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Meeting Report

Symmetry, Probabiliy, Entropy: Synopsis of the Lecture at MAXENT 2014 †

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Abstract: In this discussion, we indicate possibilities for (homological and non-homological) linearization of basic notions of the probability theory and also for replacing the real numbers as values of probabilities by objects of suitable combinatorial categories.

Keywords: entropy; Bernoulli approximation; homology measures

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The success of the probability theory decisively, albeit often invisibly, depends on symmetries of systems this theory applies to. For instance:

- The symmetry group of *a single round of gambling with three dice* has order $288 = 6 \times 6 \times 8$: it is a semidirect product of the permutation group S_3 of order 6 and the symmetry group of the $3d$ cube, that is, in turn, is a semidirect product of S_3 and $\{\pm 1\}^3$.

- *The Bernoulli spaces* $(\blacksquare_p, \blacklozenge_{1-p})^{\mathbb{Z}}$, $0 < p < 1$, of $(\blacksquare, \blacklozenge)$-sequences indexed by integers $z \in \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ are acted upon by a semidirect product of the infinite permutation group

$$S_{\infty = \mathbb{Z}} \supset \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$

and the (compact) group $\{\pm 1\}^{\mathbb{Z}} = \{\blacksquare \leftrightarrow \blacklozenge\}^{\mathbb{Z}}$, with the role of the latter being essential even for $p \neq \frac{1}{2}$ where the probability measure is not preserved.

- The system of *identical point-particles* $\bullet_i$ *in the Euclidean* 3*-space* $\mathbb{R}^3$, that are indexed by a countable set $I \ni i$, is acted upon by the isometry group of $\mathbb{R}^3$ times the infinite permutation group $S_{\infty = I}$.

- *Buffon's probabilistic needle formula for* $\pi = 3.141592653589793\cdots$ relies on the invariance of the Haar measure on the circle.

I. What happens if the symmetry is enhanced, e.g., from the permutation group $S_{\infty=I}$ to the group $GL_{\mathbb{F}}(\infty)$ of liner transformations of the vector space $\mathbb{F}^I$ (formally) spanned by symbols $[i]$, $i \in I$, regarded as (linearly independent) vectors over a filed $\mathbb{F}$?

II. What could you do if your system is *inherently heterogeneous*, such as *a folding polypeptide chain* or a *natural language*, for instance?

Hilbertisation/unitarisation/quantization of set categories brought along a development of several magnificent *non-commutative probability theories*, e.g., of those under the headings of *von-Neumann algebras, von Neumann entropy* [1,2], *free probabilities* [3].

By comparison, the achievements of the *non-unitary linearisation* of probability theory are modest—just a few amusing observations.

Example 1. *Linearized Loomis-Whitney-Shannon-Shearer Submultiplicativity Inequality* [4,5].

Let $\Phi = \Phi(x_1, x_2, x_3, x_4)$ *be a 4-linear function (form) over some field (where the variables* x_i *run over some vector spaces* X_i*). Then the ranks of the following four* ***bilinear*** *forms* $\Phi(x_1, x_2 \otimes x_3 \otimes x_4)$, $\Phi(x_1 \otimes x_2, x_3 \otimes x_4)$, $\Phi(x_1 \otimes x_3, x_2 \otimes x_4)$ *and* $\Phi(x_1 \otimes x_4, x_2 \otimes x_3)$ *satisfy*

$$(rank[1,234])^2 \leq rank[12,34] \cdot rank[13,24] \cdot rank[14,23].$$

Example 2. *Homology Measures* [6].

Homologies $H_*(X) = \oplus_i H_i(X)$ *of topological spaces* X *and natural subgroups in* H_* *are graded Abelian groups: their ranks are properly represented not by individual numbers* r_i*, but by Poincaré polynomials* $P_X(t) = \sum_i r_i \cdot t^i$.

The polynomial valued set function $U \mapsto P_U$, $U \subset X$*, has some measure/entropy-like properties that become more pronounced for the ideal valued function that assigns the kernels*

$$Ker_{X \smallsetminus U} \subset H^*(X; A)$$

of the inclusion/restriction cohomology homomorphisms for the complements $X \smallsetminus U \subset X$ *for subsets* $U \subset X$,

$$U \mapsto \mu^*(U) =_{def} Ker_{X \smallsetminus U} =_{def} Ker\big[H^*(X; A) \to H^*(X \smallsetminus U; A)\big],$$

for some Abelian (cohomology coefficient) group A.

The basic properties of this μ^* (stated slightly differently in topology textbooks) have an attractive measure theoretic flavour. Namely,

$\mu^*(U)$ *is* **additive** *for the sum-of-subsets in the group* $H^*(X; A)$ *and, if* A *is a commutative ring, then* μ^* *is* **super-multiplicative** *for the the* $\smile$*-product of ideals:*

$$\mu^*(U_1 \cup U_2) = \mu^*(U_i) + \mu^*(U_2)$$

for *disjoint* open subsets U_1 and U_2 in A, and

$$\mu^*(U_1 \cap U_2) \supset \mu^*(U_1) \smile \mu^*(U_2)$$

for all open $U_1, U_2 \subset A$.

Next, given a linear subspace $\Theta \subset H^*(X; A)$, let

$$\mu_\Theta(U) = \Theta \cap Ker_{X \smallsetminus U}$$

and, assuming A is (the additive group of) a field, denote the rank of $\mu_\Theta(U)$ over this field by $|\mu_\Theta(U)| = |\mu_\Theta(U)|_A$.

Linearized Matsumoto-Tokushige Separation Inequality in the N-torus.

Let $U_1, U_2 \subset \mathbb{T}^N$ be *non-intersecting* (closed or open) subsets and let

$$\Theta_1 = H^{n_1}(\mathbb{T}^N; A), \text{and } \Theta_2 = H^{n_2}(\mathbb{T}^N; A)$$

for $n_i \leq N/2$, $i = 1, 2$, and some field A. Then

$$|\mu_{\Theta_1}(U_1)| \cdot |\mu_{\Theta_2}(U_2)| \leq c \cdot |\Theta_1| \cdot |\Theta_2|$$

for $c = n_1 n_2 / N^2$ and where, observe, $\left|\Theta_i = \wedge^{n_i} A\right| = \binom{N}{n_i}$.

If we think of the torus $\mathbb{T}^N$ as a physical system of N uncoupled linear oscillators then the "measures" $\mu^*(U)$ and/or $\mu_\Theta(U)$ may be interpreted as

"the numbers of persistent degrees of freedom" of this system that are observable from U.

Probabilistic/entropic interpretation of homology, which is kind of "dual" to "homological interpretation of entropy-like invariants" by Bennequin [7], and also by Drummond-Cole *et al.* [8,9], is also possible for "coupled systems" [10] where particularly attractive ones are systems of moving *disjoint* balls in space where the configuration spaces of these systems support rich homology structures that are induced from the classifying spaces of (subgroups of) infinite symmetric groups $S_{\infty = I}$ [11], that is expanded/corrected in [12].

A mathematical study of "loose structures" such as what you find in biology and linguistics needs generalisations that would allow a use of relaxed, rather than enhanced, symmetries.

For instance, just to warm up, one may start by elaborating on the category theoretic definition of the entropy suggested "In a Search for a Structure, Part 1: On Entropy" [13], where the entropy of a finite probability space $P = \{p_i\}$, $p_i > 0$, $\sum_i p_i = 1$, comes as the class $[P]_{Gro}$ of P in the *Grothendieck group* $Gro(\mathcal{P})$ of the topological category $\mathcal{P}$ of finite probability spaces P and probability/measure preserving maps $P \to Q$ with a properly defined topological structure in $\mathcal{P}$.

Since *the group $Gro(\mathcal{P})$ is isomorphic to the multiplicative group of positive real numbers* [13]—this is a reformulation of the Bernoulli law of large numbers – the Grothendieck class $[P]_{Gro}$ can be identified with $\exp ent(P)$.

In general, such a Grothendieck-style entropy would be not a *number valued* function of any kind, but (not quite) a functor from an elaborate combinatorial (not quite) category, e.g., comprised of fragments of a natural language with some (not always composable) "morphisms/arrows" between them, to some "simple category" e.g., the category of weighted trees.

The so modified probability/entropy theory is badly needed for designing algorithms that would model what we call *(ego)learning* described in " Ergostructures, Ergodic and the Universal Learning Problem" [14] and in " Understanding Languages and Making Dictionaries" [15], (in preparation) but I have not progressed much in pursuing this direction yet.

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Conflicts of Interest

The author declares no conflict of interest.

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