Entropy **2015**, *17*, 6925-6936; doi:10.3390/e17106925

OPEN ACCESS

entropy

ISSN 1099-4300

www.mdpi.com/journal/entropy

Article

Modified Legendre Wavelets Technique for Fractional Oscillation Equations

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Academic Editor: Carlo Cattani

Received: 22 July 2015 / Accepted: 15 September 2015 / Published: 16 October 2015

Abstract: Physical Phenomena's located around us are primarily nonlinear in nature and their solutions are of highest significance for scientists and engineers. In order to have a better representation of these physical models, fractional calculus is used. Fractional order oscillation equations are included among these nonlinear phenomena's. To tackle with the nonlinearity arising, in these phenomena's we recommend a new method. In the proposed method, Picard's iteration is used to convert the nonlinear fractional order oscillation equation into a fractional order recurrence relation and then Legendre wavelets method is applied on the converted problem. In order to check the efficiency and accuracy of the suggested modification, we have considered three problems namely: fractional order force-free Duffing–van der Pol oscillator, forced Duffing–van der Pol oscillator and higher order fractional Duffing equations. The obtained results are compared with the results obtained via other techniques.

Keywords: Legendre wavelets method; Picard's iteration; nonlinear problems; fractional oscillation equations

PACS Codes: 02.30.Hq; 02.60.Lj; 02.70.Jn

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1. Introduction

Importance of fractional calculus [1–3] has increased a lot especially over the past few decades. Physical phenomena, describing fractional oscillation equations [4–6], are mainly nonlinear in nature. In general, exact solutions of these governing fractional oscillation equations are not available. Therefore, different techniques for finding approximate analytical solutions of such problems were developed. Recent commonly used techniques are Adomian's Decomposition Method [7,8], Homotopy Perturbation Method [9], Exp-function Method [10], Rational Homotopy Perturbation Method [11], Variational Iteration Method [12] and Wavelets Techniques [13–25]. Wavelet techniques, one of the relatively new techniques, employed for solving wide range of problems related to various branches of engineering and applied sciences. Wavelet techniques are used in image processing, flow injection analysis, infrared spectrometry, chromatography, mass spectrometry, ultraviolet-visible spectrometry and voltammetry. Wavelets are also used to solve certain problems in quantum chemistry and chemical physics, see [13–25] and the references therein. With the passage of time, lots of developments have been taking place in this area, which are helpful in increasing the accuracy of these schemes. The most common related schemes are Haar Wavelets [14], Harmonic Wavelets of successive approximation [14], Legendre Wavelets [15,16,21], CAS Wavelets [17], Wavelet Collocation [18,19,22–25] and Chebyshev Wavelets [20]. It is to be highlighted that Abd-Elhameed and Youssri [22] introduced new spectral solutions of multi-term fractional order initial value problems with error analysis in the recent past. Moreover, Abd-Elhameed *et al.* [23] extended new spectral second kind Chebyshev wavelets algorithm for solving linear and nonlinear second order differential equations involving singular and Bratu type equations. It is worth mentioning that Youssri *et al.* and Doha *et al.* [24–25] developed an excellent scheme which is called Ultraspherical wavelets method and applied the same on Lane–Emden type equations, some other initial and boundary value problems and hence calculated extremely accurate results. Inspired and motivated by ongoing research in this area, we propose Legendre Wavelet-Picard Method (LWPM) to solve the nonlinear fractional oscillation equations. The obtained results are highly encouraging and reflect an excellent level of accuracy. Finally, solutions obtained by LWPM are compared with Variational Iterational Method (VIM) using exact Lagrange multiplied and Ultraspherical Wavelets Collocation Method (UWCM) [24]. It is observed that wavelets basis of the suggested scheme may be obtained as a direct case of Ultraspherical wavelets, see [22–25] and the references therein.

The fractional order forced Duffing–van der Pol oscillator is given by the following second order differential equation [3]:

$$D^{\alpha}y(t) - \mu\big(1 - y^2(t)\big)y'(t) + ay(t) + by^3(t) = g(f,\omega,t), 1 < \alpha \le 2,$$

where D^{α} is the Caputo derivative, $g(f,\omega,t) = f cos(\omega t)$ represents the periodic driving function of time with period $T = 2\pi/\omega$, where ω is the angular frequency of the driving force, f is the forcing strength and $\mu > 0$ is the damping parameter of the system. Duffing–van der Pol oscillator equations can be expressed in three physical situations:

(1) single-well $a > 0, b > 0$;
(2) double-well $a < 0, b > 0$;
(3) double-hump $a > 0, b < 0$.

Caputo's fractional derivative of order α is given by

$$D_a^{\alpha} y(x) = \frac{1}{\Gamma(n-a)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n y(t)dt$$

For $a < x \leq b$, where $n-1 < \alpha \leq n, n\epsilon\mathbb{N}$.

2. Legendre Wavelets and Picard's Iteration

2.1. Legendre Wavelets

Wavelets [20] are defined by the following formula, where a and b are dilation and translation parameters

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a, b \in R, a \neq 0.$$

By restricting the parameters a and b to discrete values as $a = {a_0}^{-k}, b = nb_0{a_0}^{-k}$, $a_0 > 1$, $b_0 > 0$, we have

$$\psi_{k,n}(t) = |a|^{\frac{k}{2}} \psi\left(a_0^k t - nb_0\right), k, n \in \mathbb{Z},$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. Legendre wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ involve four parameters in which $n = 1,2, \cdots, 2^{k-1}, k$ is any positive integer, m is the degree of the Legendre polynomials and t is normalized time. They are defined on the interval $(-1, 1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{L}_m(2^k t - 2n + 1), \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0, \qquad \text{otherwise} \end{cases} \tag{1}$$

where

$$\tilde{L}_m(t) = \sqrt{\tfrac{2}{2m+1}} L_m(t), \tag{2}$$

$m = 0,1,2,\cdots, M-1$. For orthonormality, coefficients are used which are given in Equation (2). Here $L_m(t)$ are the Legendre polynomials of degree m and satisfy the following recursive formula

$$L_0(t) = 1, L_1(t) = t, (m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t), m = 1,2,3, \cdots.$$

Legendre polynomial's are also a special case of Ultrasperical harmonic polynomials [22–25] and can also be derived from these directly.

The solution obtained by Legendre wavelets is of the form

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t)$$

where $\psi_{n,m}(t)$ is given by the Equation (1). We approximate $y(t)$ by the truncated series

$$y_{k,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t). \tag{3}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients $c_{10}, c_{11}, \cdots, c_{1M-1}, c_{20}, c_{21}, \cdots, c_{2M-1}, \cdots, c_{2^{k-1}0}, c_{2^{k-1}1}, \cdots, c_{2^{k-1}M-1}$.

Some equations are furnished by the initial or boundary conditions, while for remaining equations we replace $y_{k.M}(t)$ in our differential equation to recover the unknown coefficients $c_{n,m}$.

Convergence

Convergence of Legendre wavelet method is discussed in [21]. The statement of theorem is as follows:

Theorem 1. The series solution (3) converges to $y(\mathrm{t})$, when $2^{k-1}, M \to \infty$.

2.2. Picard's Iteration

Picard technique is used for solving nonlinear differential equations. Consider the following nonlinear, second order differential equation:

$$\frac{d^2y}{dt^2} = f\left(y, \frac{dy}{dt}\right) + g\left(y, \frac{dy}{dt}\right) + h(t),$$

where $f\left(y, \frac{dy}{dt}\right)$ consists of linear term and $g\left(y, \frac{dy}{dt}\right)$ consists of nonlinear terms only, with conditions

$$y(t_0) = a, \frac{dy(t_1)}{dt} = b.$$

Applying Picard technique to Equation (7) converts it into the form

$$\frac{d^2y_{n+1}}{dt^2} = f\left(y_{n+1}, \frac{dy_{n+1}}{dt}\right) + g\left(y_n, \frac{dy_n}{dt}\right) + h(t)$$

with conditions

$$y_{n+1}(t_0) = a, \frac{dy_{n+1}(t_1)}{dt} = b.$$

3. Applications

Problem 1. Consider the following fractional order forced Duffing–van Der Pol oscillator equation [4]

$$D^\alpha y(t) - \mu\big(1 - y^2(t)\big)y'(t) + ay(t) + by^3(t) = f\cos(\omega t) \ , 1 < \alpha \le 2,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Firstly applying Picard Technique, we have

$$D^\alpha y_{n+1}(t) - \mu {y'}_{n+1}(t) + \mu y_n^2(t)y_n'(t) + ay_{n+1}(t) + by_n^3(t) = f\cos(\omega t) \ , 1 < \alpha \le 2,$$

with initial conditions $y_{n+1}(0) = 1$ and $y_{n+1}'(0) = 0$.

Now applying Legendre wavelets method on the above equation, we have

$$D^{\alpha}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)-\mu\frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+\mu y_n^2(t)y_n'(t)$$
$$+a\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+by_n^3(t)=f\cos(\omega t)\ ,1<\alpha\le 2,$$

with initial approximations $y_1(0)=1$ and $y_1'(0)=0$.

(1) (Single-well $a>0, b>0$). Consider $a=0.5, b=0.5, \mu=0.1, f=0.5, \omega=0.79$. (See Table 1 and Figure 1).

Table 1. Comparison of Single-well solution at Picard's 8th iterationobtained by Legendre Wavelet-Picard Method (LWPM) with Variational Iterational Method (VIM) and RK-4, for M = 6 and $\alpha=2$.

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.0×10^{-9}	1.2×10^{-10}
0.1	0.99750286	0.99750276	0.99750274	0.99750272	4.0×10^{-8}	2.1×10^{-8}
0.2	0.99004534	0.99004513	0.99004508	0.99004504	9.0×10^{-8}	4.3×10^{-8}
0.3	0.97772778	0.97772579	0.97772572	0.97772567	1.2×10^{-7}	5.2×10^{-8}
0.4	0.96071284	0.96070262	0.96070255	0.96070236	2.6×10^{-7}	1.9×10^{-7}
0.5	0.93922114	0.93918360	0.93918327	0.93918299	6.1×10^{-7}	2.8×10^{-7}
0.6	0.91352389	0.91341578	0.91341532	0.91341497	8.1×10^{-7}	3.5×10^{-7}
0.7	0.88393496	0.88367502	0.88367483	0.88367344	1.6×10^{-6}	1.4×10^{-6}
0.8	0.85080112	0.85025195	0.85025145	0.85024907	2.8×10^{-6}	2.1×10^{-6}
0.9	0.81449135	0.81343957	0.81343786	0.81343631	3.3×10^{-6}	2.5×10^{-6}
1.0	0.77538351	0.77352648	0.77352488	0.77352238	4.1×10^{-6}	3.2×10^{-6}

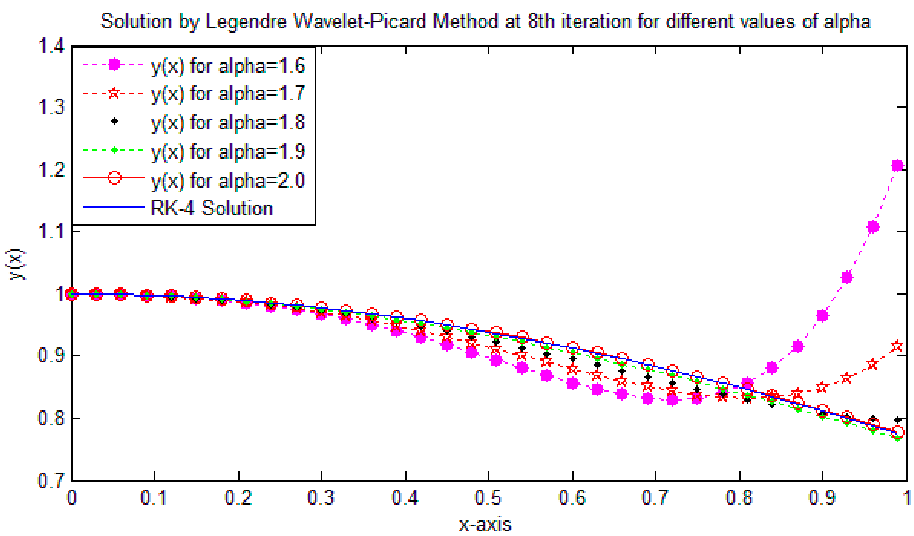

Figure 1. Comparison of solutions for different fractional values by RK-4 solution for single well case.

(2) (Double-well $a < 0, b > 0$). $a = -0.5, b = 0.5, \mu = 0.1, f = 0.5, \omega = 0.79$. (See Table 2 and Figure 2)

Table 2. Comparison of Double-well solution at Picard's 8th iteration obtained by Legendre Wavelet-Picard Method (LWPM) with Variational Iterational Method (VIM) and RK-4, when M = 6 and $\alpha = 2$.

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.0×10^{-10}	1.2×10^{-13}
0.1	1.00249660	1.00249669	1.00249669	1.00249670	1.1×10^{-8}	1.0×10^{-8}
0.2	1.00994530	1.00994541	1.00994542	1.00994545	4.0×10^{-8}	3.1×10^{-8}
0.3	1.02222113	1.02222170	1.02222174	1.02222179	9.0×10^{-8}	5.9×10^{-8}
0.4	1.03911114	1.03911438	1.03911446	1.03911459	2.1×10^{-7}	1.3×10^{-7}
0.5	1.06030866	1.06032195	1.06032214	1.06032231	3.6×10^{-7}	1.7×10^{-7}
0.6	1.08540584	1.08544861	1.08544887	1.08544906	4.5×10^{-7}	1.9×10^{-7}
0.7	1.11388470	1.11400052	1.11400072	1.11400108	5.6×10^{-7}	3.6×10^{-7}
0.8	1.14510669	1.14538393	1.14538393	1.14538468	7.5×10^{-7}	7.5×10^{-7}
0.9	1.17830101	1.17890549	1.17890570	1.17890664	1.1×10^{-6}	9.4×10^{-7}
1.0	1.21255189	1.21377602	1.21377710	1.21377819	2.1×10^{-6}	1.1×10^{-6}

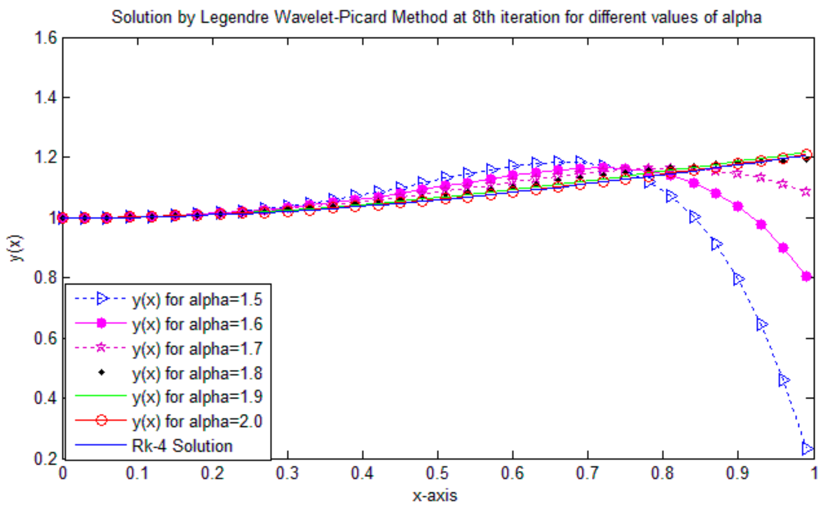

Figure 2. Comparison of solutions for different fractional values by RK-4 solution for double well case.

(3) (Double-hump $a > 0, b < 0$). $a = 0.5, b = -0.5, \mu = 0.1, f = 0.5, \omega = 0.79$. (See Table 3 and Figure 3)

Table 3. Comparison of Double-Hump solution at Picard's 8th iteration obtained by Legendre Wavelet-Picard Method (LWPM) with Variational Iterational Method (VIM) and RK-4, when M = 6 and $\alpha = 2$.

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.0×10^{-9}	1.0×10^{-11}
0.1	1.00250077	1.00250088	1.00250086	1.00250078	1.0×10^{-7}	8.0×10^{-8}
0.2	1.01001232	1.01001263	1.01001258	1.01001240	2.3×10^{-7}	1.8×10^{-7}
0.3	1.02256255	1.02256359	1.02256352	1.02256311	4.8×10^{-7}	4.1×10^{-7}
0.4	1.04019982	1.04020342	1.04020320	1.04020266	7.6×10^{-7}	5.4×10^{-7}
0.5	1.06299669	1.06300891	1.06300878	1.06300754	1.4×10^{-6}	1.2×10^{-6}
0.6	1.09105590	1.09109135	1.09109104	1.09108901	2.3×10^{-6}	2.0×10^{-6}
0.7	1.12451829	1.12460876	1.12460856	1.12460496	3.8×10^{-6}	3.6×10^{-6}
0.8	1.16357278	1.16377998	1.16377964	1.16377494	5.0×10^{-6}	4.7×10^{-6}
0.9	1.20846809	1.20890678	1.20890608	1.20890103	5.8×10^{-6}	5.1×10^{-6}
1.0	1.25952626	1.26040318	1.26040254	1.26039413	9.1×10^{-6}	8.4×10^{-6}

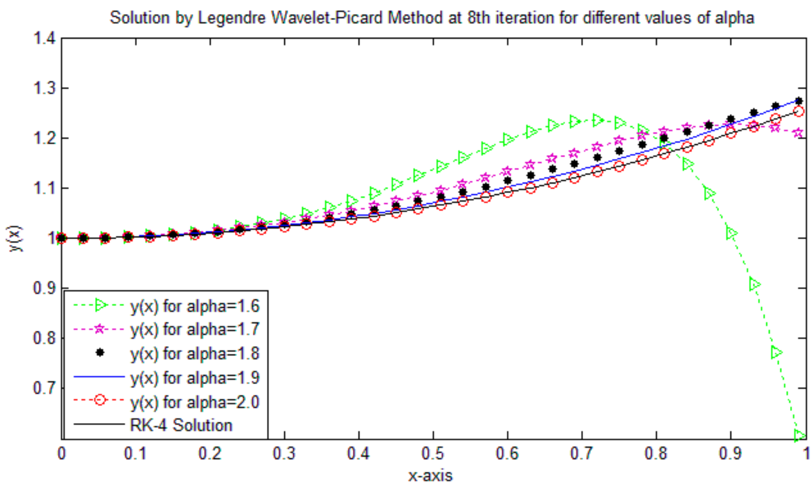

Figure 3. Comparison of solutions for different fractional values by RK-4 solution for double hump case.

Problem 2. *Consider the α-th order fractional force-free Duffing-Van der Pol oscillator equation* [6]

$$D^{\alpha}y(t) - \mu\left(1 - y^2(t)\right)y'(t) + ay(t) + by^3(t) = 0\,, 1 < \alpha \leq 2,$$

subject to the initial conditions $y(0) = 1$ *and* $y'(0) = 0$.

Firstly applying Picard Technique, we have

$$D^{\alpha}y_{n+1}(t) - \mu {y'}_{n+1}(t) + \mu y_n^2(t)y_n'(t) + ay_{n+1}(t) + by_n^3(t) = 0, 1 < \alpha \leq 2,$$

with initial conditions $y_{n+1}(0) = 1$ and $y_{n+1}'(0) = 0$.

Now applying Legendre wavelets method on the above equation, we have

$$D^{\alpha}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)-\mu\frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+\mu y_n^2(t)y_n'(t)$$
$$+a\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+by_n^3(t)=0\,,1<\alpha\leq 2,$$

with initial approximations$y_1(0) = 1$ and $y_1'(0) = 0$. Solutions are given in Table 4 and Figure 4.

Table 4. Comparison of Force-Free Duffing equation solution at Picard's 8th iteration obtained by Legendre Wavelet-Picard Method (LWPM) with Variational Iterational Method (VIM) and RK-4, when M = 6 and $\alpha = 2$.

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.0×10^{-9}	1.2×10^{-9}
0.1	0.99495428	0.99495428	0.99495427	0.99495427	1.0×10^{-8}	7.0×10^{-9}
0.2	0.97986771	0.97986769	0.97986765	0.97986761	8.0×10^{-8}	4.1×10^{-8}
0.3	0.95488865	0.95488784	0.95488778	0.95488770	1.4×10^{-7}	8.3×10^{-8}
0.4	0.92025739	0.92025265	0.92025256	0.92025243	2.2×10^{-7}	1.3×10^{-7}
0.5	0.87630062	0.87628321	0.87628308	0.87628280	4.1×10^{-7}	2.8×10^{-7}
0.6	0.82342666	0.82337721	0.82337705	0.82337636	8.5×10^{-7}	6.9×10^{-7}
0.7	0.76212192	0.76200270	0.76200234	0.76200157	1.1×10^{-6}	7.7×10^{-7}
0.8	0.69294873	0.69269483	0.69269436	0.69269338	1.5×10^{-6}	9.8×10^{-7}
0.9	0.61654510	0.61605312	0.61605226	0.61604996	3.1×10^{-6}	2.3×10^{-6}
1.0	0.53362658	0.53274003	0.53273896	0.53273066	9.3×10^{-6}	8.3×10^{-6}

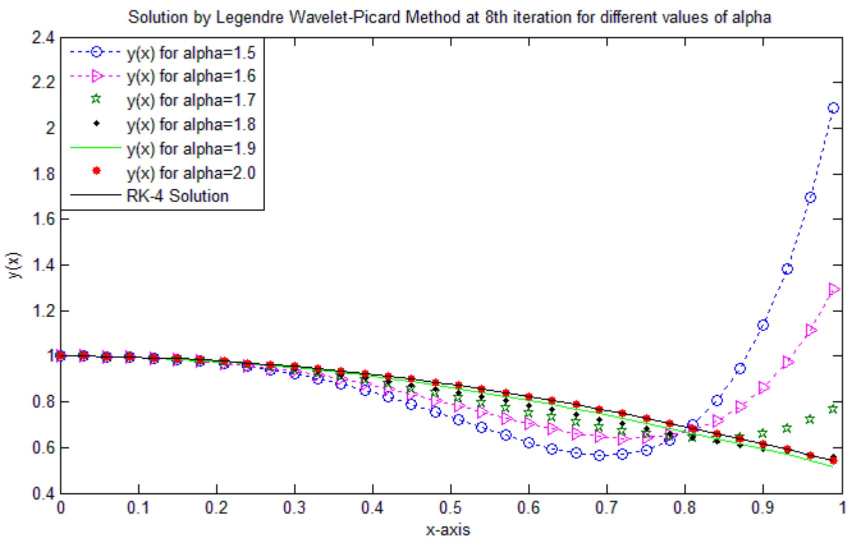

Figure 4. Comparison of solutions for different fractional values by RK-4 solution for Problem 2.

Problem 3. Consider the higher order fractional Duffing equation [5]

$$D^{\alpha}y(t)+5y''(t)+4y(t)-\frac{1}{6}y^3(t)=0, 3<\alpha\leq 4.$$

subject to the initial conditions:

$$y(0)=0, y'(0)=1.91103$$
$$y''(0)=0, y'''(0)=-1.15874.$$

The exact solution, when $\alpha = 4$, is given by

$$y(t)=2.1906\sin(0.9\,x)-0.02247\sin(2.7x)+0.000045\sin(4.5x)$$

Applying Picard's method to above considered problem, we have

$$D^{\alpha}y_{n+1}(t)+5y''_{n+1}(t)+4y_{n+1}(t)-\frac{1}{6}y_n^3(t)=0, \qquad 3<\alpha\leq 4.$$

with initial conditions:

$$y_{n+1}(0)=0, y'_{n+1}(0)=1.91103$$
$$y''_{n+1}(0)=0, y'''_{n+1}(0)=-1.15874.$$

Implementing Legendre Wavelet method to above equation, we have

$$D^{\alpha}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+5\frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)+4\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right)-\frac{1}{6}y^3(t)$$
$$=0, 3<\alpha\leq 4.$$

with initial conditions:

$$y_0(0)=0, y'_0(0)=1.91103$$
$$y''_0(0)=0, y'''_0(0)=-1.15874.$$

Solutions are given in Table 5 and Figure 5.

Table 5. Comparison of higher order Duffing equation solution at Picard's 8th iteration obtained by Legendre Wavelet-Picard Method (LWPM) with Variational Iterational Method (VIM) and RK-4, for M = 6 and $\alpha = 4$.

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.0	0.00000000	0.00000000	0.00000000	0.00000000	1.0×10^{-9}	2.1×10^{-12}
0.1	0.19090972	0.19090974	0.19090975	0.19090978	4.0×10^{-8}	3.0×10^{-8}
0.2	0.38065613	0.38065651	0.38065653	0.38065658	7.0×10^{-8}	5.0×10^{-8}
0.3	0.56805809	0.56805941	0.56805942	0.56805959	1.8×10^{-7}	1.7×10^{-7}
0.4	0.75190092	0.75190395	0.75190402	0.75190437	4.2×10^{-7}	3.5×10^{-7}
0.5	0.93092454	0.93092998	0.93093003	0.93093096	9.8×10^{-7}	9.3×10^{-7}
0.6	1.10381762	1.10382601	1.10382611	1.10382756	1.6×10^{-6}	1.4×10^{-6}

Table 5. *Cont.*

t	VIM Solution	UWCM [24] Solution at M = 6	LWPM Solution at M = 6	RK-4 Solution	Error in UWCM [24]	Error in LWPM
0.7	1.26921875	1.26922885	1.26922919	1.26923100	2.1×10^{-6}	1.9×10^{-6}
0.8	1.42572628	1.42573213	1.42573263	1.42573479	2.7×10^{-6}	2.1×10^{-6}
0.9	1.57191725	1.57190182	1.57190210	1.57190495	3.1×10^{-6}	2.8×10^{-6}
1.0	1.70637581	1.70629821	1.70630014	1.70630325	5.0×10^{-6}	3.1×10^{-6}

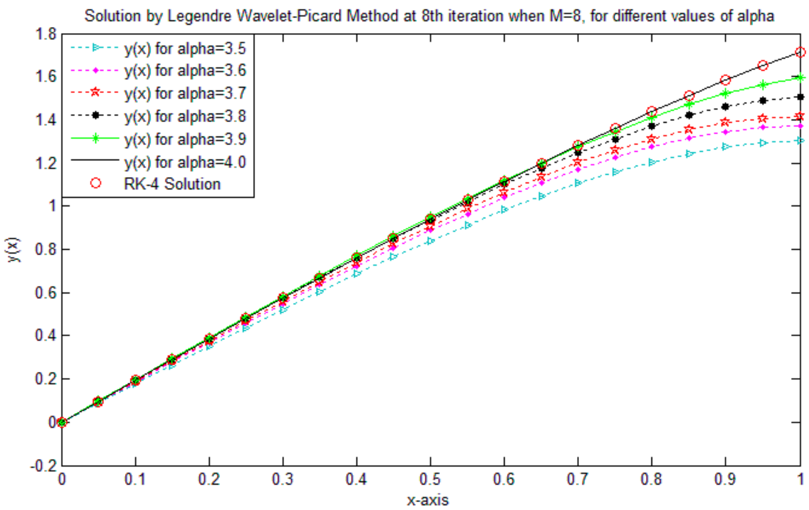

Figure 5. Comparison of solutions for different fractional values by RK-4 solution for Problem 3.

4. Conclusions

In this paper, a systematic technique, is employed and executed successfully to solve the emerging problems modeled from nonlinear fractional oscillation phenomena. The results are also obtained via LWPM, VIM, UWCM and RK-4 method. Comparison with VIM, UWCM of the approximate solutions show that UWCM is more accurate as compared to VIM. Moreover, proposed LWPM shows slightly better results as compare to UWCM and VIM which is mainly due to the insertion of Picard's iteration technique with the nonlinear part. It is also observed that in certain cases [24], UWCM has some edge over LWPM. It is also concluded that suggested scheme (LWPM) may be extended for some other nonlinear problems of diversified physical nature.

Acknowledgments

The authors are highly grateful to the unknown referees for their valuable useful suggestions. This project was funded by the National Plane for Science, Technology and Innovation (MAARIF), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, Award Number (15-MAT4688-02).

Author Contributions

Author S.T. Mohyud-Din developed the problem and its MAPLE code. M.A. Iqbal, in collaboration with first author, did the literature review, developed and implemented the computer code, and interpreted the subsequently obtained results. S.M. Hassan, in consultation of rest of the Authors, did the literature review, re-confirmed the credibility of obtained solutions and also removed grammatical mistakes and typing errors. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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