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Article

Information Geometry on the κ-Thermostatistics

Tatsuaki Wada [1,*] and Antonio M. Scarfone [2]

[1] Department of Electrical and Electronic Engineering, Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan

[2] Istituto dei Sistemi Complessi (ISC-CNR) c/o Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy; E-Mail: antonio.scarfone@polito.it

* Author to whom correspondence should be addressed; E-Mail: wada@mx.ibaraki.ac.jp.

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Abstract: We explore the information geometric structure of the statistical manifold generated by the κ-deformed exponential family. The dually-flat manifold is obtained as a dualistic Hessian structure by introducing suitable generalization of the Fisher metric and affine connections. As a byproduct, we obtain the fluctuation-response relations in the κ-formalism based on the κ-generalized exponential family.

Keywords: κ-entropy; κ-exponential; κ-logarithm; information geometry; Fisher metric; dually-flat; fluctuation-response relation

1. Introduction

Information geometry [1] is a powerful framework for studying the family of probability distributions by applying the geometric tools developed in affine differential geometry. It has a dualistic structure of affine connections, which appears to be very useful and has been applied to many scientific fields, such as information theory, statistics, neural networks, statistical physics, and so on. Information geometry was introduced in 1980s and has been mainly applied to the exponential family of probability distributions. Recently, it has attracted great attention for studying some deformed exponential families of probability distributions. Indeed, the α-geometry introduced by Amari [1] is deeply related to the q-deformed exponential family [2], which plays a fundamental role in Tsallis generalization [3] of thermostatistics. The q-deformed relative entropy is related to the α-geometry on the statistical manifold with a constant curvature [2]. Ohara *et al.* [4] have obtained a dually-flat structure on the space of the escort probabilities by applying ± 1-conformal transformation to the α-geometry. On the other hand, Naudts [5] has generalized Callen's thermostatistics [6], named generalized thermostatistics, by introducing the so-called ϕ-exponential function, which is his generalization of the standard exponential function. In [7,8], he studied the information geometric structure for the ϕ-exponential family. The ϕ-exponential function is defined by the inverse of the ϕ-logarithm that is a generalization of the standard logarithm and is defined by:

$$\ln_\phi x \equiv \int_1^x \frac{1}{\phi(s)} ds, \tag{1}$$

where $\phi(s)$ is a strictly increasing function. When we choose $\phi(s) = s$, the ϕ-logarithm reduces to the standard logarithm $\ln x$. In addition, Matsuzoe and Henmi [9] have considered the Hessian structure on Naudts' ϕ-deformed exponential family. For notational reasons, they used the term "χ-deformed functions" in their paper instead of "ϕ-deformed functions". They showed that a deformed exponential family has two kinds of dualistic Hessian structures and conformal structures of Hessian metrics. Remarkably, the ϕ-exponential function includes the q-exponential [3] and the κ-exponential [10,11] as a special case. In particular, the κ-exponential is defined by:

$$\exp_\kappa(x) \equiv \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}, \tag{2}$$

for a real deformed parameter $\kappa \in (-1, 1)$. We studied the information geometric structures based not directly on the ϕ-deformed functions, but by means of a different method, both for the q-exponential function [12] and for the κ-exponential function [13].

In this contribution, we explore further the information geometric structures of the statistical manifold generated by the κ-deformed probability distribution:

$$p(x;\boldsymbol{\theta}) = \alpha \exp_\kappa\left[\frac{1}{\lambda}\left(\sum_{m=1}^{M} \theta^m f_m(x) - \gamma(\boldsymbol{\theta})\right)\right]. \tag{3}$$

A key point of our construction is to choose the appropriate κ-deformed functions that are consistent with the κ-generalized MaxEnt principle. The Legendre structures in both the information geometry and the κ-thermostatistics are shown to be consistent with each other and play a fundamental role. The next section shows some preliminaries on the basics of the information geometry, especially focused on the exponential family. As an example of the exponential families in statistical mechanics, we study the grand-canonical ensemble and derive the fluctuation-response relations for the thermal equilibrium systems in the present framework. In Section 3, we explore the information geometry for the κ-deformed exponential family. We explore the Hessian structure associated with the Legendre relations for the κ-entropy. The final section is devoted to the conclusions.

2. Preliminaries

Information geometry is a powerful framework for studying a family:

$$\mathcal{S} = \left\{ p(x;\boldsymbol{\theta}) \,\middle|\, p(x;\boldsymbol{\theta}) > 0, \int dx\, p(x;\boldsymbol{\theta}) = 1 \right\}, \tag{4}$$

of probability distribution functions (pdfs) $p(x;\boldsymbol{\theta})$ of a stochastic variable that takes a real value x and is characterized by a set of the real parameters $\boldsymbol{\theta} = (\theta^1, \theta^2 \ldots, \theta^M)$. $\mathcal{S}$ is called a (M-dimensional) statistical model. Under the appropriate conditions, $\mathcal{S}$ can be regarded as a differential manifold $\mathcal{M}$ with local coordinates $\{\theta^i\}$, endowed with a Fisher information matrix g^{F}_{ij} [1]:

$$g^{\mathrm{F}}_{ij}(\boldsymbol{\theta}) = \mathrm{E}_p\left[\partial_i \ell_\theta(x)\, \partial_j \ell_\theta(x)\right], \quad i, j = 1, 2, \ldots, M, \tag{5}$$

as a Riemannian metric and an affine connections, where $\ell_\theta(x) \equiv \ln p(x;\boldsymbol{\theta})$. Here and hereafter, $\mathrm{E}_p[\cdot]$ stands for the linear expectation with respect to the pdf $p(x;\boldsymbol{\theta})$ and $\partial_i = \partial/\partial\theta^i$. Though the Fisher information matrix is generally semi-positive definite, we assume g^{F} to be positive definite, and all of its components are assumed to be finite.

A manifold $\mathcal{M}$ is called e-flat (exponential-flat) if a set of coordinates system $\boldsymbol{\theta}$ satisfies:

$$\mathrm{E}_p\left[\partial_i \partial_j \ell_\theta(x)\, \partial_k \ell_\theta(x)\right] = 0, \quad \forall\, i, j, k, \tag{6}$$

identically. Any set of coordinates $\boldsymbol{\theta}$ satisfying (6) is called e-affine coordinates. A well-known example of e-flat manifolds is the exponential family:

$$\mathcal{S}_{\mathrm{exp}} = \left\{ p(x;\boldsymbol{\theta}) \,\middle|\, p(x;\boldsymbol{\theta}) = \exp\left[\sum_{m=1}^{M} \theta^m f_m(x) - \Psi(\boldsymbol{\theta})\right], \int dx\, p(x;\boldsymbol{\theta}) = 1 \right\}, \tag{7}$$

where $f_m(x)$ are given functions of a random value x and $\Psi(\boldsymbol{\theta})$ is the normalization factor. The condition (6) is satisfied for the exponential family because

$$\mathrm{E}_p\left[\partial_i \ell_\theta(x)\right] = 0, \tag{8}$$

which is due to the normalization of the pdf and

$$\partial_i \partial_j \ell_\theta(x) = -\partial_i \partial_j \Psi(\boldsymbol{\theta}), \tag{9}$$

does not depend on x.

A manifold $\mathcal{M}$ is said m-flat (mixture-flat) if a coordinate system $\boldsymbol{\eta}$ satisfies:

$$\mathrm{E}_p\left[\frac{1}{p(x;\boldsymbol{\eta})} \partial^i \partial^j p(x;\boldsymbol{\eta})\, \partial^k \ln p(x;\boldsymbol{\eta})\right] = 0, \quad \forall\, i, j, k, \tag{10}$$

identically, where $\partial^i = \partial/\partial\eta_i$, and in this case, the set of coordinates $\boldsymbol{\eta}$ is called m-affine coordinates. A well-known example of m-flat manifolds is the mixture family:

$$\mathcal{S}_{\mathrm{mix}} = \left\{ p(x;\boldsymbol{\theta}) \,\middle|\, p(x;\boldsymbol{\theta}) = \sum_{j=1}^{n} \eta_j r_j(x) + \left(1 - \sum_{j=1}^{n} \eta_j\right) r_{n+1}(x), \int dx\, p(x;\boldsymbol{\theta}) = 1 \right\}, \tag{11}$$

where $r_j(x), j = 1, \ldots, n+1$ are given probability distributions for a random variable taking a value x, and $\eta_j \geq 0, \sum_{j=1}^{n} \eta_j \leq 1$.

For the exponential family, we have:

$$\partial_i \ell_\theta(x) = f_i(x) - \partial_i \Psi(\boldsymbol{\theta}). \tag{12}$$

Taking the expectation of the both sides and using Equation (8), we see that the m-affine coordinates ($\boldsymbol{\eta}$-coordinates) of the exponential family are given by:

$$\eta_i = \mathrm{E}_p\left[f_i(x)\right] = \partial_i \Psi(\boldsymbol{\theta}). \tag{13}$$

Accounting for Equations (12) and (13), from Definition (5), we obtain

$$g^{\mathrm{F}}_{ij}(\boldsymbol{\theta}) = \mathrm{E}_p\left[\left(f_i - \mathrm{E}_p\left[f_i\right]\right)\left(f_j - \mathrm{E}_p\left[f_j\right]\right)\right], \quad i,j = 1,2,\ldots,M, \tag{14}$$

which is the covariance matrix for the statistical model $\mathcal{S}_{\mathrm{exp}}$.

In a dully flat structure, the relationship between $\boldsymbol{\theta}$- and $\boldsymbol{\eta}$-coordinates is given by the Legendre transformation:

$$\Psi(\boldsymbol{\theta}) + \Psi^*(\boldsymbol{\eta}) - \boldsymbol{\theta}\cdot\boldsymbol{\eta} = 0, \tag{15}$$

$$\theta^i = \partial^i \Psi^*(\boldsymbol{\eta}), \tag{16}$$

$$\eta_i = \partial_i \Psi(\boldsymbol{\theta}), \tag{17}$$

where $\Psi(\boldsymbol{\theta})$ and $\Psi^*(\boldsymbol{\eta})$ are Legendre dual to each other and are called θ- and η-potential functions, respectively. In other words, when $\mathcal{S}$ is a dually-flat manifold, both the e-affine and m-affine coordinates ($\boldsymbol{\theta}$ and $\boldsymbol{\eta}$) are connected by the Legendre transformation, and the tangent vectors $\boldsymbol{e}_i$ of the coordinate curves θ^i and those $\boldsymbol{e}^j$ of the coordinate curves η_j are orthonormal at every point on the manifold:

$$\langle \boldsymbol{e}_i, \boldsymbol{e}^j \rangle = \mathrm{E}_p\left[\partial_i \ln p(x;\boldsymbol{\theta})\, \partial^j \ln p(x;\boldsymbol{\eta})\right] = \delta_i^j. \tag{18}$$

As is well known, maximizing the Boltzmann–Gibbs–Shannon (BGS) entropy:

$$S^{\mathrm{BGS}} \equiv -\int dx\, p(x) \ln p(x) = \mathrm{E}_p\left[-\ln p\right], \tag{19}$$

under the M-constraints:

$$\mathrm{E}_p\left[f_m(x)\right] = U_m, \quad m = 1,2,\cdots,M, \tag{20}$$

for a given set of U_m and the normalization $\int dx\, p(x) = 1$, leads to the optimized pdf:

$$p(x;\boldsymbol{\theta}) = \exp\left(\sum_{m=1}^{M} \theta^m f_m(x) - \Psi(\boldsymbol{\theta})\right), \tag{21}$$

which belongs to the exponential family $\mathcal{S}_{\mathrm{exp}}$. The control parameters $\{\theta^m\}$ are the Lagrange multipliers for the above M-constraints. From the normalization of the pdf (21), we readily obtain the θ-potential function $\Psi(\boldsymbol{\theta})$ as:

$$\Psi(\boldsymbol{\theta}) = \ln\left(\int dx\, \exp\left[\sum_{m=1}^{M} \theta^m f_m(x)\right]\right). \tag{22}$$

At this point, we observe that, in addition to Equation (5), the Fisher metric g^{F} can be written equivalently in other different expressions:

$$g^{\mathrm{F}}_{ij} = \int dx\, \partial_i p(x;\boldsymbol{\theta})\, \partial_j \ell_\theta(x) \tag{23}$$

$$= -\int dx\, p(x;\boldsymbol{\theta})\, \partial_i\, \partial_j \ell_\theta(x) \tag{24}$$

$$= \int dx\, \frac{1}{p(x;\boldsymbol{\theta})}\, \partial_i p(x;\boldsymbol{\theta}) \partial_j p(x;\boldsymbol{\theta}). \tag{25}$$

In particular, combining Equations (9) with (24), we readily confirm the important relation:

$$g^{\mathrm{F}}_{ij} = \partial_i \partial_j \Psi(\boldsymbol{\theta}), \tag{26}$$

that is, the Fisher metric coincides with the Hessian matrix of the θ-potential function $\Psi(\boldsymbol{\theta})$. It is known that an exponential family naturally has the dualistic Hessian structures, and their canonical divergences coincide with the Kullback–Leibler divergences. Furthermore, using Equation (13), the Fisher matrix can be also rewritten as:

$$g^{\mathrm{F}}_{ij} = \partial_i \eta_j = \partial_i \mathrm{E}_p \left[f_j \right], \tag{27}$$

which holds for the exponential family $\mathcal{S}_{\mathrm{exp}}$.

In general, the dual affine connections are induced by the metric. By applying ∂_i to Equation (23) for g^{F}, we see that the next relation holds:

$$\partial_i g^{\mathrm{F}}_{jk} = \Gamma^{(e)}_{ij,k} + \Gamma^{(m)}_{ij,k}, \tag{28}$$

where the Christoffel symbol of the first kind for the e-affine connection and that for the m-affine connection are defined by:

$$\Gamma^{(e)}_{ij,k} \equiv \int dx\, \partial_k p(x;\boldsymbol{\theta}) \partial_i \partial_j \ell_\theta(x) = \mathrm{E}_p \left[\partial_k \ell_\theta \, \partial_i \partial_j \ell_\theta \right], \tag{29}$$

$$\Gamma^{(m)}_{ij,k} \equiv \int dx\, \partial_i \partial_j p(x;\boldsymbol{\theta}) \partial_k \ell_\theta(x) = \mathrm{E}_p \left[\frac{1}{p(x;\boldsymbol{\theta})} \partial_i \partial_j p(x;\boldsymbol{\theta}) \, \partial_k \ell_\theta \right], \tag{30}$$

respectively. In addition, we can introduce a cubic form:

$$C_{ijk} \equiv \Gamma^{(m)}_{ij,k} - \Gamma^{(e)}_{ij,k}, \tag{31}$$

which characterizes the difference between the affine connection $\nabla^{(e)}$ (or $\nabla^{(m)}$) and the Levi–Civita connection $\nabla^{(0)}$ through the relations:

$$\Gamma^{(e)}_{ij,k} = \Gamma^{(0)}_{ij,k} - \frac{1}{2} C_{ijk}, \tag{32}$$

$$\Gamma^{(m)}_{ij,k} = \Gamma^{(0)}_{ij,k} + \frac{1}{2} C_{ijk}. \tag{33}$$

A well-known example of the exponential family in statistical physics is given by the grand-canonical ensemble, which describes an equilibrium thermal system characterized by a constant temperature T and a constant chemical potential μ. The pdf of the grand-canonical ensemble [6] is given by:

$$p^{\mathrm{G}}(x) = \frac{1}{Z^{\mathrm{G}}(\beta,\mu)} \exp\left[-\beta E_N(x) + \beta\mu N \right], \tag{34}$$

with the grand-canonical partition function:

$$Z^{\mathrm{G}}(\beta,\mu) = \sum_{N=0}^{\infty} \int dx\, \exp[-\beta E_N(x) + \beta\mu N]. \tag{35}$$

Here, $\beta = 1/(k_B T)$ stands for inverse temperature, k_B for the Boltzmann constant, N for the number of particles, and $E_N(x)$ is the energy of the system with N particles for a given configuration x.

The pdf (34) can be cast into the exponential form (21) by choosing $\theta^1 = -\beta, \theta^2 = \beta\mu, f_1(x) = E_N(x), f_2(x) = N$ and:

$$\Psi(\boldsymbol{\theta}) = \ln Z^{\mathrm{G}}(\beta,\mu) = \Phi(\beta,\mu), \tag{36}$$

where $\Phi(\beta,\mu)$ is the Massieu potential [6].

From (26), we see that the corresponding Fisher information matrix g^{FG} for the grand canonical ensemble is the Hessian matrix of $\ln Z^G(\beta,\mu)$ (or that of $\Phi(\beta,\mu)$):

$$g_{ij}^{\mathrm{FG}} = \partial_i\partial_j \ln Z^G(\beta,\mu) = \partial_i\partial_j\Phi(\beta,\mu), \tag{37}$$

whereas from Relation (13), we see that the η-coordinates are given by:

$$\eta_1^G = -\frac{\partial}{\partial\beta}\ln Z^G(\beta,\mu) = \mathrm{E}_p\left[E_N\right], \tag{38}$$

$$\eta_2^G = \frac{\partial}{\partial(\beta\mu)}\ln Z^G(\beta,\mu) = \mathrm{E}_p\left[N\right], \tag{39}$$

which are nothing but the thermodynamic Legendre relations in statistical mechanics. Consistently, the corresponding η-potential function is the negative of the BGS entropy:

$$\Psi^*(\boldsymbol{\eta}) = -\beta\mathrm{E}_p\left[E_N\right] + \beta\mu\mathrm{E}_p\left[N\right] - \ln Z_G(\beta,\mu) = -S^{\mathrm{BGS}}. \tag{40}$$

Finally, from Relation (27), we have

$$g^{\mathrm{FG}} = \begin{pmatrix} -\frac{\partial}{\partial\beta}\mathrm{E}_p\left[E_N\right] & -\frac{\partial}{\partial\beta}\mathrm{E}_p\left[N\right] \\ \frac{\partial}{\partial(\beta\mu)}\mathrm{E}_p\left[E_N\right] & \frac{\partial}{\partial(\beta\mu)}\mathrm{E}_p\left[N\right] \end{pmatrix}, \tag{41}$$

that coincides with the susceptibilities matrix. From Relation (14), the same Fisher metric can be written in:

$$g^{\mathrm{FG}} = \begin{pmatrix} \mathrm{E}_p\left[E_N^2\right] - \mathrm{E}_p\left[E_N\right]^2 & \mathrm{E}_p\left[E_N\,N\right] - \mathrm{E}_p\left[E_N\right]\mathrm{E}_p\left[N\right] \\ \mathrm{E}_p\left[E_N\,N\right] - \mathrm{E}_p\left[E_N\right]\mathrm{E}_p\left[N\right] & \mathrm{E}_p\left[N^2\right] - \mathrm{E}_p\left[N\right]^2 \end{pmatrix}, \tag{42}$$

which describes the square correlations (or fluctuations) of the stochastic variables E_N and N. From this expression, we see that g^{FG} is actually a positive definite matrix, because of Jensen's inequalities:

$$\mathrm{E}_p\left[E_N^2\right] > \mathrm{E}_p\left[E_N\right]^2, \quad \mathrm{E}_p\left[N^2\right] > \mathrm{E}_p\left[N\right]^2, \tag{43}$$

and others. We remark that the thermodynamic stability of the grand-canonical ensemble is due to the convexity of the relevant potential function $\ln Z^G(\beta,\mu)$.

In addition, equating the two different expressions, (41) and (42), we obtain the fluctuation-response relation for systems in equilibrium [6], which is a well-known statement in statistical mechanics relating the spontaneous thermodynamic fluctuations to thermodynamic responses (or susceptibility).
We also observe that the grand-canonical Fisher matrix is symmetric $g_{ij}^{\mathrm{FG}} = g_{ji}^{\mathrm{FG}}$, as is obvious from Equation (42). Then, accounting for Equation (41), we obtain the relation:

$$-\frac{\partial}{\partial\beta}\mathrm{E}_p\left[N\right] = \frac{\partial}{\partial(\beta\mu)}\mathrm{E}_p\left[E_N\right], \tag{44}$$

which is one among the many Maxwell relations that are well known in the thermodynamics theory.

3. Information Geometric Structures in the κ-Thermostatistics

The κ-deformed entropy [10,11] is defined by:

$$S_\kappa \equiv -\int dx\, p(x) \ln_\kappa p(x) = \mathrm{E}_p\left[-\ln_\kappa p\right], \tag{45}$$

which mimics the BGS entropy by replacing the standard logarithm with the κ-logarithm:

$$\ln_\kappa x \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa} = \frac{1}{\kappa}\sinh\big(\kappa \ln x\big), \tag{46}$$

for $x > 0$ and a real-parameter $\kappa \in (-1, 1)$. The inverse function of $\ln_\kappa x$ is the κ-exponential function $\exp_\kappa(x)$ introduced in Equation (2). In the $\kappa \to 0$ limit, the κ-exponential and the κ-logarithm reduce to the standard exponential $\exp(x)$ and logarithm $\ln x$, respectively.

There exists another κ-deformed function:

$$u_\kappa(x) \equiv \frac{x^\kappa + x^{-\kappa}}{2} = \cosh\big(\kappa \ln x\big), \tag{47}$$

which is conjugate to $\ln_\kappa x$. In the $\kappa \to 0$ limit, this function $u_\kappa(x)$ reduces to the unit constant function $u_0(x) = 1$. As with the case that the κ-entropy S_κ defined as the expectation of $-\ln_\kappa p(x)$, we introduce the function:

$$\mathcal{I}_\kappa \equiv \int dx\, p(x) u_\kappa\big(p(x)\big) = \mathrm{E}_p\left[u_\kappa(p)\right], \tag{48}$$

as the expectation of $u_\kappa(p(x))$. We will see later that the function $\mathcal{I}_\kappa$ plays an important role in the Legendre structures concerning the κ-entropy S_κ.

Let us consider the maximum κ-entropy problem [14] under the M-constraints:

$$\mathrm{E}_p\left[f_m\right] = U_m, \quad m = 1, 2, \cdots, M, \tag{49}$$

for a given set of U_m and the normalization $\int dx\, p(x) = 1$

$$\max_{p(x)} \left(S_\kappa - \sum_{m=1}^{M} \theta^m \int dx\, f_m(x) p(x) - \gamma \int dx\, p(x) \right), \tag{50}$$

where $\{\theta^m\}$ and γ are Lagrange multipliers. In order to solve this problem, we introduce two constants α and λ, which satisfy the condition:

$$\frac{d}{dx}\Big(x \ln_\kappa x\Big) = \lambda \ln_\kappa\left(\frac{x}{\alpha}\right). \tag{51}$$

This relation holds for any $x > 0$ if

$$\alpha = \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{1}{2\kappa}}, \quad \lambda = \sqrt{1-\kappa^2}, \tag{52}$$

which are related to each other according to

$$\ln_\kappa\left(\frac{1}{\alpha}\right) = \frac{1}{\lambda}. \tag{53}$$

Using Relation (51), we obtain the optimal solution of the MaxEnt problem as:

$$p(x;\boldsymbol{\theta}) = \alpha \exp_\kappa \left[\frac{1}{\lambda} \left(\sum_{m=1}^{M} \theta^m f_m(x) - \gamma(\boldsymbol{\theta}) \right) \right]. \tag{54}$$

We hence consider the statistical manifold generated by the κ-exponential family [13],

$$\mathcal{S}_{\kappa\text{-exp}} = \left\{ p(x,\boldsymbol{\theta}) \,\middle|\, p(x,\boldsymbol{\theta}) = \alpha \exp_\kappa \left[\frac{1}{\lambda} \left(\sum_{m=1}^{M} \theta^m f_m(x) - \gamma(\boldsymbol{\theta}) \right) \right], \int dx\, p_\kappa(x,\boldsymbol{\theta}) = 1 \right\}. \tag{55}$$

We comment on the relations between Naudts' ϕ-deformed functions and the κ-deformed functions in this study. They are related in

$$\begin{aligned} \exp_\phi(x) &\leftrightarrow \alpha \exp_\kappa \left(\frac{x}{\lambda} \right), \\ \ln_\phi x &\leftrightarrow \lambda \ln_\kappa \left(\frac{x}{\alpha} \right). \end{aligned} \tag{56}$$

From the useful identity [13]:

$$\lambda \ln_\kappa \left(\frac{x}{\alpha} \right) = \ln_\kappa x + u_\kappa(x), \tag{57}$$

we see that

$$\begin{aligned} \mathcal{I}_\kappa - S_\kappa &= \mathrm{E}_p \left[u_\kappa \big(p(x;\boldsymbol{\theta}) \big] + \mathrm{E}_p \left[\ln_\kappa \big(p(x;\boldsymbol{\theta}) \big) \right] = \mathrm{E}_p \left[\lambda \ln_\kappa \left(\frac{p(x;\boldsymbol{\theta})}{\alpha} \right) \right] \\ &= \sum_{m=1}^{M} \theta^m \, \mathrm{E}_p \left[f_m \right] - \gamma(\boldsymbol{\theta}). \end{aligned} \tag{58}$$

By comparing this last relation with Equations (15)-(17), we realize that Equation (58) corresponds to the Legendre relation:

$$\Psi_\kappa(\boldsymbol{\theta}) = \boldsymbol{\theta} \cdot \boldsymbol{\eta} - \Psi^\star_\kappa(\boldsymbol{\eta}), \tag{59}$$

for the κ-deformed exponential family with $\eta_m = \mathrm{E}_p\left[f_m \right]$ and the θ- and η- potential functions given by:

$$\Psi_\kappa(\boldsymbol{\theta}) = \mathcal{I}_\kappa(\boldsymbol{\theta}) + \gamma(\boldsymbol{\theta}), \qquad \Psi^\star_\kappa(\boldsymbol{\eta}) = -S_\kappa(\boldsymbol{\eta}). \tag{60}$$

Next, we introduce the κ-escort distribution $P(x)$ [13] with respect to a pdf $p(x)$ by:

$$P(x) \equiv \frac{1}{\mathcal{U}_\kappa} \frac{p(x)}{\lambda u_\kappa \left(\frac{p(x)}{\alpha} \right)}, \tag{61}$$

where $\mathcal{U}_\kappa$ is the normalization factor:

$$\mathcal{U}_\kappa \equiv \int dx \, \frac{p(x)}{\lambda u_\kappa \left(\frac{p(x)}{\alpha} \right)}, \tag{62}$$

and the corresponding κ-escort expectation $\mathbb{E}_P\left[A \right]$ of a function $A(x)$ is defined by:

$$\mathbb{E}_P\left[A \right] \equiv \int dx \, P(x) \, A(x). \tag{63}$$

By using the relation:

$$\frac{d}{dx}\exp_\kappa(x) = \frac{\exp_\kappa(x)}{u_\kappa\big(\exp_\kappa(x)\big)}. \tag{64}$$

and accounting for the normalization of the κ-exponential pdf (54), we have:

$$\begin{aligned} 0 = \partial_i \int dx\, p(x;\boldsymbol{\theta}) &= \int dx \frac{1}{\lambda}\big(f_i(x) - \partial_i\gamma(\boldsymbol{\theta})\big) \frac{\alpha \exp_\kappa\left[\frac{1}{\lambda}\big(\sum_m \theta^m f_m(x) - \gamma\big)\right]}{u_\kappa\left(\exp_\kappa\left[\frac{1}{\lambda}\big(\sum_m \theta^m f_m(x) - \gamma\big)\right]\right)} \\ &= \int dx \big(f_i(x) - \partial_i\gamma(\boldsymbol{\theta})\big) \frac{p(x;\boldsymbol{\theta})}{\lambda u_\kappa\left(\frac{p(x;\boldsymbol{\theta})}{\alpha}\right)}. \end{aligned} \tag{65}$$

We thus obtain

$$\partial_i\gamma(\boldsymbol{\theta}) = \mathbb{E}_P\left[f_i\right], \tag{66}$$

i.e., the Lagrange multiplier $\gamma(\boldsymbol{\theta})$ associated with the normalization condition is the θ-potential function associated with the κ-escort expectations $\{\mathbb{E}_P\left[f_i\right]\}$. Note that, since

$$\partial_i\left(\lambda \ln_\kappa\left(\frac{p(x;\boldsymbol{\theta})}{\alpha}\right)\right) = f_i(x) - \partial_i\gamma(\boldsymbol{\theta}), \tag{67}$$

Relation (66) implies

$$\mathbb{E}_P\left[\partial_i\left(\lambda \ln_\kappa\left(\frac{p(x;\boldsymbol{\theta})}{\alpha}\right)\right)\right] = 0. \tag{68}$$

At this point, we introduce the κ-generalization of $\ell_\theta(x)$ given by:

$$\ell_\theta^{(\kappa)} \equiv \lambda \ln_\kappa\left(\frac{p(x;\boldsymbol{\theta})}{\alpha}\right), \tag{69}$$

which reduces to $\ell_\theta(x) + 1$ in the limit of $\kappa \to 0$. Due to Equation (68), the standard expectation of $\partial_i\ell_\theta^{(\kappa)}$ does not vanish:

$$\mathrm{E}_p\left[\partial_i\ell_\theta^{(\kappa)}\right] \neq 0, \tag{70}$$

and consequently $\ell_\theta^{(\kappa)}$ is not an appropriate candidate for the κ-generalized representation. We then consider a modified κ-representation given by:

$$\tilde{\ell}_\theta^{(\kappa)} \equiv \ell_\theta^{(\kappa)} - \mathcal{I}_\kappa, \tag{71}$$

which fulfills the relation:

$$\mathrm{E}_p\left[\partial_i\tilde{\ell}_\theta^{(\kappa)}\right] = 0, \tag{72}$$

as similar as Equation (8) in the standard exponential case. In addition, since $\lim_{\kappa\to 0}\mathcal{I}_\kappa = 1$, we see that $\tilde{\ell}_\theta^{(\kappa)}$ reduces to $\ell_\theta(x)$ in the limit of $\kappa \to 0$.

With the help of Equation (59), we can derive the following relations:

$$\tilde{\ell}_{\theta}^{(\kappa)} = \sum_{m=1}^{M} \theta^m f_m(x) - \gamma - \mathcal{I}_\kappa = \sum_{m=1}^{M} \theta^m f_m(x) - \Psi_\kappa(\boldsymbol{\theta}), \tag{73}$$

$$\partial_i \, \tilde{\ell}_{\theta}^{(\kappa)} = f_i(x) - \partial_i \, \Psi_\kappa(\boldsymbol{\theta}), \tag{74}$$

$$\partial_i \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)} = -\partial_i \partial_j \, \Psi_\kappa(\boldsymbol{\theta}), \tag{75}$$

and from Equations (72) and (74), we obtain

$$0 = \mathrm{E}_p \left[\partial_i \tilde{\ell}_{\theta}^{(\kappa)} \right] = \mathrm{E}_p \left[f_i \right] - \partial_i \, \Psi_\kappa(\boldsymbol{\theta}). \tag{76}$$

Since $\Psi_\kappa(\boldsymbol{\theta})$ is the θ-potential function, we have

$$\eta_i \equiv \partial_i \Psi_\kappa(\boldsymbol{\theta}) = \mathrm{E}_p \left[f_i \right], \tag{77}$$

in agreement with Equation (13), and consequently, they are the $\boldsymbol{\eta}$-coordinates for the κ-formalism.

We now introduce the κ-generalized metric as the Hessian matrix of the κ-deformed θ-potential function:

$$g_{ij}^{(\kappa)} \equiv \partial_i \partial_j \, \Psi_\kappa(\boldsymbol{\theta}). \tag{78}$$

Since Equation (75) does not depend on x, we can write

$$\begin{aligned} g_{ij}^{(\kappa)} &= -\mathrm{E}_p \left[\partial_i \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)} \right] = -\int dx \, p(x; \boldsymbol{\theta}) \, \partial_i \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)} \\ &= \int dx \, \partial_i \, p(x; \boldsymbol{\theta}) \, \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)}, \end{aligned} \tag{79}$$

where the last expression follows from the relation:

$$\partial_i \mathrm{E}_p \left[\partial_j \tilde{\ell}_{\theta}^{(\kappa)} \right] = \int dx \, \partial_i p(x; \boldsymbol{\theta}) \, \partial_j \tilde{\ell}_{\theta}^{(\kappa)} + \int dx \, p(x; \boldsymbol{\theta}) \partial_i \partial_j \tilde{\ell}_{\theta}^{(\kappa)} = 0. \tag{80}$$

Equation (79) corresponds to Equation (24) for the standard Fisher matrix.

The κ-deformed Christoffel symbol of the first kind for the e-connection is defined by:

$$\Gamma_{ij,k}^{(\kappa e)} \equiv \int dx \, \partial_i \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)} \, \partial_k p(x; \boldsymbol{\theta}) = \mathrm{E}_p \left[\partial_i \partial_j \, \tilde{\ell}_{\theta}^{(\kappa)} \, \partial_k \, \ell_\theta \right], \tag{81}$$

so that the κ-exponential family $\mathcal{S}_{\mathrm{exp}}^{\kappa}$ is e-flat, because from Equation (75):

$$\Gamma_{ij,k}^{(\kappa e)} = -\partial_i \partial_j \, \Psi_\kappa(\boldsymbol{\theta}) \int dx \, \partial_k p(x; \boldsymbol{\theta}) = 0. \tag{82}$$

Similarly, the inverse of the κ-generalized metric $g_{ij}^{(\kappa)}$ is obtained from the Hessian matrix of the κ-deformed η-potential function:

$$g^{(\kappa)\, ij} = \partial^i \partial^j \, \Psi_\kappa^*(\boldsymbol{\eta}). \tag{83}$$

This can be confirmed as follows. By utilizing the Legendre relation (59) and $\theta^m = \partial^m \Psi_\kappa^*(\boldsymbol{\eta})$, we can express $\tilde{\ell}_{\theta}^{(\kappa)}$ as a function of $\boldsymbol{\eta}$:

$$\begin{aligned} \tilde{\ell}_{\theta}^{(\kappa)} &= \sum_{m=1}^{M} \theta^m f_m(x) - \Psi_\kappa(\boldsymbol{\theta}) = \sum_{m=1}^{M} \theta^m \eta_m - \Psi_\kappa(\boldsymbol{\theta}) + \sum_{m=1}^{M} \theta^m \Big(f_m(x) - \eta_m \Big) \\ &= \Psi_\kappa^*(\boldsymbol{\eta}) + \sum_{m=1}^{M} \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \Big(f_m(x) - \eta_m \Big). \end{aligned} \tag{84}$$

Then, the derivative of $\tilde{\ell}_\theta^{(\kappa)}$ with respect to η_i becomes

$$\begin{aligned}\partial^i \tilde{\ell}_\theta^{(\kappa)} &= \partial^i \Psi_\kappa^*(\boldsymbol{\eta}) + \sum_{m=1}^M \partial^i \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \Big(f_m(x) - \eta_m \Big) - \sum_{m=1}^M \partial^m \Psi_\kappa^*(\boldsymbol{\eta})\, \partial^i \eta_m \\ &= \sum_{m=1}^M \partial^i \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \Big(f_m(x) - \eta_m \Big),\end{aligned} \tag{85}$$

where we used $\partial^i \eta_m = \delta_m^i$. Taking the linear expectation of the both sides of (85), we see $\mathrm{E}_p \Big[\partial^i \tilde{\ell}_\theta^{(\kappa)} \Big] = 0$. Then, we have

$$\partial^i \mathrm{E}_p \Big[\partial^j \tilde{\ell}_\theta^{(\kappa)} \Big] = \int dx\, \partial^i p(x;\boldsymbol{\theta})\, \partial^j \tilde{\ell}_\theta^{(\kappa)} + \mathrm{E}_p \Big[\partial^i \partial^j \tilde{\ell}_\theta^{(\kappa)} \Big] = 0. \tag{86}$$

By using this relation and Equation (85), we thus confirm that:

$$\begin{aligned} g^{(\kappa)\, ij} &= -\mathrm{E}_p \Big[\partial^i \partial^j \tilde{\ell}_\theta^{(\kappa)} \Big] = \int dx\, \partial^i p(x;\boldsymbol{\theta})\, \partial^j \tilde{\ell}_\theta^{(\kappa)} = \sum_{m=1}^M \partial^j \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \int dx\, \partial^i p(x;\boldsymbol{\eta}) \Big(f_m(x) - \eta_m \Big) \\ &= \partial^j \partial^m \Psi_\kappa^*(\boldsymbol{\eta})\, \partial^i \mathrm{E}_p [f_m] = \partial^i \partial^j\, \Psi_\kappa^*(\boldsymbol{\eta}).\end{aligned} \tag{87}$$

Next, we consider the κ-deformed Christoffel symbol of the first kind for the m-connection, which is defined by:

$$\Gamma^{(\kappa m)\, ij,k} \equiv \int dx\, \partial^i \partial^j p(x;\boldsymbol{\theta})\, \partial^k\, \tilde{\ell}_\theta^{(\kappa)} = \mathrm{E}_p \left[\frac{1}{p(x;\boldsymbol{\theta})} \partial^i \partial^j p(x;\boldsymbol{\theta})\, \partial^k\, \tilde{\ell}_\theta^{(\kappa)} \right]. \tag{88}$$

Taking the derivative of the metric $g^{(\kappa)\, jk}$ with respect to η_i, we have

$$\begin{aligned} \partial^i g^{(\kappa)\, jk} &= \partial^i \Big(\int dx \partial^j p(x;\boldsymbol{\theta}) \partial^k \tilde{\ell}_\theta^{(\kappa)} \Big) = \int dx\, \partial^i \partial^j p(x;\boldsymbol{\theta}) \partial^k \tilde{\ell}_\theta^{(\kappa)} + \int dx\, \partial^j p(x;\boldsymbol{\theta}) \partial^i \partial^k \tilde{\ell}_\theta^{(\kappa)} \\ &= \mathrm{E}_p \left[\frac{1}{p(x;\boldsymbol{\theta})} \partial^i \partial^j p(x;\boldsymbol{\theta})\, \partial^k \tilde{\ell}_\theta^{(\kappa)} \right] + \mathrm{E}_p \Big[\partial^j \ell_\theta\, \partial^i \partial^k \tilde{\ell}_\theta^{(\kappa)} \Big] \\ &= \Gamma^{(\kappa m)\, ij,k} + \Gamma^{(\kappa e)\, ik,j},\end{aligned} \tag{89}$$

which is a similar relation of (28). Note that the expression of $\Gamma^{(\kappa e)\, ik,j}$ is obtained by rising the indexes of Equation (81).

From Relation (85), we have

$$\begin{aligned} \Gamma^{(\kappa e)\, ik,j} &= \int dx\, \partial^j p(x;\boldsymbol{\theta}) \partial^i \sum_{m=1}^M \partial^k \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \Big(f_m(x) - \eta_m \Big) \\ &= \int dx\, \partial^j p(x;\boldsymbol{\theta}) \sum_{m=1}^M \Big\{ \partial^i \partial^k \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \big(f_m(x) - \eta_m \big) - \partial^i \partial^m \Psi_\kappa^*(\boldsymbol{\eta}) \partial^i \eta_m \Big\} \\ &= \sum_{m=1}^M \partial^i \partial^k \partial^m \Psi_\kappa^*(\boldsymbol{\eta})\, \partial^j \mathrm{E}_p [f_m] = \partial^i \partial^j \partial^k \Psi_\kappa^*(\boldsymbol{\eta}) = \partial^i g^{(\kappa)\, jk},\end{aligned} \tag{90}$$

where we used $\partial^i \mathrm{E}_p [f_m] = \partial^i \eta_m = \delta_m^i$. Substituting Relation (90) into Equation (89), we obtain $\Gamma^{(\kappa m)\, ij,k} = 0$, *i.e.*, the κ-exponential family $\mathcal{S}_{\kappa\text{-exp}}$ is also m-flat. Therefore, the κ-deformed statistical manifold $(\mathcal{S}_{\kappa\text{-exp}}, g^\kappa, \nabla^\kappa)$ has a dually-flat structure.

We can further elaborate the expression of the metric $g^{(\kappa)}$. In fact, accounting for Equation (77), we can rewrite Equation (78) in:

$$g_{ij}^{(\kappa)} = \partial_i \, \mathrm{E}_p \left[f_j \right] , \tag{91}$$

and taking into account the relation:

$$\partial_i p(x;\boldsymbol{\theta}) = \frac{p(x;\boldsymbol{\theta})}{\lambda u_\kappa\left(\frac{p(x;\boldsymbol{\theta})}{\alpha}\right)} \left(f_i(x) - \partial_i \gamma\right) = \mathcal{U}_\kappa P(x;\boldsymbol{\theta}) \left(f_i(x) - \partial_i \gamma\right) , \tag{92}$$

we obtain

$$\begin{aligned} g_{ij}^{(\kappa)} &= \int dx\, \partial_i p(x;\boldsymbol{\theta}) f_j(x) = \mathcal{U}_\kappa \int dx\, P(x;\boldsymbol{\theta})\, \left(f_i(x) - \partial_i \gamma\right)\, f_j(x) \\ &= \mathcal{U}_\kappa \int dx\, P(x;\boldsymbol{\theta})\, \left(f_i(x) - \mathbb{E}_P\left[f_i\right]\right)\, f_j(x) = \mathcal{U}_\kappa\, \mathbb{E}_P\left[\left(f_i(x) - \mathbb{E}_P\left[f_i\right]\right) f_j(x)\right] \\ &= \mathcal{U}_\kappa\, \mathbb{E}_P\left[\left(f_i - \mathbb{E}_P\left[f_i\right]\right)\left(f_j - \mathbb{E}_P\left[f_j\right]\right)\right] . \end{aligned} \tag{93}$$

This expression has the following meaning: the response function $\partial_i \, \mathrm{E}_p \left[f_j \right]$ associated with the standard expectation $\mathrm{E}_p \left[f_j \right]$ is related to the fluctuation associated with the κ-escort expectation, which states the κ-generalization of the standard fluctuation-response relation, as pointed out firstly by Naudts [7]. It is also clear, from the final result in Equation (93) that the metric g_{ij}^{κ} is symmetric in its indexes, and accounting for Equation (91), we see that:

$$\partial_i \, \mathrm{E}_p \left[f_j \right] = \partial_j \, \mathrm{E}_p \left[f_i \right] . \tag{94}$$

In concluding this section, let us specify our results to the grand-canonical ensemble described by the pdf:

$$p_\kappa^{\mathrm{G}}(x) = \alpha \, \exp_\kappa \left[\frac{1}{\lambda} \Big(- \beta E_N(x) + \beta \mu N - \gamma(\beta,\mu) \Big) \right] . \tag{95}$$

It belongs to the κ-exponential family (55) with $\theta^1 = -\beta$, $\theta^2 = \beta\mu$ and $f_1(x) = E_N(x)$, $f_2(x) = N$. The grand-canonical potential coincides with $\Psi_\kappa(\boldsymbol{\theta})$ and also corresponds to the κ-generalized Massieu potential:

$$\Psi_\kappa(\boldsymbol{\theta}) = \mathcal{I}_\kappa(\beta,\mu) + \gamma(\beta,\mu) = \Phi_\kappa^G(\beta,\mu), \tag{96}$$

whereas its dual, $\Psi_\kappa^*(\boldsymbol{\eta})$, is the negative of the κ-entropy:

$$\Psi_\kappa^*(\boldsymbol{\eta}) = -S_\kappa(\mathrm{E}_p\left[E_N\right], \mathrm{E}_p\left[N\right]). \tag{97}$$

The η-coordinates are given by the relations:

$$\eta_1^G = -\frac{\partial}{\partial \beta} \Phi_\kappa^G(\beta,\mu), \tag{98}$$

$$\eta_2^G = -\frac{\partial}{\partial (\beta\mu)} \Phi_\kappa^G(\beta,\mu), \tag{99}$$

and coincide with the energy average $\mathrm{E}_p[E_N]$ and the particle number average $\mathrm{E}_p[N]$, respectively, as can be verified by a direct calculation. All of Relations (96)–(99) fulfill consistently the Legendre transformations (15).

Finally, the corresponding κ-generalized metric becomes

$$\begin{aligned} g^{(\kappa)\,\mathrm{G}} &= \begin{pmatrix} -\frac{\partial}{\partial\beta}\mathrm{E}_p[E_N] & -\frac{\partial}{\partial\beta}\mathrm{E}_p[N] \\ \frac{\partial}{\partial(\beta\mu)}\mathrm{E}_p[E_N] & \frac{\partial}{\partial(\beta\mu)}\mathrm{E}_p[N] \end{pmatrix} \\ &= \mathcal{U}_\kappa \begin{pmatrix} \mathbb{E}_P[E_N^2] - \mathbb{E}_P[E_N]^2 & \mathbb{E}_P[E_N N] - \mathbb{E}_P[E_N]\,\mathbb{E}_P[N] \\ \mathbb{E}_P[E_N N] - \mathbb{E}_P[E_N]\,\mathbb{E}_P[N] & \mathbb{E}_P[N^2] - \mathbb{E}_P[N]^2 \end{pmatrix}, \end{aligned} \tag{100}$$

from which we can promptly read the energy-particle fluctuation-response relations in the κ-formalism. Again, from this last expression, we see that the κ-generalized metric is actually a positive definite matrix, because of Jensen's inequalities.

4. Conclusions

We have studied the information geometric structures of the statistical manifold generated by the κ-deformed exponential family $\mathcal{S}_{\kappa\text{-exp}}$. Our construction of the κ-statistical manifold is based on the appropriate κ-deformed functions (56), which are consistent with the κ-generalized MaxEnt principle for the κ-entropy S_κ. We have constructed the κ-deformed statistical manifold $(\mathcal{S}_{\kappa\text{-exp}}, g^{(\kappa)}, \nabla^{(\kappa)})$, which has a dually-flat structure. As a byproduct, we obtained the κ-generalized fluctuation-response relations (100) based on our κ-generalized exponential family.

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Author Contributions

Both authors have contributed to the study and preparation of the article. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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