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Article

Stochastic Processes via the Pathway Model

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Abstract: After collecting data from observations or experiments, the next step is to analyze the data to build an appropriate mathematical or stochastic model to describe the data so that further studies can be done with the help of the model. In this article, the input-output type mechanism is considered first, where reaction, diffusion, reaction-diffusion, and production-destruction type physical situations can fit in. Then techniques are described to produce thicker or thinner tails (power law behavior) in stochastic models. Then the pathway idea is described where one can switch to different functional forms of the probability density function through a parameter called the pathway parameter. The paper is a continuation of related solar neutrino research published previously in this journal.

Keywords: data analysis; model building; input-output type stochastic models; thicker or thinner-tailed models; pathway idea; pathway model, solar neutrinos

1. Introduction

After collecting data from experiments or from observations, the next step is to analyze the data and make inference out of the data. This can be achieved by using mathematical methods and models if the physical situation is deterministic in nature, otherwise create stochastic models if the physical situation is non-deterministic in nature. If the underlying phenomenon, which produced the data, is unknown, possibly deterministic but the underlying processes and the ways in which these processes act are unknown, thereby the situation becomes random in nature. Then, we go for stochastic models or non-deterministic type models. One has to know or speculate about the underlying processes as well as the ways in which these processes act so that one can decide which type of models are appropriate. If the observations are available over time then a time series type of model may be appropriate. If the time series shows periodicities then each cycle can be analyzed by using specific types of stochastic models.

Here we will consider models to describe short-term behavior of data or behavior within one cycle if a cyclic behavior is noted. As an example, when monitoring solar phenomena, specifically solar neutrinos, it is seen that there is likely to be an annual cycle and within each cycle the behavior of the graph is something like slow increase with several local peaks to a maximum peak and slow decrease with humps back to normal level [1]. In such situations, what is observed is not really what is actually produced. What is observed is the residual part of what is produced minus what is consumed or converted and thus the actual observation is made on the residual part only. Many of natural phenomena belong to this type of behavior of the form $u = x - y$ where x is the input or production variable and y is the output or consumption or destruction variable and u represents the residual part which is observed. A general analysis of input-output situation may be seen from [2]. In many situations one can assume that x and y are statistically independently distributed and that $u \geq 0$ means production dominates over destruction or input dominates over the output.

In reaction rate theory, when particles react with each other producing new particles, for example neutrinos, we may have the following type of situations. Certain particles may react with each other in short-span or short-time periods and produce small number of particles, others may take medium time intervals and produce larger numbers of particles and yet others may react over a long span and produce larger number of particles. For describing such types of situations in the production of particles the present authors considered creating mathematical models by erecting triangles whose ares are proportional to the neutrinos produced, see [1,3–5].

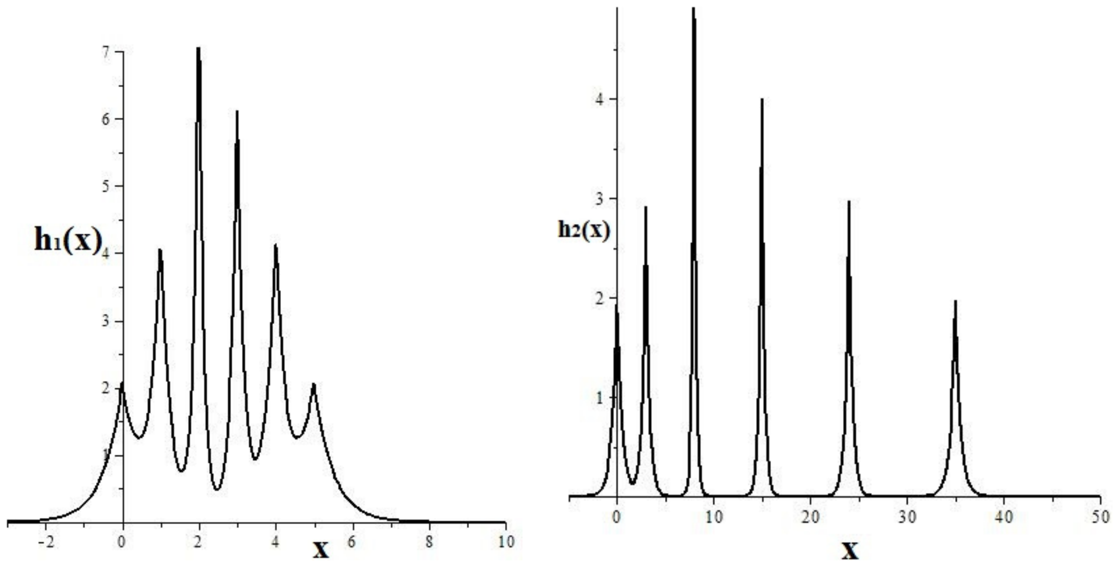
Another approach we adopted was to assume x and y as independently distributed random variables, then work out the density of the residual variable under the assumption that $x - y \geq 0$. The simplest such situation is an exponential type input and an exponential type output. Then we can look at the sum of such independently distributed residual type variables. This is a reasonable type of assumption. Then the input-output model has the Laplace density, when x and y are identically and independently distributed and the density is given by,

$$f_1(u) = \frac{\beta_1}{2} \mathrm{e}^{-\beta_1 |u - \alpha_1|}, 0 \leq u < \infty, \beta_1 > 0, \tag{1}$$

and $f_1(u) = 0$ elsewhere, where α_1 is a location parameter. Note that β_1 can act as a scale parameter or as a dispersion or scatter parameter. Suppose that this situation is repeated at successive locations and with the scale parameter $\beta = \beta_1, \beta_2, \ldots$. Then the nature of the graph will be that of a sum of Laplace densities. If the location parameters are sufficiently farther apart then the graph will look like that in Figure 1b. If such blips are occurring sufficiently close together then we have a graph of the type in Figure 1a. In these graphs we have taken only five to six locations for simplicity. However, by taking successive locations we can generate many of the phenomena that are seen in nature, especially in time series data. When the locations are sufficiently closer we get the graph with several local maxima/spikes and a continuous curve. This is the type of behavior seen in solar neutrino observations as discussed under various physical aspects [1,3–7]. Cyclic patterns can also arise depending upon the location and scale parameters. Here β_1 measures the intensity of the blip and α_1 the location where it happens, and each blip is the residual effect of an exponential type input and an independent exponential type output of the same strength. If $\alpha_1, \alpha_2, ...$ are farther apart then the contributions coming from other blips will be negligible and if $\alpha_1, \alpha_2, ...$ are close together then there will be contributions from other blips. The function will be of the following form:

$$f(u) = \sum_{j=1}^{k} \frac{\beta_j}{2} \mathrm{e}^{-\beta_j |u-\alpha_j|}, 0 \le u < \infty, \beta_j > 0, j = 1, ..., k < \infty. \quad (2)$$

If one requires $f(u)$ to be a density within a number of spikes then divide the sum by k so that we have a convex combination of Laplace densities, which will again be a density. The model does not require that we create a density out of the pattern. If the arrival of the location points (α_j) is governed by a Poisson process then we will have a Poisson mixture of Laplace densities.



(a) **(b)**

Figure 1. Sample paths of sums of Laplace densities.

A symmetric Laplace density will be of the following form:

$$f_2(u) = \frac{1}{2\beta} \mathrm{e}^{-\frac{|u|}{\beta}}, \quad -\infty < u < \infty \quad (3)$$

and the graph is of the form shown in Figure 2.

This is the symmetric case where $u < 0$ behaves the same way as $u \ge 0$. If the behavior of u is different for $u < 0$ and $u \ge 0$ then we get the asymmetric Laplace case which can be written as (graphic shown in Figure 3)

$$g(u) = \begin{cases} \frac{1}{(\beta_1+\beta_2)} \mathrm{e}^{\frac{u}{\beta_1}}, & -\infty < u < 0, \\ \frac{1}{(\beta_1+\beta_2)} \mathrm{e}^{-\frac{u}{\beta_2}}, & 0 \le u < \infty. \end{cases} \quad (4)$$

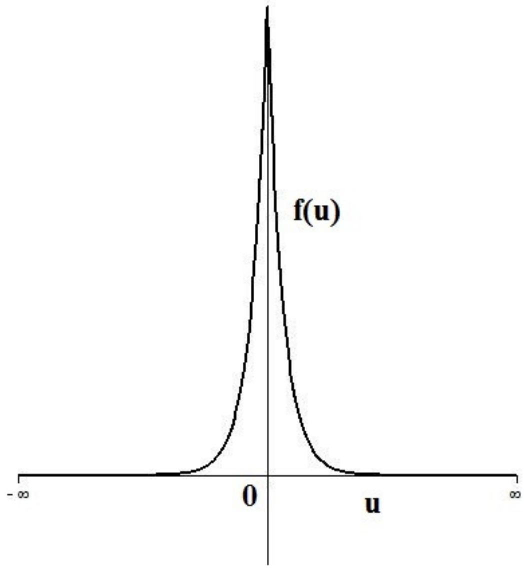


Figure 2. Symmetric Laplace density.

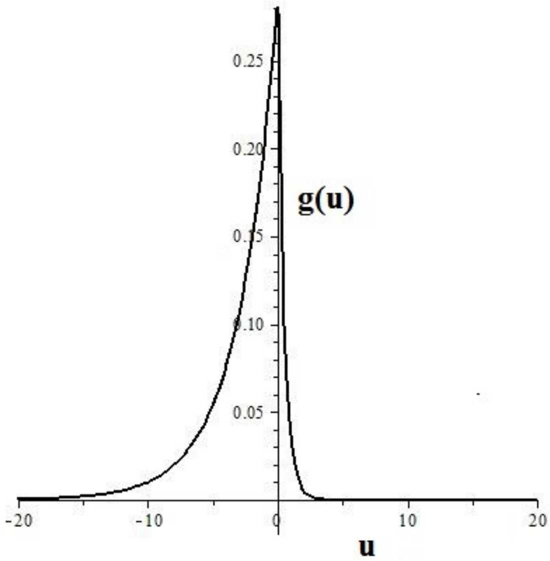


Figure 3. Asymmetric Laplace case.

When $\alpha_1 > 1, \alpha_2 > 1, \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta$ we have independently and identically distributed gamma random variables for x and y and $u = x - y$ is the difference between them. Then $g_1(u)$ can be seen to be the following:

$$g_1(u) = \frac{u^{2\alpha-1}\mathrm{e}^{-\frac{u}{\beta}}}{\beta^{2\alpha}\Gamma^2(\alpha)} \int_{z=0}^{\infty} (1+z)^{\alpha-1} z^{\alpha-1} \mathrm{e}^{-\frac{1}{\beta}(2uz)} \mathrm{d}z \tag{5}$$

for $u \geq 0$, $\alpha > 0, \beta > 0$. This behaves like a gamma density and provides a symmetric model for $u \geq 0$ and $u < 0$. The nature of the graph is shown in Figure 4.

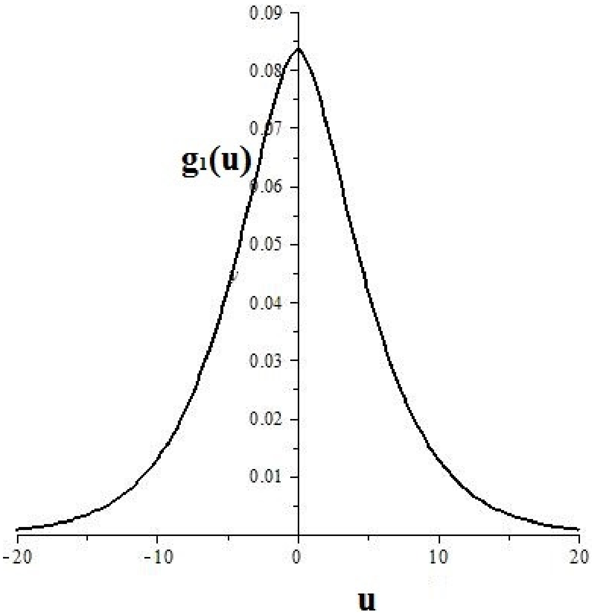


Figure 4. $g_1(u)$ in the symmetric gamma type input-output variables.

2. Models with Thicker and Thinner Tails

For a large number of processes a gamma type model may be appropriate. A two parameter gamma density has the form

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\frac{x}{\beta}}, x \geq 0, \alpha > 0, \beta > 0. \tag{6}$$

Sometimes a member from this parametric family of functions may be appropriate to describe a data set. Sometimes the data require a slightly thicker-tailed model due to chances of higher probabilities or more area under the curve in the tail. Two of such models developed by the authors' groups will be described here. One type is where the model in Equation (6) is appended with a Mittag-Leffler series and another type is where Equation (6) is appended with a Bessel series, see also [8–11].

2.1. Gamma Model with Appended Mittag-Leffler Function

Consider a gamma density of the type

$$g_3(x) = c_1\, x^{\gamma-1}\mathrm{e}^{-\frac{x}{\delta}},\ \delta > 0,\ \gamma > 0,\ x \geq 0.$$

Suppose that we append this $g_3(x)$ with a Mittag-Leffler function $E^{\beta}_{\alpha,\gamma}(-ax^{\alpha})$, where

$$E^{\beta}_{\alpha,\gamma}(-ax^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}(-a)^k \frac{x^{\alpha k}}{\Gamma(\gamma+\alpha k)},\ \alpha > 0,\ \gamma > 0.$$

Consider the function

$$f^*(x) = c \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}(-a)^k \frac{x^{\alpha k+\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}}{\Gamma(\gamma+\alpha k)},\ x \geq 0,$$

where c is the normalizing constant. Let us evaluate c. Since the total integral is 1,

$$1=\int_0^\infty f^*(x)\mathrm{d}x=c\sum_{k=0}^\infty\frac{(\beta)_k}{k!}(-a)^k\int_0^\infty\frac{x^{\alpha k+\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}}{\Gamma(\gamma+\alpha k)}\mathrm{d}x$$
$$=c\sum_{k=0}^\infty\frac{(\beta)_k}{k!}(-a)^k\delta^{\alpha k+\gamma}=c\,\delta^\gamma(1+a\delta^\alpha)^{-\beta},\ |a\delta^\alpha|<1$$

for $\beta>0,\alpha>0,\delta>0,a\beta\delta^\alpha<1,|a\delta^\alpha|<1$. Therefore the density is

$$f^*(x)=\frac{(1+a\delta^\alpha)^\beta}{\delta^\gamma}x^{\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}\sum_{k=0}^\infty\frac{(\beta)_k}{k!}\frac{(-a)^k\delta^{\alpha k}}{\Gamma(\gamma+\alpha k)}$$

for $0\le x<\infty,\alpha>0,\gamma>0,\delta>0,\beta>0,|a\delta^\alpha|<1,a\beta\delta^\alpha<1$. That is,

$$f^*(x)=\frac{(1+a\delta^\alpha)^\beta}{\delta^\gamma}x^{\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}[\frac{1}{\Gamma(\gamma)}+\sum_{k=1}^\infty\frac{(\beta)_k}{k!}\frac{(-1)^k\delta^{\alpha k}}{\Gamma(\gamma+\alpha k)}].$$

Note that $a=0$ corresponds to the original gamma density. Figure 5 shows graphs of the appended Mittag-Leffler-gamma density. When $a<0$ we have thinner tail and when $a>0$ we have thicker tails compared to the gamma tail.

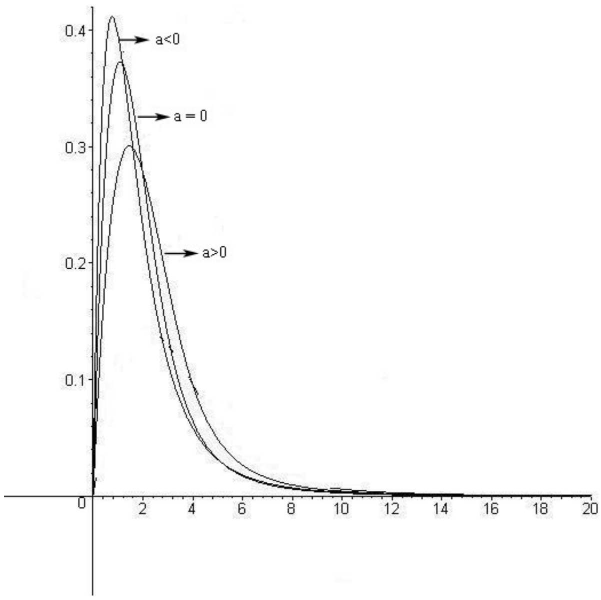


Figure 5. Gamma density with Mittag-Leffler function appended.

2.2. Bessel Appended Gamma Density

Consider the model of the type of a basic gamma density appended with a Bessel function, see also [8–10].

$$\tilde{f}(x)=c\,x^{\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}\sum_{k=0}^\infty\frac{x^k(-a)^k}{k!\Gamma(\gamma+k)},\ \delta>0,\gamma>0,x\ge0,$$

where c is the normalizing constant. The appended function is of the form

$$\frac{1}{\Gamma(\gamma)}{}_0F_1(\ ;\gamma:-ax)$$

which is a Bessel function. Let us evaluate c.

$$\begin{aligned}1 &= c\sum_{k=0}^{\infty}\frac{(-a)^k}{k!}\int_0^{\infty}\frac{x^{\gamma+k-1}}{\Gamma(\gamma+k)}\mathrm{e}^{-\frac{x}{\delta}}\mathrm{d}x\\ &= c\,\delta^{\gamma}\sum_{k=0}^{\infty}\frac{(-a)^k\delta^k}{k!} = c\,\delta^{\gamma}\mathrm{e}^{-a\delta}.\end{aligned}$$

Hence the density is of the form, also shown in Figure 6,

$$\tilde{f}(x) = \frac{\mathrm{e}^{a\delta}}{\delta^{\gamma}}x^{\gamma-1}\mathrm{e}^{-\frac{x}{\delta}}\sum_{k=0}^{\infty}\frac{x^k(-a)^k}{k!\Gamma(\gamma+k)}, x\geq 0,\ \gamma>0, \delta>0.$$

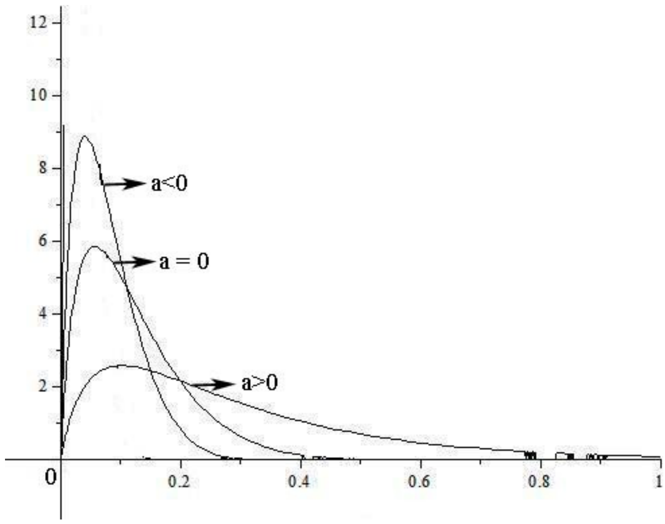


Figure 6. Gamma appended with Bessel function.

Note: *Instead of appending with Bessel function one could have appended with a general hypergeometric series. However, a general hypergeometric series does not simplify into a convenient form. We have chosen specialized parameters as well as suitable functions so that the normalizing constants simplify to convenient forms thereby general computations will be much easier and simpler.*

3. Pathway Idea

Here we consider a model which can switch to three functional forms covering almost all statistical densities in current use, see [9,10,12]. Let

$$f_1^*(x) = c_1^*|x|^{\gamma}[1-a(1-\alpha)|x|^{\delta})^{\frac{\eta}{1-\alpha}}, \alpha<1, \eta>0, a>0, \delta>0, \tag{7}$$

and $1-a(1-\alpha|x|^{\delta}>0$, where c_1^* is the normalizing constant. When $\alpha<1$ the model in Equation (7) stays as the generalized type-1 beta family, extended over the real line. When $\alpha>1$ write $1-\alpha=-(\alpha-1)$ with $\alpha>1$. Then the functional form in Equation (7) changes to

$$f_2^*(x) = c_2^*|x|^{\gamma}[1+a(\alpha-1)|x|^{\delta}]^{-\frac{\eta}{\alpha-1}} \tag{8}$$

for $\alpha > 1, a > 0, \eta > 0, -\infty < x < \infty$. Note that Equation (8) is the extended generalized type-2 beta family of functions. When $\alpha \to 1$ then both Equations (7) and (8) go to

$$f_3^*(x) = c_3^* \, |x|^{\gamma} \mathrm{e}^{-a\eta|x|^{\delta}}, a > 0, \eta > 0, \delta > 0, -\infty < x < \infty. \tag{9}$$

Equation (9) is the extended generalized gamma family of functions. Thus Equation (7) is capable of switching to three families of functions. This is the pathway idea and α is the pathway parameter. Through this parameter α one can reach the three families of functions in Equations (7)–(9). The normalizing constants can be seen to be the following:

$$c_1^* = \frac{\delta}{2} \frac{[a(1-\alpha)]^{\frac{\gamma+1}{\delta}} \Gamma(\frac{\gamma+1}{\delta} + \frac{\eta}{1-\alpha} + 1)}{\Gamma(\frac{\gamma+1}{\delta}) \Gamma(\frac{\eta}{1-\alpha} + 1)} \tag{10}$$

for $a > 0, \alpha < 1, \delta > 0, \gamma > -1, \eta > 0$,

$$c_2^* = \frac{\delta}{2} \frac{[a(\alpha-1)]^{\frac{\gamma+1}{\delta}} \Gamma(\frac{\eta}{\alpha-1})}{\Gamma(\frac{\gamma+1}{\delta}) \Gamma(\frac{\eta}{\alpha-1} - \frac{\gamma+1}{\delta})} \tag{11}$$

for $\alpha > 1, a > 0, \delta > 0, \eta > 0, \delta > 0, \frac{\eta}{\alpha-1} - \frac{\gamma+1}{\delta} > 0$,

$$c_3^* = \frac{\delta}{2} \frac{(a\eta)^{\frac{\gamma+1}{\delta}}}{\Gamma(\frac{\gamma+1}{\delta})}, a > 0, \delta > 0, \eta > 0, \gamma > -1. \tag{12}$$

Note that Equation (7) is a finite range model, suitable to describe processes where the tails are cut off. When α comes closer and closer to 1 then the cut-off point moves away from the origin and eventually goes to $\pm\infty$. When $\alpha \to 1$ then model Equation (7) goes to model Equation (9) which is an extended generalized gamma model. The model in Equation (8) is type-2 beta form, spreads out over the whole real line and the shape will be closer to that of a gamma type model when α approaches 1. Thus the pathway models in Equations (7)–(9) cover all types of processes where the tails are cut off, tails are made thinner or thicker compared to a gamma type model. The extended gamma type model in Equation (9) also contains the Gaussian model, Brownian motion, Maxwell-Boltzmann density *etc*. If Gaussian or Maxwell-Boltzmann is the stable or required form in a physical process then the unstable neighborhoods are covered by Equations (7) and (8) or the paths leading to this stable form is described by Equations (7) and (8).

Here it is important to note that Equation (7) for $x > 0, \gamma = 0, a = 1, \delta = 1, \eta = 1$ is the Tsallis statistics of non-extensive statistical mechanics [13,14]. Also note that Equation (8) for $a = 1$, $\delta = 1, \eta = 1$ is superstatistics. This superstatistics can also be derived as the unconditional density when both the conditional density of x given a parameter θ and the marginal density of θ are gamma densities or exponential type densities, the details may be seen from Mathai and Haubold [15–19].

4. Reaction Rate Probability Integral Model

Starting in the 1980s, the authors had pursued mathematical models for reaction-rate theory for processes such as non-resonant reactions and resonant reactions under conditions such as depletion and high energy tail cut off, see [1,20–25]. The basic model is an integral of the following form:

$$I_{(1)} = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-ax^{\delta} - zx^{-\rho}}, a > 0, z > 0, \rho > 0, \delta > 0. \tag{13}$$

For $\rho = \frac{1}{2}, \delta = 1$ one has the basic probability integral in the non-resonant case, see [22]. For $\gamma = 0, \rho = 1$ one has the Krätzel integral [26]. For $\gamma = 0, \delta = 1, \rho = 1$ one has the inverse Gaussian density. Computational aspect of Equation (13) is discussed in [27] and related material may be seen from [28]. Since the integral in Equation (13) is a product of integrable functions one can evaluate the integral in Equation (13) with the help of Mellin convolution of a product because the integrand can be written as

$$\int_0^\infty \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v, f_1(x) = x^\gamma \mathrm{e}^{-ax^\delta}, f_2(y) = \mathrm{e}^{-y^\rho} \tag{14}$$

for $u = z^{\frac{1}{\rho}}$, $u = xy$. Then the Mellin convolution of the integral in Since the integral in Equation (13), denoting the Mellin transform of a function f with Mellin parameter s as $M_f(s)$, we have from Equation (13)

$$M_{I_{(1)}}(s) = M_{f_1}(s) M_{f_2}(s), \tag{15}$$

where

$$M_{f_1}(s) = \int_0^\infty x^{s-1} f_1(x) \mathrm{d}x = \int_0^\infty x^{\gamma+s-1} \mathrm{e}^{-ax^\delta} \mathrm{d}x = \frac{1}{\delta} \frac{\Gamma(\frac{s+\gamma}{\delta})}{a^{\frac{s+\gamma}{\delta}}}, \quad \Re(s+\gamma) > 0$$

$$M_{f_2}(s) = \int_0^\infty y^{s-1} \mathrm{e}^{-y^\rho} \mathrm{d}y = \frac{1}{\rho} \Gamma(\frac{s}{\rho}), \quad \Re(s) > 0.$$

Hence

$$M_{I_{(1)}}(s) = M_{f_1}(s) M_{f_2}(s) = \frac{1}{\rho\delta\, a^{\frac{\gamma}{\delta}}} \frac{\Gamma(\frac{s+\gamma}{\delta})\Gamma(\frac{s}{\rho})}{a^{\frac{s}{\delta}}}.$$

Therefore the integral in Equation (13) is the inverse Mellin transform of Equation (15). That is,

$$\begin{aligned} I_{(1)} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\rho\delta a^{\frac{\gamma}{\delta}}} \Gamma(\frac{s+\gamma}{\delta}) \Gamma(\frac{s}{\rho}) (u a^{\frac{1}{\delta}})^{-s} \mathrm{d}s, \quad u = z^{\frac{1}{\rho}}, i = \sqrt{-1} \\ &= \frac{1}{\rho\delta a^{\frac{\gamma}{\delta}}} H_{0,2}^{2,0}[z^{\frac{1}{\rho}} a^{\frac{1}{\delta}} \big|_{(0,\frac{1}{\rho}),(\frac{\gamma}{\delta},\frac{1}{\delta})}] \end{aligned} \tag{16}$$

where $H(\cdot)$ is the H-function, see [9,10]. From the basic result in Equation (16) we can evaluate the reaction-rate probability integrals in the other cases of non-relativistic reactions.

4.1. Generalization of Reaction-Rate Models

A companion integral corresponding to Equation (13) is the integral

$$I_{(2)} = \int_0^\infty x^\gamma \mathrm{e}^{-ax^\delta - zx^\rho} \mathrm{d}x, a > 0, \delta > 0, \rho > 0, z > 0. \tag{17}$$

In Equation (13) we had $x^{-\rho}$ with $\rho > 0$ whereas in Equation (17) we have x^ρ with $\rho > 0$. For $\delta = 1$, Equation (17) corresponds to the Laplace transform or moment generating function of a generalized gamma density in statistical distribution theory. The integral in Equation (17) can be written in the form of an integral as follows

$$\int_0^\infty v f_1(v) f_2(uv) \mathrm{d}v, f_1(x) = x^{\gamma-1} \mathrm{e}^{-ax^\delta}, f_2(y) = \mathrm{e}^{-y^\rho} \tag{18}$$

for $u = z^{\frac{1}{\rho}}$. The integral in Equation (18) is in the structure of a Mellin transform of a ratio $u = \frac{y}{x}$, so that the Mellin transform of $I_{(2)}$ is then

$$M_{I_{(2)}}(s) = M_{f_2}(s)M_{f_1}(2-s). \tag{19}$$

The inverse Mellin transform in Equation (19) gives the integral $I_{(2)}$. The pair of integrals $I_{(1)}$ and $I_{(2)}$ belong to a particular case of a general versatile model considered by the authors earlier [25].

A generalization of $I_{(1)}$ and $I_{(2)}$ is the pathway generalized model, which results in the versatile integral. The pathway generalization is done by replacing the two exponential functions by the corresponding pathway form. Consider the integrals of the following types:

$$I_p = \int_0^\infty x^\gamma [1 + a(q_1 - 1)x^\delta]^{-\frac{1}{q_1-1}} [1 + b(q_2 - 1)x^\rho]^{-\frac{1}{q_2-1}} \mathrm{d}x, \tag{20}$$

where $q_1 > 1, q_2 > 1, a > 0, b > 0$. We will keep ρ free, could be negative or positive. Note that

$$\lim_{q_1 \to 1} [1 + a(q_1 - 1)x^\delta]^{-\frac{1}{q_1-1}} = \mathrm{e}^{-ax^\delta}$$

and

$$\lim_{q_2 \to 1} [1 + b(q_2 - 1)x^\rho]^{-\frac{1}{q_2-1}} = \mathrm{e}^{-bx^\rho}.$$

Hence

$$\lim_{q_1 \to 1, q_2 \to 1} I_p = \int_0^\infty x^\gamma \mathrm{e}^{-ax^\delta - bx^\rho} \mathrm{d}x$$

which is the integral in Equation (17) and if $\rho < 0$ then it is the integral in Equation (13). The general integral in Equation (20) belongs to the general family of versatile integrals. The factors in the integrand in Equation (20) are of the generalized type-2 beta form. We could have taken each factor in type-1 beta or type-2 beta form, thus providing 6 different combinations. For each case, we could have the situation of $\rho > 0$ or $\rho < 0$. The whole collection of such models is known as the versatile integrals. Integral transforms, known as P-transforms, are also associated with the integrals in Equation (20), see for example [29,30].

4.2. Fractional Calculus Models

In a series of papers the authors [9,15,31–35] have shown recently that fractional integrals can be classified into the forms in Equations (14) and (18) or fractional integral operators of the second kind or right-sided fractional integral operators can be considered as Mellin convolution of a product as in Equation (14) and left-sided or fractional integral operators of the first kind can be considered as Mellin convolution of a ratio where the functions f_1 and f_2 are of the following forms:

$$f_1(x) = \phi_1(x)(1-x)^{\alpha-1}, 0 \le x \le 1, f_2(y) = \phi_2(y)f(y) \tag{21}$$

where ϕ_1 and ϕ_2 are pre-fixed functions, $f(y)$ is arbitrary and $f_1(x) = 0$ outside the interval $0 \le x \le 1$. Thus, essentially, all fractional integral operators belong to the categories of Mellin convolution of a product or ratio where one function is a multiple of type-1 beta form and the other is arbitrary. The right-sided or type-2 fractional integral of order α is denoted by $D_{2,u}^{-\alpha} f$ and defined as

$$D_{2,u}^{-\alpha} f = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v \tag{22}$$

and the left-sided or type-1 fractional integral of order α is given by

$$D_{1,u}^{-\alpha} f = \int_v \frac{v}{u^2} f_1(\frac{v}{u}) f_2(v) \mathrm{d}v \tag{23}$$

where f_1 and f_2 are as given in Equation (21). Let n be a positive integer such that $\Re(n - \alpha) > 0$. The smallest such n is $[\Re(\alpha)] + 1 = m$ where $[\Re(\alpha)]$ denotes the integer part of $\Re(\alpha)$. Here $D_{2,u}^{-\alpha} f$ and $D_{1,u}^{-\alpha} f$ are defined as in Equations (22) and (23) respectively. Let $D = \frac{\mathrm{d}}{\mathrm{d}u}$ the integer order derivative with respect to u and D^n be the n-th order derivative. Then the fractional derivative of order α is defined as

$$D^\alpha f = D^n[D_{i,u}^{-(n-\alpha)} f] \text{ in the Riemann-Liouville sense} \tag{24}$$

$$D^\alpha f = [D_{i,u}^{-(n-\alpha)} D^n f] \text{ in the Caputo sense} \tag{25}$$

for $i = 1, 2$, see also [36,37].

The input-output model that we started with, when applied to reaction-diffusion problems can result in fractional order reaction-diffusion differential equations. Such fractional order differential equations are seen to provide solutions which are more relevant to practical situations compared to the solutions coming from differential equations in the conventional sense or involving integer-order derivatives. Some of the relevant papers in this direction may be seen from [31,32,35,38–40].

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Author Contributions

Both authors contributed to the manuscript. Both authors have read and approved the final manuscript.

Conflict of Interest

The authors declare no conflict of interest.

References

1. Sakurai, K. *Solar Neutrino Problems—How They Were Solved*; TERRAPUB: Tokyo, Japan, 2014.
2. Mathai, A.M. The residual effect of a growth-decay mechanism and the covariance structures. *Can. J. Stat.* **1993**, *21*, 277–283.
3. Haubold, H.J.; Mathai, A.M. A heuristic remark on the periodic variation in the number of solar neutrinos detected on Earth. *Astrophys. Space Sci.* **1995**, *228*, 113–134.

4. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Analysis of solar neutrino data from Super-Kamiokande I and II. *Entropy* **2014**, *16*, 1414–1425.
5. Mathai, A.M.; Haubold, H.J. On a generalized entropy measure leading to the pathway model with a preliminary application to solar neutrino data. *Entropy* **2013**, *15*, 4011–4025.
6. Bonventre, R.; LaTorre, A.; Klein, J.R.; Orebi Gann, G.D.; Seibert, S.; Wasalski, O. Nonstandard models, solar neutrinos, and large Θ_{13}. *Phys. Rev. D* **2013**, *88*, 053010.
7. Jacobsson, B.; Ohlsson, T.; Snellman, H.; Winter, W. Effects of random matter density fluctuations on the neutrino oscillation transition probabilities in the Earth. *Phys. Lett. B* **2002**, 259–266.
8. Mathai, A.M. Stochastic models under power transformations and exponentiation. *J. Indian Soc. Probab. Stat.* **2012**, *13*, 1–19.
9. Mathai, A.M.; Haubold, H.J. *Special Functions for Applied Scientists*; Springer: New York, NY, USA, 2008.
10. Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-Function: Theory and Applications*; Springer: New York, NY, USA, 2010.
11. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin, Germany, 2014.
12. Mathai, A.M. A pathway to matrix-variate gamma and normal densities. *Linear Algebra Appl.* **2005**, *396*, 317–328.
13. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479–487.
14. Tsallis, C. *Introduction to Nonextensive Statistical Mechanics—Approaching a Complex World*; Springer: New York, NY, USA, 2009.
15. Haubold, H.J.; Mathai, A.M. Pathway model, superstatistics, Tsallis statistics and a measure of entropy. *Physica A* **2007**, *375*, 110–122.
16. Mathai, A.M.; Haubold, H.J. On generalized entropy measure and pathways. *Physica A* **2007**, *385*, 493–500.
17. Mathai, A.M.; Haubold, H.J. Pathway parameter and thermonuclear functions. *Physica A* **2008**, *387*, 2462–2470.
18. Mathai, A.M.; Haubold, H.J. Mittag-Leffler functions to pathway model to Tsallis statistics. *Integr. Transf. Spec. Funct.* **2011**, *21*, 867–875.
19. Mathai, A.M.; Haubold, H.J. A pathway for Bayesian statistical analysis to superstatistics. *Appl. Math. Comput.* **2011**, *218*, 799–804.
20. Haubold, H.J.; Mathai, A.M. On the nuclear energy generation rate in a simple analytic stellar model. *Ann. Phys.* **1984**, *41*, 372–379.
21. Haubold, H.J.; Mathai, A.M. On nuclear reaction rate theory. *Ann. Phys.* **1984**, *41*, 380–396.
22. Mathai, A.M.; Haubold, H.J. *Modern Problems in Nuclear and Neutrino Astrophysics*; Akademie-Verlag: Berlin, Germany, 1988.
23. Haubold, H.J.; Mathai, A.M. On thermonuclear reaction rates. *Astrophys. Space Sci.* **1998**, *258*, 185–189.
24. Haubold, H.J.; Mathai, A.M. An integral arising frequently in astronomy and physics. *SIAM Rev.* **1998**, *40*, 995–997.

25. Mathai, A.M.; Haubold, H.J. A versatile integral in physics and astronomy. **2011**, arXiv: 1109.5173.
26. Mathai, A.M. Generalized Krätzel integrals associated with statistical densities. *Int. J. Math. Anal.* **2012**, *6*, 2501–2510.
27. Anderson, W.J.; Haubold, H.J.; Mathai, A.M. Astrophysical thermonuclear functions. *Astrophysi. Space Sci.* **1994**, *214*, 49–70.
28. Mathai, A.M. On a system of differential equations connected with the gravitational instability in a multi-component medium in Newtonian cosmology. *Stud. Appl. Math.* **1989**, *80*, 75–03.
29. Kumar, D.; Kilbas, A.A. Fractional calculus of P-transform. *Fract. Calc. Appl. Anal.* **2010**, *13*, 317–328.
30. Kumar, D. P-transform. *Integr. Transf. Spec. Funct.* **2011**, *22*, 603–611.
31. Haubold, H.J.; Mathai, A.M. The fractional kinetic equation and thermonuclear functions. *Astrophys. Space Sci.* **2000**, *273*, 53–63.
32. Haubold, H.J.; Mathai, A.M. On fractional kinetic equations. *Astrophys. Space Sci.* **2002**, *282*, 281–287.
33. Haubold, H.J.; Mathai, A.M. A certain class of Laplace transform with application to reaction and reaction-diffusion equations. *Astrophys. Space Sci.* **2005**, *305*, 283–288.
34. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Further solutions of fractional reaction-diffusion equations in terms of the H-function. *J. Comput. Appl. Math.* **2011**, *235*, 1311–1316.
35. Mathai, A.M. Fractional integrals in the matrix-variate cases and connection to statistical distributions. *Integr. Transforms Spec. Funct.* **2009**, *20*, 871–882.
36. Mathai, A.M.; Haubold, H.J. Erdélyi-Kober fractional integral operators from a statistical perspective I–IV. **2013**, arXiv: 1303.3978-3981.
37. Luchko, Y.; Kiryakova, V. The Mellin integral transform in fractional calculus. *Fract. Calc. Appl. Anal.* **2013**, *16*, 405–430.
38. Haubold, H.J.; Mathai, A.M., Saxena, R.K. Solution of certain fractional kinetic equations and a fractional diffusion equation. *J. Math. Phys.* **2010**, *51*, 103506.
39. Sebastian, N. *Statistical Distributions and Time Series Model Connection to Bessel and Mittag-Leffler Functions via Fractional Calculus*. Ph.D. Thesis, Banaras Hindu University, Varanasi, India, 2012.
40. Nair, S.S. *Statistical Distributions Connected with Pathway Model and Generalized Fractional Integral Operators*. Ph.D. Thesis, Banaras Hindu University, Varanasi, India, 2012.