Cessa, H.; Tombesi, P. Filtering number states of the vibrational motion of an ion. *Phys. Rev. A* **2000**, *61*, doi:10.1103/PhysRevA.61.025401.
10. Moya-Cessa, H.; Soto-Eguibar, F.; Vargas-Martinez, J.M.; Juarez-Amaro, R.; Zuñiga-Segundo, A. Ion-laser interactions: The most complete solution. *Phys. Rep.* **2012**, *513*, 229–261.
11. Eleuch, H.; Rostovtsev, Y.V.; Scully, M.O. New analytic solution of Schrödinger's equation. *Europhys. Lett.* **2010**, *89*, doi:10.1209/0295-5075/89/50004 .
12. London, F. Winkelvariable und kanonische Transformationen in der Undulationsmechanik. *Z. Phys.* **1927**, *40*, 193–210.
13. Susskind, L.; Glogower, J. Quantum mechanical phase and time operator. *Physics* **1964**, *1*, 49–61.
14. Moya-Cessa, H.; Knight, P.L.; Rosenhouse-Dantsker, A. Photon amplification in a two-photon lossless micromaser. *Phys. Rev. A* **1994**, *50*, 1814–1821.
15. Eleuch, H.; Guérin, S.; Jauslin, H.R. Effects of an environment on a cavity-quantum-electrodynamics system controlled by bichromatic adiabatic passage. *Phys. Rev. A* **2012**, *85*, doi:10.1103/PhysRevA.85.013830.
16. Giacobino, E.; Karr, J.-P.; Messin, G.; Eleuch, H.; Baas, A. Quantum optical effects in semiconductor microcavities. *C. R. Phys.* **2002**, *3*, 41–52.