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*Article*

# Applications of the Fuzzy Sumudu Transform for the Solution of First Order Fuzzy Differential Equations

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**Abstract:** In this paper, we study the classical Sumudu transform in fuzzy environment, referred to as the fuzzy Sumudu transform (FST). We also propose some results on the properties of the FST, such as linearity, preserving, fuzzy derivative, shifting and convolution theorem. In order to show the capability of the FST, we provide a detailed procedure to solve fuzzy differential equations (FDEs). A numerical example is provided to illustrate the usage of the FST.

**Keywords:** fuzzy number; fuzzy Sumudu transform; generalized differentiability; fuzzy differential equation; Sumudu transform; integral transform

## 1. Introduction

Integral transforms constitute fundamental tools in operational calculus. They are mathematical operators that have been used widely in solving many practical problems in applied mathematics, physics and engineering [1–4]. The precursor of the integral transforms is the Fourier transform, which is used to express functions in a finite interval. Thenceforward, there are a number of works on the theories and applications of integral transforms, some of which are Laplace, Mellin and Hankel transforms [5–7]. Subsequently, the concept of integral transforms was expanded to remove the necessity of finite intervals. Watugala [8,9] has proposed a new integral transform called the Sumudu transform. The Sumudu transform has been used to solve ordinary differential equations in control engineering problems.

Weerakon [10,11] has extended the Sumudu transform on partial differential equations. The work was then continued by Asiru [12], who studied the convolution theorem of the Sumudu transform, which can be expressed in terms of polynomial and convergent infinite series. Next, Belgacem *et al.* [13] have emphasized the Laplace–Sumudu duality, which is a vital step in establishing new results on the Sumudu transform. For instance, the duality property has been used to invoke a complex inverse of the Sumudu transform, as a Bromwich contour integral formula [14]. Furthermore, in [15], the applications of the Sumudu transform on Bessel functions and equations have been explored. The theories and applications of the Sumudu transform have been examined and explored by many authors (see in [16–21]).

In general, the Sumudu transform is considered a popular integral transform for solving differential equations. This is due to its unity property, which eases the process of finding solutions. It is also more powerful compared to other integral transforms, as the function transformed is a similitude of the resulting function.

Many real-world problems are modeled by differential equations. However, we cannot be sure that the models is perfect. For example, the initial value of the models might not be known accurately. The initial value may contain some uncertainty quantities, such as "less than $x_0$", "about $x_0$" or "more than $x_0$". If this is the case, the classical differential equations cannot be used to handle this situation. Therefore, it is necessary to study other theories in order to overcome this problem. One of the most popular theories for describing this situation is the fuzzy set theory [22]. By incorporating fuzziness into classical mathematics, many authors studied fuzzy derivatives [23–28], fuzzy differential equations (FDEs) [29–31] and fuzzy fractional differential equations (FFDEs) [32–34].

Recently, Allahviranloo and Ahmadi [35] have proposed the fuzzy Laplace transform (FLT) and showed its applications to solve FDEs. The FLT is then used to solve second order linear FDEs and the state-space description of fuzzy linear continuous time systems [36]. This work has motivated many authors to expand the theories and applications of FLT in the mathematics and engineering fields [37–41]. The work has also motivated a few researchers to study the classical Sumudu transform in the fuzzy setting. The first effort was initiated by Ahmad and Abdul Rahman [42] and further studied by Alam Khan *et al.* [43]. In this paper, we add some new results on the Sumudu transform in the fuzzy setting, especially on the linearity and preserving properties. Some other results may parallel the ones proposed in [43]. However, our definition of the Sumudu transform in the fuzzy setting is quite general, and the results are presented in different ways.

This paper is organized as follows. In Section 2, we recall several basic definitions and concepts of fuzzy numbers. In Section 3, we provide a general definition of the fuzzy Sumudu transform (FST) and investigate the duality property between FST and FLT. We also provide some theorems and properties regarding the FST. In Section 4, we construct detailed procedures to solve FDEs. Later, in Section 6, we give conclusions.

## 2. Preliminaries

In this section, we recall some definitions and theorems needed in order to understand the contribution in this paper. The definition of a fuzzy number is as follows.

**Definition 1.** [44] *By $\mathbb{R}$, we denote the set of all real numbers. A fuzzy number is a mapping $U : \mathbb{R} \to [0, 1]$ with the following properties:*

*(1) $U$ is upper semi-continuous,*
*(2) $U$ is fuzzy convex, i.e., $U(\lambda x + (1 - \lambda)y) \geq min\{U(x), U(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$,*
*(3) $U$ is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $U(x_0) = 1$,*
*(4) supp $U = \{x \in \mathbb{R} | U(x) > 0\}$ is the support of the $U$, and its closure, i.e. cl(supp $U$) is compact.*

The following definition is the $\alpha$-level set of fuzzy numbers.

**Definition 2.** [44] *Let $\mathcal{F}(\mathbb{R})$ be the set of all fuzzy numbers on $\mathbb{R}$. The $\alpha$-level set of a fuzzy number $U \in \mathcal{F}(\mathbb{R})$, $\alpha \in [0, 1]$, denoted by $U_\alpha$, is defined as:*

$$U_\alpha = \begin{cases} \{x \in \mathbb{R} | U(x) \geq \alpha\}, & \textit{if } 0 \leq \alpha \leq 1, \\ \textit{cl(supp } U\textit{)}, & \textit{if } \alpha = 0. \end{cases}$$

It is clear that the $\alpha$-level set of a fuzzy number is a closed and bounded interval, *i.e.*, $[\underline{u}_\alpha, \overline{u}_\alpha]$, where $\underline{u}_\alpha$ and $\overline{u}_\alpha$ denote the lower bound and the upper bound of $U$, respectively. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\widetilde{y}$ defined by:

$$\widetilde{y}(t) = \begin{cases} 1, & \text{if } t = y, \\ 0, & \text{if } t \neq y, \end{cases}$$

$\mathbb{R}$ can be embedded in $\mathcal{F}(\mathbb{R})$.

**Remark.** [45] *Let $X$ be the Cartesian product of universes $X = X_1 \times \ldots \times X_n$ and $A_1, \ldots, A_n$ be $n$ fuzzy numbers in $X_1, \ldots, X_n$, respectively. A fuzzy function $f$ maps from $X$ to a universe $Y$, $y = f(x_1, \ldots, x_n)$. Then, the extension principle allows us to define a fuzzy set $B$ in $Y$ by:*

$$B = \{(y, U_B(y)) | y = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X\},$$

*where:*

$$U_B(y) = \begin{cases} \sup_{(x_1,\ldots,x_n) \in f^{-1}(y)}, & \\ \min\{U_{A_1}(x_1), \ldots, U_{A_n}(x_n)\}, & \textit{if } f^{-1} \neq 0, \\ 0, & \textit{if otherwise,} \end{cases}$$

*where $f^{-1}$ is the inverse of $f$.*

*For $n = 1$, the extension principle is reduced to:*

$$B = \{(y, U_B(y)) | y = f(x), x \in X\},$$

*where:*

$$U_B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} U_A(x), & \textit{if } f^{-1} \neq 0, \\ 0, & \textit{if otherwise.} \end{cases}$$

Referring to the extension principle, addition on $\mathcal{F}(\mathbb{R})$ can be defined by:

$$(U \oplus V)(x) = \sup_{y \in \mathbb{R}} \min\{U(y), V(x - y)\}, x \in \mathbb{R},$$

and scalar multiplication is defined by:

$$(k \odot U)(x) = \begin{cases} U(x/k) & k > 0, \\ \widetilde{0}, & k = 0, \end{cases}$$

where $\widetilde{0} \in \mathcal{F}(\mathbb{R})$.

Furthermore, for all $\alpha$-levels,

$$[U_\alpha \oplus V_\alpha] = U_\alpha + V_\alpha,$$

and:

$$[k \odot U]_\alpha = kU_\alpha,$$

is true.

In this paper, the notation $U_\alpha$ represents the $\alpha$-level set of a fuzzy number.

We may conclude that the fuzzy number is determined by the endpoints of the intervals $U_\alpha$. This leads to other representation of a fuzzy number, which will be defined by two endpoint functions $\underline{u}_\alpha$ and $\overline{u}_\alpha$. Friedman *et al.* [46] and Ma *et al.* [47] defined the representation as:

**Definition 3.** *A fuzzy number $U$ in parametric form is a pair $[\underline{u}_\alpha, \overline{u}_\alpha]$ of functions $\underline{u}_\alpha$ and $\overline{u}_\alpha$, $\alpha \in [0, 1]$, which satisfy the following requirements:*

*(1) $\underline{u}_\alpha$ is a bounded non-decreasing left continuous function in $(0, 1]$ and right continuous at zero,*
*(2) $\overline{u}_\alpha$ is a bounded non-increasing left continuous function in $(0, 1]$ and right continuous at zero,*
*(3) $\underline{u}_\alpha \leq \overline{u}_\alpha$.*

A fuzzy number can be represented as a fuzzy membership function. One of the most commonly-used fuzzy membership functions in the literature is the fuzzy triangular membership function. It is defined as follows.

**Definition 4.** *Let $U \in \mathcal{F}(\mathbb{R})$. $U$ is called a triangular fuzzy number if its membership function has the following form:*

$$U(x) = \begin{cases} 0, & \text{if } x < a, \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x < b, \\ \dfrac{c-x}{c-b}, & \text{if } b \leq x \leq c, \\ 0, & \text{if } x > c, \end{cases}$$

*and its $\alpha$-level is simply $U_\alpha = [a + (b - c)\alpha, c - (c - b)\alpha]$, for any $\alpha \in [0, 1]$.*

**Definition 5.** [46] *For arbitrary $U_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $V_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ and $k > 0$, we define addition, subtraction and multiplication by $k$ for $U_\alpha$ and $V_\alpha$ as:*

*(1) addition,*

$$U_\alpha \oplus V_\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha],$$

(2) *subtraction,*

$$U_\alpha \ominus V_\alpha = [\underline{u}_\alpha - \overline{v}_\alpha, \overline{u}_\alpha - \underline{v}_\alpha],$$

(3) *scalar multiplication,*

$$k \odot U_\alpha = \begin{cases} [k\underline{u}_\alpha, k\overline{u}_\alpha], & k \geq 0, \\ [k\overline{u}_\alpha, k\underline{u}_\alpha], & k < 0. \end{cases}$$

If $k = -1$, then $k \odot U_\alpha = -U_\alpha$.

**Definition 6.** [48] *The distance $D(U,V)$ between two fuzzy intervals $U$ and $V$ is defined as:*

$$D(U,V) = \sup_{\alpha \in [0,1]} d_H(U_\alpha, V_\alpha),$$

*where:*

$$d_H(U_\alpha, V_\alpha) = \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}$$

*is the Hausdorff distance between $U_\alpha$ and $V_\alpha$.*

Thus, we can conclude that $D$ is a metric space and has the following properties:

(1) $D(U \oplus W, V \oplus W) = D(U,V), \forall U,V,W \in \mathcal{F}(\mathbb{R})$,
(2) $D(k \odot U, k \odot V) = |k|D(U,V), \forall k \in \mathbb{R}, U,V \in \mathcal{F}(\mathbb{R})$,
(3) $D(U \oplus V, W \oplus E) \leq D(U,W) + D(V,E), \forall U,V,W,E \in \mathcal{F}(\mathbb{R})$,
(4) $(D, \mathcal{F}(\mathbb{R}))$ is a complete metric space.

**Definition 7.** [49] *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$. The function $f$ is called continuous if for every $x_0 \in \mathbb{R}$ and every $\epsilon > 0$, there exists $\delta > 0$, such that if $|x - x_0| < \delta$, then $D(f(x), f(x_0)) < \epsilon$.*

**Theorem 1.** [50] *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, and it is represented by $[\underline{f}_\alpha(x), \overline{f}_\alpha(x)]$. For any fixed $\alpha \in [0,1]$, assume that $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume that there are two positive $\underline{M}_\alpha$ and $\overline{M}_\alpha$, such that $\int_a^b |\underline{f}_\alpha(x)|dx \leq \underline{M}_\alpha$ and $\int_a^b |\overline{f}_\alpha(x)|dx \leq \overline{M}_\alpha$ for every $b \geq a$. Then, $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$, and the improper fuzzy Riemann-integrable is a fuzzy number. Furthermore, we have:*

$$\int_a^\infty f(x)dx = \left[\int_a^\infty \underline{f}_\alpha(x)dx, \int_a^\infty \overline{f}_\alpha(x)dx\right].$$

**Proposition 1.** [51] *If each of $f(x)$ and $g(x)$ is a fuzzy-valued function and fuzzy Riemann-integrable on $[a, \infty)$, then $f(x) \oplus g(x)$ is fuzzy Riemann-integrable on $[a, \infty)$. Moreover, we have:*

$$\int_I (f(x) \oplus g(x))dx = \int_I f(x)dx \oplus \int_I g(x)dx.$$

The next definition is Hukuhara's differentiability, also known as H-derivatives. The definition is about H-differences of sets, and it is introduced as follows.

**Definition 8.** [27] *Let $x, y \in \mathcal{F}(\mathbb{R})$. If there exists $z \in \mathcal{F}(\mathbb{R})$, such that $x = y \oplus z$, then $z$ is called the H-difference of $x$ and $y$, and it is denoted by $x -^H y$.*

**Definition 9.** [52,53] *Let $f : (a, b) \to \mathcal{F}(\mathbb{R})$ and $x_0 \in (a, b)$. We say that $f$ is strongly generalized differentiable at $x_0$, if there exists an element $f'(x_0) \in \mathcal{F}(\mathbb{R})$, such that:*

*(1) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^H f(x_0)$, $f(x_0) -^H f(x_0 - h)$ and the limits (in the metric $D$):*

$$\lim_{h \to 0} \frac{f(x_0 + h) -^H f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) -^H f(x_0 - h)}{h} = f'(x_0),$$

*or*

*(2) for all $h > 0$ sufficiently small, there exist $f(x_0) -^H f(x_0 + h)$, $f(x_0 - h) -^H f(x_0)$ and the limits (in the metric $D$):*

$$\lim_{h \to 0} \frac{f(x_0) -^H f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) -^H f(x_0)}{-h} = f'(x_0).$$

*($h$ and $-h$ in the denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively.)*

**Note.** *In this paper, we only consider Cases (1) and (2) in the strongly generalized differentiability proposed by Bede and Gal* [52]. *Chalco-Cano and Roman-Florés* [54] *stated that Cases (1) and (2) are more important, since Cases (3) and (4) occur only on a discrete set of points.*

**Theorem 2.** [54] *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and denote $f(x) = [\underline{f}_\alpha(x), \overline{f}_\alpha(x)]$, for each $\alpha \in [0, 1]$. Then:*

*(1) if $f$ is (1)-differentiable, then $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f'(x) = [\underline{f}'_\alpha(x), \overline{f}'_\alpha(x)]$,*
*(2) if $f$ is (2)-differentiable, then $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f'(x) = [\overline{f}'_\alpha(x), \underline{f}'_\alpha(x)]$.*

## 3. Fuzzy Sumudu Transform

In order to establish results, some definitions are needed. $G(u)$ and $\mathcal{S}[f(x)]$ will be used as the notations for the fuzzy Sumudu transform throughout this paper.

**Definition 10.** *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function. Suppose that $f(ux) \odot e^{-x}$ is improper fuzzy Riemann-integrable on $[0, \infty)$, then $\int_0^\infty f(ux) \odot e^{-x} dx$ is called the fuzzy Sumudu transform and is denoted by:*

$$G(u) = \mathcal{S}[f(x)] = \int_0^\infty f(ux) \odot e^{-x} dx, \qquad (u \in [-\tau_1, \tau_2]), \tag{1}$$

*where the variable $u$ is used to factor the variable $x$ in the argument of the fuzzy-valued function.*

From Theorem 1, we obtain:

$$\int_0^\infty f(ux) \odot e^{-x} dx = \left[ \int_0^\infty \underline{f}_\alpha(ux) e^{-x} dx, \int_0^\infty \overline{f}_\alpha(ux) e^{-x} dx \right].$$

From the classical Sumudu transform, we have:

$$s[\underline{f}_\alpha(x)] = \int_0^\infty \underline{f}_\alpha(ux) e^{-x} dx,$$

and:

$$s[\overline{f}_\alpha(x)] = \int_0^\infty \overline{f}_\alpha(ux) e^{-x} dx.$$

Finally, we have:

$$\mathcal{S}[f(x)] = [s[\underline{f}_\alpha(x)], s[\overline{f}_\alpha(x)]].$$

### *3.1. Duality Properties of the Fuzzy Laplace and Fuzzy Sumudu Transform*

FLT has a close relationship with FST. It is necessary for us to be able to link between the two transforms in order to prove theorems and properties of the FST. The definition for FLT is given as follows.

**Definition 11.** [35] *Let $f(x)$ be a continuous fuzzy-valued function. Suppose that $f(x) \odot e^{-px}$ is improper fuzzy Riemann-integrable on $[0, \infty)$, then $\int_0^\infty f(x) \odot e^{-px} dx$ is called the fuzzy Laplace transform and is denoted by:*

$$F(p) = \mathcal{L}[f(x)] = \int_0^\infty f(x) \odot e^{-px} dx, \qquad (p > 0). \tag{2}$$

**Theorem 3.** *Let $f(x)$ be a continuous fuzzy-valued function. If $F$ is the fuzzy Laplace transform of $f(x)$ and $G$ is the fuzzy Sumudu transform of $f(x)$, then:*

$$G(u) = \frac{F(1/u)}{u}. \tag{3}$$

**Proof.** Let $f(x) \in \mathcal{F}(\mathbb{R})$, then for $-\tau_1 < u < \tau_2$,

$$G(u) = \left[ \int_0^\infty \underline{f}_\alpha(ux) e^{-x} dx, \int_0^\infty \overline{f}_\alpha(ux) e^{-x} dx \right].$$

Substituting $w = ux$ or $x = \frac{w}{u}$, then we have:

$$\begin{aligned} G(u) &= \left[ \int_0^\infty \underline{f}_\alpha(w) e^{-w/u} \frac{dw}{u}, \int_0^\infty \overline{f}_\alpha(w) e^{-w/u} \frac{dw}{u} \right], \\ &= \frac{1}{u} \left[ \int_0^\infty \underline{f}_\alpha(w) e^{-w/u} dw, \int_0^\infty \overline{f}_\alpha(w) e^{-w/u} dw \right], \\ &= \frac{1}{u} \int_0^\infty f(w) \odot e^{-w/u} dw. \end{aligned}$$

It is clear that the $\frac{1}{u} \int_0^\infty f(w) \odot e^{-w/u} dw$ is $\frac{F(1/u)}{u}$. $\square$

In the following corollary, we show that the roles of $F$ and $G$ in Equation (3) can be interchanged.

**Corollary 1.** *Let $f(x) \in \mathcal{F}(\mathbb{R})$, having F and G for the fuzzy Laplace transform and fuzzy Sumudu transform, respectively. Then:*

$$F(p) = \frac{G(1/p)}{p}. \tag{4}$$

**Proof.** The proof of Equation (4) can be obtained by changing $u$ to $\frac{1}{p}$ in Equation (3). □

Equations (3) and (4) form the fuzzy Laplace–Sumudu duality and serve as a mean of changing between those two transforms when needed.

### *3.2. Fundamental Theorems and Properties of the Fuzzy Sumudu Transform*

In this section, we provide some theorems and properties associated with FST. Please note that the theorems and properties proposed in this section are extension of the classical Sumudu transform, as studied in [13,14].

**Theorem 4.** *Let $f, g : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be two continuous fuzzy-valued functions. Suppose that $c_1$ and $c_2$ are arbitrary constants, then:*

$$\mathcal{S}[(c_1 \odot f(x)) \oplus (c_2 \odot g\,(x))] = (c_1 \odot \mathcal{S}[f\,(x)]) \oplus (c_2 \odot \mathcal{S}[g\,(x)])\,.$$

**Proof.** Assume that $f(x) = [\underline{f}_\alpha(x), \overline{f}_\alpha(x)]$ and $g(x) = [\underline{g}_\alpha(x), \overline{g}_\alpha(x)]$. First, we proof for the lower bound of $f(x)$ and $g(x)$.

$$\begin{aligned} s[(c_1\underline{f}_\alpha(x)) + (c_2\underline{g}_\alpha(x))] &= \int_0^\infty ((c_1\underline{f}_\alpha(ux)) + (c_2\underline{g}_\alpha(ux)))e^{-x}dx, \\ &= \int_0^\infty (c_1\underline{f}_\alpha(ux))e^{-x}dx + \int_0^\infty (c_2\underline{g}_\alpha(ux))e^{-x}dx, \\ &= c_1\int_0^\infty \underline{f}_\alpha(ux)e^{-x}dx + c_2\int_0^\infty \underline{g}_\alpha(ux)e^{-x}dx, \\ &= c_1 s[\underline{f}_\alpha(x)] + c_2 s[\underline{g}_\alpha(x)]. \end{aligned}$$

Secondly, we proof for the upper bound of $f(x)$ and $g(x)$.

$$\begin{aligned} s[(c_1\overline{f}_\alpha(x)) + (c_2\overline{g}_\alpha(x))] &= \int_0^\infty ((c_1\overline{f}_\alpha(ux)) + (c_2\overline{g}_\alpha(ux)))e^{-x}dx, \\ &= \int_0^\infty (c_1\overline{f}_\alpha(ux))e^{-x}dx + \int_0^\infty (c_2\overline{g}_\alpha(ux))e^{-x}dx, \\ &= c_1\int_0^\infty \overline{f}_\alpha(ux)e^{-x}dx + c_2\int_0^\infty \overline{g}_\alpha(ux)e^{-x}dx, \\ &= c_1 s[\overline{f}_\alpha(x)] + c_2 s[\overline{g}_\alpha(x)]. \end{aligned}$$

Finally, we conclude that:

$$\mathcal{S}[(c_1 \odot f(x)) \oplus (c_2 \odot g\,(x))] = (c_1 \odot \mathcal{S}[f\,(x)]) \oplus (c_2 \odot \mathcal{S}[g\,(x)])\,.$$

The proof is complete. □

In the following theorem, we provide the first preserving theorem.

**Theorem 5.** *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and $a$ an arbitrary constant, then:*

$$\mathcal{S}[f(ax)] = G(au).$$

**Proof.** From Definition 10,

$$\begin{aligned}\mathcal{S}[f(ax)] &= [s[\underline{f}_\alpha(ax)], s[\overline{f}_\alpha(ax)]],\\ &= \left[\int_0^\infty \underline{f}_\alpha(aux)e^{-x}dx, \int_0^\infty \overline{f}_\alpha(aux)e^{-x}dx\right],\\ &= G(au).\end{aligned}$$

The theorem is proven to be true. □

Next, we provide the second preserving theorem.

**Theorem 6.** *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function, then:*

$$\mathcal{S}\left[x \odot \frac{df(x)}{dx}\right] = u\frac{dG(u)}{du}.$$

**Proof.** From the definition of FST, $G(u) = \int_0^\infty f(ux) \odot e^{-x}dx$. Then, for Case 1 in Theorem 2,

$$\begin{aligned}\frac{dG(u)}{du} &= \frac{d}{du}\left[\int_0^\infty \underline{f}_\alpha(ux)e^{-x}dx, \int_0^\infty \overline{f}_\alpha(ux)e^{-x}dx\right],\\ &= \left[\int_0^\infty \frac{d}{du}\underline{f}_\alpha(ux)e^{-x}dx, \int_0^\infty \frac{d}{du}\overline{f}_\alpha(ux)e^{-x}dx\right],\\ &= \left[\int_0^\infty xe^{-x}\frac{d\underline{f}_\alpha(ux)}{dx}dx, \int_0^\infty xe^{-x}\frac{d\overline{f}_\alpha(ux)}{dx}dx\right].\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dG(u)}{du} &= \frac{1}{u}\left[\int_0^\infty (ux)\underline{f}'_\alpha(ux)e^{-x}dx, \int_0^\infty (ux)\overline{f}'_\alpha(ux)e^{-x}dx\right],\\ &= \frac{1}{u}\mathcal{S}\left[x \odot \frac{df(x)}{dx}\right].\end{aligned}$$

Multiplying both sides with $u$, we obtain:

$$\mathcal{S}\left[x \odot \frac{df(x)}{d(x)}\right] = u\frac{dG(u)}{du}.$$

For Case 2 from Theorem 2,

$$\begin{aligned}\frac{dG(u)}{du} &= \frac{d}{du}\left[\int_0^\infty \overline{f}_\alpha(ux)e^{-x}dx, \int_0^\infty \underline{f}_\alpha(ux)e^{-x}dx\right],\\ &= \left[\int_0^\infty \frac{d}{du}\overline{f}_\alpha(ux)e^{-x}dx, \int_0^\infty \frac{d}{du}\underline{f}_\alpha(ux)e^{-x}dx\right],\\ &= \left[\int_0^\infty xe^{-x}\frac{d\overline{f}_\alpha(ux)}{dx}dx, \int_0^\infty xe^{-x}\frac{d\underline{f}_\alpha(ux)}{dx}dx\right].\end{aligned}$$

Hence,

$$\frac{dG(u)}{du} = \frac{1}{u}\left[\int_0^\infty (ux)\overline{f}'_\alpha(ux)e^{-x}dx, \int_0^\infty (ux)\underline{f}'_\alpha(ux)e^{-x}dx\right],$$
$$= \frac{1}{u}\mathcal{S}\left[x \odot \frac{df(x)}{dx}\right].$$

Multiplying both sides with $u$, we obtain:

$$\mathcal{S}\left[x \odot \frac{df(x)}{d(x)}\right] = u\frac{dG(u)}{du}.$$

Therefore, we can conclude that:

$$\mathcal{S}\left[x \odot \frac{df(x)}{d(x)}\right] = u\frac{dG(u)}{du}$$

for both cases from Theorem 2. □

Next, we provide the theorem for the first degree derivative.

**Theorem 7.** *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and $f$ the primitive of $f'$ on $[0, \infty)$. Then:*

$$\mathcal{S}[f'(x)] = \frac{G(u) -^H f(0)}{u} \quad \textit{where f is (1)-differentiable},$$

*or:*

$$\mathcal{S}[f'(x)] = \frac{(-f(0)) -^H (-G(u))}{u} \quad \textit{where f is (2)-differentiable}.$$

**Proof.** First, we assume $f$ is (1)-differentiable. Therefore,

$$\frac{G(u) -^H f(0)}{u} = \left[\frac{s[\underline{f}_\alpha(x)] - \underline{f}_\alpha(0)}{u}, \frac{s[\overline{f}_\alpha(x)] - \overline{f}_\alpha(0)}{u}\right].$$

Since:

$$s[\underline{f}'_\alpha(x)] = \frac{s[\underline{f}_\alpha(x)] - \underline{f}_\alpha(0)}{u},$$

and:

$$s[\overline{f}'_\alpha(x)] = \frac{s[\overline{f}_\alpha(x)] - \overline{f}_\alpha(0)}{u},$$

then:

$$\frac{G(u) -^H f(0)}{u} = [s[\underline{f}'_\alpha(x)], s[\overline{f}'_\alpha(x)]].$$

Since $f$ is (1)-differentiable,

$$\frac{G(u) -^H f(0)}{u} = \mathcal{S}[f'(x)].$$

Now, we assume that $f$ is (2)-differentiable. Therefore,

$$\frac{(-f(0)) -^H (-G(u))}{u} = \left[\frac{-(\overline{f}_\alpha(0)) - (-s[\overline{f}_\alpha(x)])}{u}, \frac{-(\underline{f}_\alpha(0)) - (-s[\underline{f}_\alpha(x)])}{u}\right],$$

equivalent to:

$$\frac{(-f(0)) -^H (-G(u))}{u} = \left[\frac{s[\overline{f}_\alpha(x)] - \overline{f}_\alpha(0)}{u}, \frac{s[\underline{f}_\alpha(x)] - \underline{f}_\alpha(0)}{u}\right].$$

Since:

$$s[\overline{f}'_\alpha(x)] = \frac{s[\overline{f}_\alpha(x)] - \overline{f}_\alpha(0)}{u},$$

and:

$$s[\underline{f}'_\alpha(x)] = \frac{s[\underline{f}_\alpha(x)] - \underline{f}_\alpha(0)}{u},$$

then:

$$\frac{(-f(0)) -^H (-G(u))}{u} = [s[\overline{f}'_\alpha(x)], s[\underline{f}'_\alpha(x)]].$$

Since $f$ is (2)-differentiable, it follows that:

$$\frac{(-f(0)) -^H (-G(u))}{u} = \mathcal{S}[f'(x)].$$

The proof is now complete. □

In the next theorem, we provide the first shifting theorem.

**Theorem 8.** *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and a an arbitrary constant, then:*

$$\mathcal{S}[e^{ax} \odot f(x)] = \frac{1}{1-au} G\left(\frac{u}{1-au}\right).$$

**Proof.** From Definition 10,

$$\mathcal{S}[e^{ax} \odot f(x)] = \left[\int_0^\infty \underline{f}_\alpha(ux)e^{-(1-au)x}dx, \int_0^\infty \overline{f}_\alpha(ux)e^{-(1-au)x}dx\right].$$

By using substitution $w = (1-au)x$, we then obtain:

$$\begin{aligned}
\mathcal{S}[e^{ax} \odot f(x)] &= \left[\frac{1}{1-au}\int_0^\infty \underline{f}_\alpha\left(\frac{uw}{1-au}\right)e^{-w}dw, \frac{1}{1-au}\int_0^\infty \overline{f}_\alpha\left(\frac{uw}{1-au}\right)e^{-w}dw\right], \\
&= \frac{1}{1-au}\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}dw, \\
&= \frac{1}{1-au} G\left(\frac{u}{1-au}\right).
\end{aligned}$$

□

The convolution theorem is provided below.

**Theorem 9.** *Let $f, g : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be two continuous fuzzy-valued functions. Let $F(p)$ and $G(p)$ be fuzzy Laplace transforms, and let $M(u)$ and $N(u)$ be fuzzy Sumudu transforms for $f$ and $g$, respectively. Then, the Sumudu transform of the convolution of $f$ and $g$,*

$$(f * g)(x) = \int_0^t f(\tau)g(t-\tau)d\tau,$$

*is given by:*

$$\mathcal{S}[(f * g)(x)] = uM(u)N(u).$$

**Proof.** The FLT for $(f * g)$ as in [55] is given by:

$$\mathcal{L}[(\underline{f}_\alpha * \underline{g}_\alpha)(x), (\overline{f}_\alpha * \overline{g}_\alpha)(x)] = F(p)G(p).$$

By the fuzzy Laplace–Sumudu duality relation,

$$\mathcal{S}[(\underline{f}_\alpha * \underline{g}_\alpha)(x), (\overline{f}_\alpha * \overline{g}_\alpha)(x)] = \frac{1}{u}\mathcal{L}[(\underline{f}_\alpha * \underline{g}_\alpha)(x), (\overline{f}_\alpha * \overline{g}_\alpha)(x)],$$

and since:

$$M(u) = \frac{F(1/u)}{u}, \qquad N(u) = \frac{G(1/u)}{u},$$

the FST for $(f * g)$ is as follows:

$$\begin{aligned}\mathcal{S}[(\underline{f}_\alpha * \underline{g}_\alpha)(x), (\overline{f}_\alpha * \overline{g}_\alpha)(x)] &= \frac{F(1/u)G(1/u)}{u},\\ &= u\frac{F(1/u)}{u}\frac{G(1/u)}{u},\\ &= uM(u)N(u).\end{aligned}$$

□

## 4. Procedure for Solving Fuzzy Differential Equations

We consider a crisp differential equation given by:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{5}$$

where $f : [t_0, T] \times \mathbb{R} \to \mathbb{R}$. Suppose that the initial value in Equation (5) is not precisely known and modeled with a fuzzy number, we have the following fuzzy initial value problem [56]:

$$\begin{cases} Y'(t) = f(t, Y(t)), & 0 \le t \le T, \\ Y(t_0) = [\underline{y}_\alpha(0), \overline{y}_\alpha(0)], & 0 < \alpha \leqslant 1, \end{cases} \tag{6}$$

where $f : [t_0, T] \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ is a continuous fuzzy mapping. By referring to Kaleva [57], we observe that Theorem 2 provides a procedure to solve the Equation (6). As a matter of fact, $Y_\alpha(t) = [\underline{y}_\alpha(t), \overline{y}_\alpha(t)]$.

By using FST on Equation (6), we have the following equation.

$$\mathcal{S}[Y'(t)] = \mathcal{S}[f(t, Y(t))]. \tag{7}$$

Case 1: If we consider $Y'(t)$ by using a (1)-differentiable function, then from Theorem 2, we get $Y'(t) = [\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)]$. Now, we obtain the following FDE to be solved.

$$\begin{cases} \underline{y}'_\alpha(t) = \underline{f}_\alpha(t, Y(t)), & \underline{y}_\alpha(t_0) = \underline{y}_\alpha(0), \\ \overline{y}'_\alpha(t) = \overline{f}_\alpha(t, Y(t)), & \overline{y}_\alpha(t_0) = \overline{y}_\alpha(0). \end{cases} \tag{8}$$

From Theorem 7, for Case 1,

$$\mathcal{S}[Y'(t)] = \frac{\mathcal{S}[Y(t)] -^{H} Y(t_0)}{u}.$$

Therefore,

$$\begin{cases} s[\underline{f}_{\alpha}(t, Y(t))] = \dfrac{s[\underline{y}_{\alpha}(t)] - \underline{y}_{\alpha}(0)}{u}, \\ \\ s[\overline{f}_{\alpha}(t, Y(t))] = \dfrac{s[\overline{y}_{\alpha}(t)] - \overline{y}_{\alpha}(0)}{u}. \end{cases} \tag{9}$$

To solve Equation (9), first we assume that:

$$s[\underline{y}_{\alpha}(t)] = L^1_{\alpha}(u),$$
$$s[\overline{y}_{\alpha}(t)] = U^1_{\alpha}(u),$$

where $L^1_{\alpha}(u)$ and $U^1_{\alpha}(u)$ are solutions of Equation (9). By using the inverse of FST, we then compute $\underline{y}_{\alpha}(t)$ and $\overline{y}_{\alpha}(t)$ as follows.

$$\underline{y}_{\alpha}(t) = s^{-1}[L^1_{\alpha}(u)],$$
$$\overline{y}_{\alpha}(t) = s^{-1}[U^1_{\alpha}(u)].$$

Case 2: If we consider $Y'(t)$ by using a (2)-differentiable function, then from Theorem 2, we get $Y'(t) = [\overline{y}'_{\alpha}(t), \underline{y}'_{\alpha}(t)]$. Now, we obtain the following FDE to be solved.

$$\begin{cases} \underline{y}'_{\alpha}(t) = \overline{f}_{\alpha}(t, Y(t)), & \underline{y}_{\alpha}(t_0) = \underline{y}_{\alpha}(0), \\ \overline{y}'_{\alpha}(t) = \underline{f}_{\alpha}(t, Y(t)), & \overline{y}_{\alpha}(t_0) = \overline{y}_{\alpha}(0). \end{cases} \tag{10}$$

From Theorem 7, for Case 2,

$$\mathcal{S}[Y'(t)] = \frac{-(Y(t_0)) -^{H} (-\mathcal{S}[Y(t)])}{u}.$$

Therefore,

$$\begin{cases} s[\underline{f}_{\alpha}(t, Y(t))] = \dfrac{s[\underline{y}_{\alpha}(t)] - \underline{y}_{\alpha}(0)}{u}, \\ \\ s[\overline{f}_{\alpha}(t, Y(t))] = \dfrac{s[\overline{y}_{\alpha}(t)] - \overline{y}_{\alpha}(0)}{u}. \end{cases} \tag{11}$$

To solve Equation (11), first we assume that:

$$s[\underline{y}_{\alpha}(t)] = L^2_{\alpha}(u),$$
$$s[\overline{y}_{\alpha}(t)] = U^2_{\alpha}(u),$$

where $L^2_{\alpha}(u)$ and $U^2_{\alpha}(u)$ are solutions of Equation (11). By using the inverse of the FST, we then compute $\underline{y}_{\alpha}(t)$ and $\overline{y}_{\alpha}(t)$ as follows.

$$\underline{y}_{\alpha}(t) = s^{-1}[L^2_{\alpha}(u)],$$

$$\overline{y}_{\alpha}(t) = s^{-1}[U^2_{\alpha}(u)].$$

## 5. A Numerical Example

In this section, we provide a numerical example of solving an FDE using the FST.

**Example 1.** *Consider the following initial value problem:*

$$\begin{cases} Y'(t) = -Y(t), & 0 \leq t \leq T, \\ Y(t_0) = [\underline{y}_\alpha(0), \overline{y}_\alpha(0)], & 0 < \alpha \leqslant 1. \end{cases} \tag{12}$$

*By using FST, we have:*

$$\mathcal{S}[Y'(t)] = \mathcal{S}[-Y(t)]$$

*and:*

$$\mathcal{S}[Y'(t)] = \int_0^\infty Y'(ut) \odot e^{-t} dt.$$

*First, we consider the condition where $Y'(t)$ is (1)-differentiable. Therefore, from Theorem 2,*

$$\begin{cases} \underline{y}'_\alpha(t) = -\underline{y}_\alpha(t), & \underline{y}_\alpha(t_0) = \underline{y}_\alpha(0), \\ \overline{y}'_\alpha(t) = -\overline{y}_\alpha(t), & \overline{y}_\alpha(t_0) = \overline{y}_\alpha(0). \end{cases}$$

*By Theorem 7,*

$$\mathcal{S}[Y'(t)] = \frac{\mathcal{S}[Y(t)] -^H Y(t_0)}{u}.$$

*Therefore,*

$$\mathcal{S}[-Y(t)] = \frac{\mathcal{S}[Y(t)] -^H Y(t_0)}{u},$$

$$-\mathcal{S}[Y(t)] = \frac{\mathcal{S}[Y(t)] -^H Y(t_0)}{u}.$$

*From Equation (9), we have:*

$$-s[\overline{y}_\alpha(t)] = \frac{s[\underline{y}_\alpha(t)] - (\underline{y}_\alpha(0))}{u},$$

*and:*

$$-s[\underline{y}_\alpha(t)] = \frac{s[\overline{y}_\alpha(t)] - (\overline{y}_\alpha(0))}{u}.$$

*Then, we obtain*

$$s[\underline{y}_\alpha(t)] = \underline{y}_\alpha(0)\left(\frac{1}{1-u^2}\right) - \overline{y}_\alpha(0)\left(\frac{u}{1-u^2}\right),$$

*and:*

$$s[\overline{y}_\alpha(t)] = \overline{y}_\alpha(0)\left(\frac{1}{1-u^2}\right) - \underline{y}_\alpha(0)\left(\frac{u}{1-u^2}\right).$$

*Thus,*

$$\underline{y}_\alpha(t) = \underline{y}_\alpha(0)s^{-1}\left(\frac{1}{1-u^2}\right) - \overline{y}_\alpha(0)s^{-1}\left(\frac{u}{1-u^2}\right),$$

*and:*

$$\overline{y}_\alpha(t) = \overline{y}_\alpha(0)s^{-1}\left(\frac{1}{1-u^2}\right) - \underline{y}_\alpha(0)s^{-1}\left(\frac{u}{1-u^2}\right).$$

*Therefore,*

$$\underline{y}_\alpha(t) = e^t\left(\frac{\underline{y}_\alpha(0) - \overline{y}_\alpha(0)}{2}\right) + e^{-t}\left(\frac{\underline{y}_\alpha(0) + \overline{y}_\alpha(0)}{2}\right),$$

*and:*

$$\overline{y}_\alpha(t) = e^t\left(\frac{\overline{y}_\alpha(0) - \underline{y}_\alpha(0)}{2}\right) + e^{-t}\left(\frac{\overline{y}_\alpha(0) + \underline{y}_\alpha(0)}{2}\right).$$

*Next, we consider the condition where $Y'(t)$ is (2)-differentiable. Therefore, from Theorem 2,*

$$\begin{cases} \underline{y}'_\alpha(t) = -\overline{y}_\alpha(t), & \underline{y}_\alpha(t_0) = \underline{y}_\alpha(0), \\ \overline{y}'_\alpha(t) = -\underline{y}_\alpha(t), & \overline{y}_\alpha(t_0) = \overline{y}_\alpha(0). \end{cases}$$

*By Theorem 7,*

$$\mathcal{S}[Y'(t)] = \frac{-(Y(t_0)) -^H (-\mathcal{S}[Y(t)])}{u}.$$

*Therefore,*

$$\mathcal{S}[-Y(t)] = \frac{-(Y(t_0)) -^H (-\mathcal{S}[Y(t)])}{u},$$
$$-\mathcal{S}[Y(t)] = \frac{-(Y(t_0)) -^H (-\mathcal{S}[Y(t)])}{u}.$$

*From Equation (11), we have:*

$$-s[\underline{y}_\alpha(t)] = \frac{s[\underline{y}_\alpha(t)] - (\underline{y}_\alpha(0))}{u},$$

*and:*

$$-s[\overline{y}_\alpha(t)] = \frac{s[\overline{y}_\alpha(t)] - (\overline{y}_\alpha(0))}{u}.$$

*Then, we obtain:*

$$s[\underline{y}_\alpha(t)] = \underline{y}_\alpha(0)\left(\frac{1}{1+u}\right),$$

*and:*

$$s[\overline{y}_\alpha(t)] = \overline{y}_\alpha(0)\left(\frac{1}{1+u}\right).$$

*Therefore,*

$$\underline{y}_\alpha(t) = \underline{y}_\alpha(0)e^{-t},$$

*and:*

$$\overline{y}_\alpha(t) = \overline{y}_\alpha(0)e^{-t}.$$

*Assume that $[\underline{y}_\alpha(t_0), \overline{y}_\alpha(t_0)] = [\alpha - a, a - \alpha]$. Then, the solutions of Equation (12) for Case (1) and Case (2) are as follows.*

*Case 1:*

$$\underline{y}_\alpha(t) = (\alpha - a)e^t,$$
$$\overline{y}_\alpha(t) = (a - \alpha)e^t.$$

*Case 2:*

$$\underline{y}_\alpha(t) = (\alpha - a)e^{-t},$$
$$\overline{y}_\alpha(t) = (a - \alpha)e^{-t}.$$

The results obtained using the FST for both cases proposed in this paper are shown in Figures 1 and 2, respectively. We can see that for Case 1, the result diverges as $t$ increases; while for Case 2, the result implicates that the solution converges as $t$ increases.

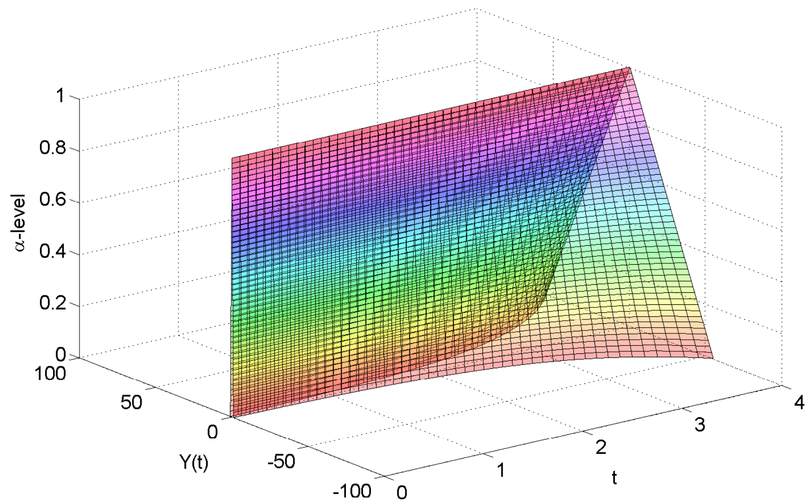


**Figure 1.** The solution of Equation (12) for Case 1 when $Y(t_0) = (-1, 0, 1)$.

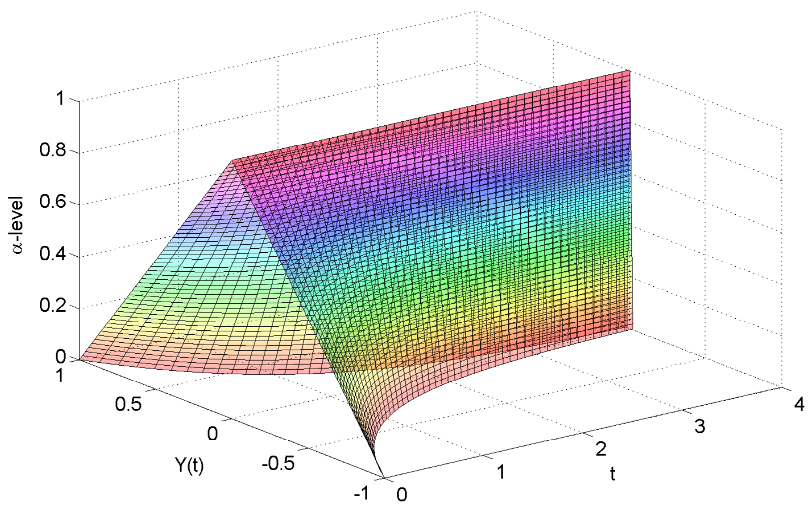


**Figure 2.** The solution of Equation (12) for Case 2 when $Y(t_0) = (-1, 0, 1)$.

**Remark.** *From the example, we notice that the solutions depend on the differential equation we chose. For Case 1, the solution has the property that the diameter, i.e. diam(supp $y(t)$) $= 2ae^t$, which is unbounded as $t$ approaches infinity. Comparing to Case 2, the diam(supp $y(t)$) $= 2ae^{-t} \to 0$ as $t$ approaches infinity, which leads to much more intuitive results.*

## 6. Conclusions

In this paper, we have studied the classical Sumudu transform in the fuzzy setting. We have also proposed detailed procedures to solve FDEs. In the last part, we have conducted a numerical example of solving first order linear FDE using the FST.

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## Author Contributions

Both authors have read and agreed on the submission of this paper. Furthermore, both authors have contributed to every part of this paper. Both authors have read and approved the final manuscript.

## Conflicts of Interest

The authors declare no conflict of interest.

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