Supplementary Materials: The Bogdanov-Takens Normal Form: A Minimal Model For Single Neuron Dynamics

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Contents

1. Jordan Block and Jordan Basis for CB Models in the Neighborhood of a Double Zero Bifurcation

1.1. A Jordan Block Always Arise

We will prove that in any CB model, if there is a double zero bifurcation, a Jordan block always will arise and therefore one has a Bogdanov-Takens bifurcation. If we meet the double zero eigenvalue condition for CB models (Equations (16) and (17) in the main text), the critical linear matrix is written as

$$\mathbb{L}^c = \begin{pmatrix} 0 & \beta_0 M_1 & \dots & \beta_0 M_N \\ 0 & \alpha_1 + \beta_1 M_1 & \dots & \beta_1 M_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta_N M_1 & \dots & \alpha_N + \beta_N M_N \end{pmatrix}$$

It is immediate to see that the vector $\underline{\chi}^{(0)}$ below is an eigenvector for the eigenvalue 0.

$$\underline{\chi}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

With a little bit of algebra, it can be shown that the second condition in the main text (Equation (17)) is equivalent to impose that the submatrix $\mathbb{L}^c_{N\times N}$ has a determinant equal to zero. If this holds, it always exists an N component vector $\underline{\tilde{\chi}}^{(1)} = (\tilde{x}_1, \tilde{x}_2 \dots, \tilde{x}_{N-1}, \tilde{x}_N)$ such that

$$\mathbb{L}^c_{N\times N}\underline{\tilde{\chi}}^{(1)} = 0$$

Then if we consider the vector (ν is arbitrary)

$$\underline{\chi}^{(1)} \equiv \frac{1}{\beta_0 \sum_{j=1}^{N} M_j \tilde{x}_j} \begin{pmatrix} \nu \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_N \end{pmatrix} \quad \text{with} \qquad \nu \in \mathbb{R}$$

this vector $\underline{\chi}^{(1)}$ is the second vector of the Jordan basis of the critical matrix $\mathbb{L}^c$ as the following equation trivially shows

$$\mathbb{L}^c\underline{\chi}^{(1)} = \begin{pmatrix} 0 & \beta_0 M_1 & \dots & \beta_0 M_N \\ 0 & \alpha_1 + \beta_1 M_1 & \dots & \beta_1 M_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta_N M_1 & \dots & \alpha_N + \beta_N M_N \end{pmatrix} \begin{pmatrix} \nu \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \frac{\beta_0 \sum_{j=1}^{N} M_j \tilde{x}_j}{\beta_0 \sum_{j=1}^{N} M_j \tilde{x}_j} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{\chi}^{(0)}$$

We have then proved that the critical matrix of a generic double zero bifurcation in CB models will always have two vectors such that

$$\begin{aligned} \mathbb{L}^c\underline{\chi}^{(0)} &= 0 \\ \mathbb{L}^c\underline{\chi}^{(1)} &= \underline{\chi}^{(0)} \end{aligned}$$

which is the definition of a Jordan block for a double zero bifurcation. Therefore, we proved that when any CB model undergoes a double zero bifurcation this bifurcation always will be a Bogdanov-Takens bifurcation.

1.2. Analytical Expression for the Jordan Basis

In Section 1.1 we show that the first vector of the Jordan basis is

$$\underline{\chi}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

Therefore, the equation that must be solved to explicitly find the second vector of the Jordan basis is

$$\mathbb{L}^c \underline{\chi}^{(1)} = \underline{\chi}^{(0)}$$

Then, the explicit linear equations read

$$\begin{array}{cccccccc} \beta_0 M_1 x_1 & + & \beta_0 M_2 x_2 + & \cdots & \cdots & +\beta_0 M_N x_N & = & 1 \\ (\alpha_1 + \beta_1 M_1) x_1 & + & \beta_1 M_2 x_2 + & \cdots & \cdots & +\beta_1 M_N x_N & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & = & \vdots \\ \beta_j M_1 x_1 & + & \beta_j M_2 x_2 + & \cdots & (\alpha_j + \beta_j M_j) x_j \cdots & +\beta_j M_N x_N & = & 0 \\ \vdots & \cdots & \vdots + & \vdots & \vdots & \vdots & = & \vdots \\ \beta_N M_1 x_1 & + & \beta_N M_2 x_2 + & \cdots & \cdots & +(\alpha_N + \beta_N M_N) x_N & = & 0 \end{array} \tag{2}$$

Let us number these $N + 1$ equations beginning with the first equations as the equation 0 and the last one as equation N. If we perform the following operation with the equations

$$\text{equation } j) \times \beta_0 - \text{equation } 0) \times \beta_j$$

We find that

$$\alpha_j \beta_0 x_j = -\beta_j$$

Because neither α_j ,β_0 or β_j are singular

$$x_j = -\frac{\beta_j}{\alpha_j \beta_0}$$

And if we choose $\nu = 0$ we find that the second vector of the Jordan basis, which we call $\underline{\chi}^{(1)}$, also belongs to the two dimensional Jordan block subspace and it is given by

$$\underline{\chi}^{(1)} = -\frac{1}{\beta_0} \begin{pmatrix} 0 \\ \frac{\beta_1}{\alpha_1} \\ \frac{\beta_2}{\alpha_2} \\ \vdots \\ \frac{\beta_N}{\alpha_N} \end{pmatrix} \tag{3}$$

It is important to notice that if we plug-in the Expression (3) in the original linear Equations (2) the equations are fulfilled and are written in terms of the second condition for the Bogdanov-Takens Bifurcation (Equation (17) in the main text).

Now we have an analytical expression for the two vectors $\underline{\chi}^{(0)}$ and $\underline{\chi}^{(1)}$, and they belong to the Jordan block subspace. To find the other $N - 1$ vectors of the Jordan base, let us suppose the most generic case when all the rest of the $N-1$ eigenvalues ($\{\lambda_2, \lambda_3 \ldots \lambda_N\}$) are different. Therefore, the general equation for the $N-1$ vectors is ($l = 2, 3, \ldots, N$)

$$\mathbb{L}^c \underline{\chi}^{(l)} = \lambda_l \underline{\chi}^{(l)}$$

Then the equations are

$$\begin{array}{ccccc}
-\lambda_l x_0^{(l)} + \beta_0 M_1 x_1^{(l)} + & \cdots & +\beta_0 M_N x_N^{(l)} & = & 0 \\
(\alpha_1 + \beta_1 M_1 - \lambda_l) x_1^{(l)} + & \cdots & +\beta_1 M_N x_N^{(l)} & = & 0 \\
\vdots & \vdots & \vdots & = & \vdots \\
\beta_j M_1 x_1^{(l)} + & \cdots (\alpha_j + \beta_j M_j - \lambda_l) x_j^{(l)} & +\beta_j M_N x_N^{(l)} & = & 0 \\
\vdots & \vdots & \vdots & = & \vdots \\
\beta_N M_1 x_1^{(l)} + & \cdots & +(\alpha_N + \beta_N M_N - \lambda_l) x_N^{(l)} & = & 0
\end{array} \tag{4}$$

Doing the following operation with the equations

$$\text{equation } j)\beta_0 - \text{equation } 0) \times \beta_j$$

we find that

$$x_j^{(l)} = \frac{\beta_j}{\beta_0} \frac{\lambda_l}{\lambda_l - \alpha_j} x_0^{(l)}$$

Then we find that

$$\underline{\chi}^{(l)} = \frac{\lambda_l x_0^{(l)}}{\beta_0} \begin{pmatrix} 1 \\ \frac{\beta_1}{\lambda_l - \alpha_1} \\ \frac{\beta_2}{\lambda_l - \alpha_2} \\ \vdots \\ \frac{\beta_N}{\lambda_l - \alpha_N} \end{pmatrix} \tag{5}$$

Where $x_0^{(l)}$ is a free parameter. Similar to the previous case, if we plug-in the Expression (5) in the original linear Equation (4) the equations are fulfilled. But now the expression that arises is written in terms of the analytical general expression of the characteristic polynomial (Equation (15) of the main text).

2. Two Time Scales Gating Variables

In this section we will analyse when the two time scales gating variables scenario is fullfiled. We want to prove the statement: if the two time scales gating variables condition holds, then the sum of the fast gating functions must be amplifying (negative) and the sum of the slow gating functions must be resonant (positive).

Let us assume that we have two sets of gating variables in a CB model: the fast gating variables with relaxation times of the order of τ_F (set of variables F) and the slow gating variables with relaxation times of the order of T_S (set of variables S), i.e.

$$\begin{array}{lr}
\tau_k \sim \tau_F & k \in F \\
\tau_l \sim T_S & l \in S \\
\text{with } \quad \frac{\tau_F}{T_S} \ll 1 &
\end{array} \tag{6}$$

Hence, the Bogdanov-Takens conditions (Equations (16) and (17) in the main text) are written as

$$1 + \tau_0 \left[\sum_{l \in S} \beta_l M_l + \sum_{k \in F} \beta_k M_k \right] = 0 \tag{7}$$

$$1 - T_S \sum_{l \in S} \beta_l M_l - \tau_F \sum_{k \in F} \beta_k M_k = 0 \tag{8}$$

If we define

$$\begin{aligned} BM_S &\equiv \sum_{l \in S} \beta_l M_l \\ BM_F &\equiv \sum_{k \in F} \beta_k M_k \end{aligned}$$

the Equations (7) and (8) become

$$1 + \tau_0 \left[BM_S + BM_F \right] = 0 \tag{9}$$

$$1 - T_S BM_S - \tau_F BM_F = 0 \tag{10}$$

Then, after some algebra

$$BM_S = \frac{1}{T_S} \frac{1 + \frac{\tau_F}{\tau_0}}{1 - \frac{\tau_F}{T_S}} \tag{11}$$

$$BM_F = -\frac{1}{T_S} \frac{1 + \frac{T_S}{\tau_0}}{1 - \frac{\tau_F}{T_S}} \tag{12}$$

Since $\tau_F / T_S \ll 1$ we conclude that

$$BM_S > 0 \tag{13}$$

$$BM_F < 0 \tag{14}$$

Proving our claim.

3. Quadratic Coefficient Bogdanov-Takens Normal Form

In this section we will show that the quadratic term of the *force* in the Bogdanov-Takens normal form is proportional to the second derivative of the I-V curve. Let us consider the generic equation for a conductance based model (Equation (9), main text)

$$\begin{aligned} \dot{u} &= I(t) - I^{\infty}(u; \vec{\sigma}_T, \vec{\eta}) - I^T(u, \vec{x}; \vec{\sigma}_T, \vec{\eta}) \\ \dot{x}_j &= -\frac{x_j}{\tau_j(u; \vec{\sigma}_j)} + \beta_j(u; \vec{\sigma}_j) \dot{u} \qquad j = 1, 2, \ldots, N \end{aligned} \tag{15}$$

And define

$$
\begin{aligned}
M_i^{(1)} &\equiv -\left.\frac{\partial I^T(u,\vec{x};\vec{\sigma}^T,\vec{\eta})}{\partial x_i}\right|_{u=u^*,x_i=0} \\
M_{i,j}^{(2)} &\equiv -\left.\frac{\partial^2 I^T(u,\vec{x};\vec{\sigma}^T,\vec{\eta})}{\partial x_i \partial x_j}\right|_{u=u^*,x_i=0} \\
M_{i,u}^{(2)} &\equiv -\left.\frac{\partial^2 I^T(u,\vec{x};\vec{\sigma}^T,\vec{\eta})}{\partial x_i \partial u}\right|_{u=u^*,x_i=0} \\
F^{(1)} &\equiv -\left.\frac{\partial I^\infty(u;\vec{\sigma}^T,\vec{\eta})}{\partial u}\right|_{u=u^*} \\
F^{(2)} &\equiv -\left.\frac{\partial^2 I^\infty(u;\vec{\sigma}^T,\vec{\eta})}{\partial u^2}\right|_{u=u^*} \\
\alpha_i^{(0)} &\equiv -\frac{1}{\tau_i(u^*;\vec{\sigma}_i)} \qquad i=1,\ldots,N \\
\alpha_i^{(1)} &\equiv -\left.\frac{\partial}{\partial u}\left(\frac{1}{\tau_i(u^*;\vec{\sigma}_i)}\right)\right|_{u=u^*} \\
\beta_i^{(1)} &\equiv -\left.\frac{\partial m_i^\infty(u;\vec{\sigma}_i)}{\partial u}\right|_{u=u^*} \\
\beta_i^{(2)} &\equiv -\left.\frac{\partial^2 m_i^\infty(u;\vec{\sigma}_i)}{\partial^2 u}\right|_{u=u^*}
\end{aligned}
$$

Where u^* is a Bogdanov-Takens bifurcation point and the indexes go from 1 to N. For simplicity, let us re-write Equation (15) defining $\underline{V} = (\delta u, \delta x_1, \cdots, \delta x_N)$. Then Equation (15) expanded around the bifurcation points up to the quadratic order read

$$
\begin{aligned}
\dot{V}_0 &= \sum_{j=1}^{N} M_j^{(1)} V_j - \frac{1}{2} F^{(2)} V_0^2 + \sum_{j=1}^{N} M_{j,u}^{(2)} V_0 V_j + \frac{1}{2}\sum_{j=1}^{N} M_{j,j}^{(2)} V_j^2 + \frac{1}{2}\sum_{i\neq j}^{N} M_{i,j}^{(2)} V_i V_j + \mathcal{O}(3) \\
\dot{V}_l &= \alpha_i^{(0)} V_l + \beta_j \sum_{j=1}^{N} M_j^{(1)} V_j + \sum_{j=i}^{N} \alpha_j^{(1)} V_0 V_j + \beta_i^{(2)} \sum_{j=1}^{N} M_j^{(1)} V_0 V_j + \\
&\quad \beta_l^{(1)} \left(-\frac{1}{2} F^{(2)} V_0^2 + \sum_{j=1}^{N} M_{j,u}^{(2)} V_0 V_j + \frac{1}{2}\sum_{j=1}^{N} M_{j,j}^{(2)} V_j^2 + \frac{1}{2}\sum_{i\neq j}^{N} M_{i,j}^{(2)} V_i V_j \right) + \mathcal{O}(3)
\end{aligned} \tag{16}
$$

To calculate the normal form, we need to write Equation (16) in the Jordan base. Using the results obtained in Section 1.2 we define the change of base matrix as

$$
\mathbb{S} = [\underline{\chi}^{(0)}, \underline{\chi}^{(1)}, \underline{\chi}^{(2)}, \cdots, \underline{\chi}^{(N)}]
$$

With analytical expressions for the Jordan base, we can perform all the calculations of the normal form explicitly. As Elphick, Tirapegui, Brachet, Coullet and Iooss showed in [1], using the inner product that the authors define, we can write the adjoint of the homologic operator (that depends on the critical linear matrix, $\mathcal{A}(\hat{\mathbb{J}}^c)$) of any dynamical system as the homologic operator of the adjoint of the critical linear matrix, i.e. $\mathcal{A}(\hat{\mathbb{J}}^c)^\dagger = \mathcal{A}([\hat{\mathbb{J}}^c]^\dagger)$. Therefore, using this inner product and knowing the form of the linear matrix operator in a given basis projected in the critical subspace ($\hat{\mathbb{J}}^c$), we can find all the terms of the normal form of a given bifurcation. When one works in the basis in which the original critical linear matrix is in its Jordan form, the homologic operator has the form

$$
\mathcal{A}(\hat{\mathbb{J}}^c) = \mathbb{J}^c_{\alpha,\beta} c_\beta \frac{\partial}{\partial c_\alpha} - \hat{\mathbb{J}}^c \tag{17}
$$

And the the adjoint of the homologic operator $\mathcal{A}(\hat{\mathbb{J}}^c)$ is

$$\mathcal{A}(\hat{\mathbb{J}}^c)^\dagger = (\mathbb{J}^c_{\beta,\alpha})^* c_\alpha \frac{\partial}{\partial c_\beta} - (\hat{\mathbb{J}}^c)^\dagger \tag{18}$$

Where c_α are the variables of the normal form. In the case of Bogdanov-Takens the operator $\hat{\mathbb{J}}^c$ has the form

$$\mathbb{J}^c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{19}$$

Then the adjoint of the homologic operator reads

$$\mathcal{A}(\hat{\mathbb{J}}^c)^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} c_1 \frac{\partial}{\partial c_2} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{20}$$

Therefore, if the vector $\underline{\xi}(c_1, c_2)$= $\begin{pmatrix} \xi_1(c_1, c_2) \\ \xi_2(c_1, c_2) \end{pmatrix}$ is an element of the kernel of the adjoint of the homologic operator, then

$$\begin{aligned} c_1 \tfrac{\partial \xi_1(c_1,c_2)}{\partial c_2} & = 0 \\ c_1 \tfrac{\partial \xi_2(c_1,c_2)}{\partial c_2} & -\xi_1(c1, c2) = 0 \end{aligned}$$

It is straightforward to show that at order m in the components (c_1, c_2) the vector $\underline{\xi}(c_1, c_2)$ has the form

$$\begin{aligned} \xi_1(c_1, c_2) = & \quad c_1 \psi^{(m-1)}(c_1) \\ \xi_2(c_1, c_2) = & \quad c_2 \psi^{(m-1)}(c_1) + \varphi^{(m)}(c_1) \end{aligned}$$

where $\varphi^{(m)}(c_1)$ is a monomial in c_1 of order m and $\psi^{(m-1)}(c_1)$ is a monomial in c_1 of order $m-1$. Therefore, the kernel of the homologic operator for the Bogdanov-Takens bifurcation written in the Jordan basis is

$$\text{Kernel}_{BT}\left[\mathcal{A}(\hat{\mathbb{J}}^c)^\dagger\right] = \left\{\psi^{(m-1)}(c_1)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \varphi^{(m)}(c_1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \tag{21}$$

To obtain the normal form, we must impose the general solubility condition for linear equations of the form $A\vec{x} = \vec{b}$ where A is a linear operator in a finite-dimensional vector space (a matrix), $\vec{x}$ the unknown vector and $\vec{b}$ a given vector. This condition (Fredholm alternative) is that $\vec{b}$ must be orthogonal to the adjoint A^* of A in any nondegenerate scalar product defined in the vector space. In our case we have two critical variables (c_1, c_2) and the original physical variables of the CB models $\underline{V} = (u, x_1, x_2, \ldots, x_N)$ are expressed in terms of (c_1, c_2) in a series of the form $\underline{V} = \underline{U}^{[1]}(c_1, c_2) + \underline{U}^{[2]}(c_1, c_2) + \underline{U}^{[3]}(c_1, c_2) + \ldots$. Where $\underline{U}^{[r]}(c_1, c_2)$ is a vector whose components are polynomials of order r in the variables (c_1, c_2), and at each polynomial order r we have to solve the homological equation $\mathcal{A}(\hat{\mathbb{J}}^c)\underline{U}^{[r]}(c_1, c_2) = \underline{I}^{[r]}(c_1, c_2) - \underline{f}^{[r]}(c_1, c_2)$, $r = 1, 2, \ldots$. Where $\underline{I}^{[r]}(c_1, c_2)$ is a known vector determined by the previous orders and $\underline{f}^{[r]}(c_1, c_2) = \begin{pmatrix} f_1^{[r]}(c_1, c_2) \\ f_2^{[r]}(c_1, c_2) \end{pmatrix}$ are also unknown, and they determine the r^{th} polynomial order in the differntial equations of the normal form which are

$$\partial_t c_1 = f_1^{[1]}(c_1, c_2) + f_1^{[2]}(c_1, c_2) + f_1^{[3]}(c_1, c_2) + \ldots$$

$$\partial_t c_2 = f_2^{[1]}(c_1, c_2) + f_2^{[2]}(c_1, c_2) + f_2^{[3]}(c_1, c_2) + \ldots$$

$\underline{I}^{[n]}$ and $\underline{f}^{[n]}$ have the form:

$$\underline{I}^{[n]} = \begin{pmatrix} \sigma^{(1)}_{n,0}c_1^n + \sigma^{(1)}_{n-1,1}c_1^{n-1}c_2 + \cdots\cdots + \sigma^{(1)}_{1,n-1}c_1c_2^{n-1} + \sigma^{(1)}_{0,n}c_2^n \\ \sigma^{(2)}_{n,0}c_1^n + \sigma^{(2)}_{n-1,1}c_1^{n-1}c_2 + \cdots\cdots + \sigma^{(2)}_{1,n-1}c_1c_2^{n-1} + \sigma^{(2)}_{0,n}c_2^n \end{pmatrix}$$
$$\underline{f}^{[n]} = \begin{pmatrix} \nu^{(1)}_{n,0}c_1^n + \nu^{(1)}_{n-1,1}c_1^{n-1}c_2 + \cdots\cdots + \nu^{(1)}_{1,n-1}c_1c_2^{n-1} + \nu^{(1)}_{0,n}c_2^n \\ \nu^{(2)}_{n,0}c_1^n + \nu^{(2)}_{n-1,1}c_1^{n-1}c_2 + \cdots\cdots + \nu^{(2)}_{1,n-1}c_1c_2^{n-1} + \nu^{(2)}_{0,n}c_2^n \end{pmatrix}$$

With $\sigma^{(1,2)}_{n-j,j}$ and $\nu^{(1,2)}_{n-j,j}$ coefficients of the monomial in c_1 and c_2. The functions $(f_1^{[m]}(c_1,c_2), f_2^{[m]}(c_1,c_2))$ are determined by the soulubility condition applied to the linear homological equation through the equations

$$\left\langle \underline{I}^{(m)} - \underline{f}^{(m)}, \psi^{(m-1)}(c_1) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right\rangle = 0 \tag{22}$$

$$\left\langle \underline{I}^{(m)} - \underline{f}^{(m)}, \varphi^{(m)}(c_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0 \tag{23}$$

We obtain using the inner product described in [1] that for any order we must have

$$n(\sigma^{(1)}_{n,0} - \nu^{(1)}_{n,0}) + (\sigma^{(2)}_{n-1,1} - \nu^{(2)}_{n-1,1}) = 0 \tag{24}$$

$$\sigma^{(2)}_{n,0} - \nu^{(2)}_{n,0} = 0 \tag{25}$$

These two last equations can be satisfied in more than one way and we shall use this freedom. Apart from this general feature, these two last equations leave an inherent freedom to incorporate to the normal from $\underline{f}^{(m)}$ elements that do not belong to the Kernel of the adjoint of the homologic operator; a freedom which exists in any normal form. In the Bogdanov-Takens bifurcation we have two extreme choices to write the normal form: The Arnold's choice and The Takens choice. In the Arnold's choice $\sigma^{(1)}_{n,0} = 0$ and $\sigma^{(2)}_{n-1,1} = n\nu^{(1)}_{n,0} + \nu^{(2)}_{n-1,1}$ and $\sigma^{(2)}_{n,0} = \nu^{(2)}_{n,0}$ and the normal form can be written as a perturbed hamiltonian system [1]. The other extreme choice is the Takens choice with $\sigma^{(2)}_{n-1,1} = 0$ and $\sigma^{(1)}_{n,0} - = \nu^{(2)}_{n-1,1}/n + \nu^{(2)}_{n-1,1}$ and $\sigma^{(2)}_{n,0} = \nu^{(2)}_{n,0}$. Because in the Arnold's choice we gain all the Hamiltonian intuition, in this work we will use the Arnold form for the Bogdanov-Takens normal form. Then we have that the coefficient of the quadratic term of the normal form is

$$\sigma^{(2)}_{2,0} = \nu^{(2)}_{2,0}$$

Writing Equation (16) in the Jordan base, we have that

$$\sigma^{(2)}_{2,0} = \nu^{(2)}_{2,0} = -\frac{1}{2}F^{(2)}\left[(\mathbb{S}^{-1})_{1,0} + \sum_{j=1}^{N}(\mathbb{S}^{-1})_{1,j}\beta_j^{(1)}\right]$$

Then we have shown that in the notation of the main text

$$\gamma_2 \propto \left.\frac{\partial^2 I^\infty(u;\vec{\sigma}_T,\vec{\eta})}{\partial u^2}\right|_{u=u^*}$$

4. The Morris-Lecar Model

4.1. The Model

The model has three conductances: Potassium, Calcium and a leak. In the simplest version of the model, the Calcium current depends instantaneously on the voltage. Then there is no differential equation for the variable m. The Morris-Lecar is representative of the conductance based models that have excitability class 1 and Class 2 [2], and the global and local bifurcations characteristics of this kind of neuronal dynamics [3]. The mathematical formulation of the Morris-Lecar model is

$$\dot{v} = \frac{1}{C_m}\left[I - g_K n(v-E_K) - g_{Ca} m^{\infty}(v)(v-E_{Ca}) - g_L(v-E_L)\right] \tag{26}$$

$$\dot{n} = \phi \frac{n^{\infty}(v) - n}{\tau(v)} \tag{27}$$

with

$$m^{\infty}(v) = \frac{1}{2}\left(1 + \tanh\frac{v - V1}{V2}\right) \tag{28}$$

$$n^{\infty}(v) = \frac{1}{2}\left(1 + \tanh\frac{v - V3}{V4}\right) \tag{29}$$

$$\tau(v) = \frac{1}{\cosh\frac{v-V3}{2V4}} \tag{30}$$

To transform the equations to a dimensionless form we can use the scaling

Table S1. Dimensionless parameters for the Morris-Lecar model.

I	g_1	g_2	g_3	u_1	u_2	u_3	c	a_1	b_1	a_2	b_2
$\frac{I}{g_L\|E_L\|}$	$\frac{g_K}{g_L}$	$\frac{g_{Ca}}{g_L}$	1	$\frac{E_K}{\|E_L\|}$	$\frac{E_{Ca}}{\|E_L\|}$	1	$\frac{g_L}{\phi Cm}$	$\|E_L\|/V_4$	$-V_3/V_4$	$\|E_L\|/V_2$	$-V_1/V_2$

We scale time as

$$t = \frac{C_m}{g_L}\bar{t}$$

and the variable u as

$$u = \frac{v}{|E_L|}$$

Then the new equations are

$$\dot{u} = I - g_1 n(u-u_1) - g_2 m^{\infty}(u)(u-u_2) - g_3(u-u_3) \tag{31}$$

$$\dot{n} = \frac{n^{\infty}(u) - n}{\tau(u)} \tag{32}$$

with

$$m^{\infty}(u) = \frac{1}{2}\left[1 + \tanh\left(a_2 u + b_2\right)\right] \tag{33}$$

$$n^{\infty}(u) = \frac{1}{2}\left[1 + \tanh\left(a_1 u + b_1\right)\right] \tag{34}$$

$$\tau(u) = \frac{c}{\cosh\left(\frac{a_1 u + b_1}{2}\right)} \tag{35}$$

Using the non singular change of variable proposed in the main text we have

$$x = n - n^{\infty}(u) \tag{36}$$

Using definition (8) in the main text we also have

$$I^{\infty}(u) = g_1 n^{\infty}(u)(u-u_1) + g_2 m^{\infty}(u)(u-u_2) + g_3(u-u_3) \tag{37}$$

Then doing the calculations that we developed for the general case in the main text we obtain

$$\begin{aligned}\dot{u} &= I - I^{\infty}(u) - g_1 x(u-u_1) && (38)\\ \dot{x} &= -\frac{x}{\tau(u)} - \frac{\partial n^{\infty}}{\partial u}\left[I - I^{\infty}(u) - g_1 x(u-u_1)\right] && (39)\end{aligned}$$

Using the definition from the main text

$$\beta(u) = -\frac{\partial n^{\infty}}{\partial u} = -\frac{1}{2}a_1 \mathrm{sech}^2(a_1 u + b_1)$$

the equations finally take the form

$$\begin{aligned}\dot{u} &= I - I^{\infty}(u) - g_1 x(u-u_1) && (40)\\ \dot{x} &= -\frac{x}{\tau(u)} + \beta(u)\dot{u} && (41)\end{aligned}$$

4.2. Morris-Lecar in a second order derivative form

We begin by doing the time derivative in (40), which leads to

$$\ddot{u} = -\frac{\partial f(u)}{\partial u}\dot{u} - g_1 \dot{x}(u-u_1) - g_1 x\dot{u} \qquad (42)$$

Then using Equation (41) in (42) we obtain

$$\ddot{u} = -\frac{\partial f(u)}{\partial u}\dot{u} - g_1(u-u_1)\left\{-\frac{x}{\tau(u)} + \beta(u)\dot{u}\right\} - g_1 x\dot{u} \qquad (43)$$

If we use Equation (40) we have

$$x = \frac{I - f(u)}{g_1(u-u_1)} - \frac{\dot{u}}{g_1(u-u_1)}$$

and if we plug this expression in Equation (43) and consider the definitions for the gating function of the slowest variable n, i.e.

$$G_n = \tau(u) g_2 \left.\frac{\partial n^{\infty}(\bar{u})}{\partial \bar{u}}\right|_{\bar{u}=u}(u-u_2)$$

Then we obtain Moris-Lecar model in its second derivative form

$$\ddot{u} = \frac{I - f(u)}{\tau(u)} - \dot{u}\left(\frac{I - f(u)}{u-u_1} + \frac{\partial f(u)}{\partial u} + \frac{1 - G_n(u)}{\tau(u)}\right) + \frac{\dot{u}^2}{u-u_1} \qquad (44)$$

In the main text we explore the previous equation when we neglect the term $\frac{\dot{u}^2}{u-u_1}$, which is

$$\ddot{u} = \frac{I - f(u)}{\tau(u)} - \dot{u}\left(\frac{I - f(u)}{u-u_1} + \frac{\partial f(u)}{\partial u} + \frac{1 - G_n(u)}{\tau(u)}\right) \qquad (45)$$

We also refer to the reduced model when we additionally neglect the term $-\dot{u}\left(\frac{I-f(u)}{u-u_1}\right)$, which is

$$\ddot{u} = \frac{I - f(u)}{\tau(u)} - \dot{u}\left(\frac{\partial f(u)}{\partial u} + \frac{1 - G_n(u)}{\tau(u)}\right) \qquad (46)$$

5. Simulations Parameters

5.1. Figure 3

The parameters used for simulations presented in Figure 3 are

Table S2. Parameters Figure 3 top: saddle-node homoclinc bifurcation.

α	γ_2	γ_3	β	λ_1	λ_2
$7\cdot10^{-6}$	$-7.7\cdot10^{-4}$	$-3.5\cdot10^{-3}$	$-5\cdot10^{-3}$	-0.1	1

Table S3. Parameters Figure 3 middle: big homoclinc bifurcation.

α	γ_2	γ_3	β	λ_1	λ_2
$1.5\cdot10^{-6}$	$1.9\cdot10^{-5}$	$-2.1\cdot10^{-3}$	$-8.8\cdot10^{-3}$	-0.1	0.35

Table S4. Parameters Figure 3 bottom: saddle homoclinc bifurcation.

α	γ_2	γ_3	β	λ_1	λ_2
$2.8\cdot10^{-5}$	$-1.8\cdot10^{-3}$	$-2.6\cdot10^{-3}$	$-8.3\cdot10^{-3}$	-0.1	0.73

5.2. Figure 4

The parameters used for simulations presented in Figure 4 are

Table S5. Parameters Figure 4 top: saddle-node homoclinc bifurcation.

g_1	g_2	g_3	u_1	u_2	u_3	c	a_1	b_1	a_2	b_2
4	2	1	-1.4	2	-1	1.49	3.45	-0.69	3.33	0.07

Table S6. Parameters Figure 4 middle: big homoclinc bifurcation.

g_1	g_2	g_3	u_1	u_2	u_3	c	a_1	b_1	a_2	b_2
4	2.2	1	-1.4	2	-1.03	3.13	2	-0.07	2.8	0.51

Table S7. Parameters Figure 4 bottom: saddle homoclinc bifurcation.

g_1	g_2	g_3	u_1	u_2	u_3	c	a_1	b_1	a_2	b_2
4	2	1	-1.4	2	-1	0.364	3.45	-0.69	3.36	0

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