Some New Results on the Multiple-Access Wiretap Channel

Bin Dai * and Zheng Ma

School of Information Science and Technology, Southwest JiaoTong University, Northbound Section Second Ring Road 111, Chengdu 610031, China; E-Mail: zma@home.swjtu.edu.cn

* Author to whom correspondence should be addressed; E-Mail: daibinsjtu@gmail.com; Tel.: +86-028-87634758.

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Abstract: In this paper, some new results on the multiple-access wiretap channel (MAC-WT) are provided. Specifically, first, we investigate the degraded MAC-WT, where two users transmit their corresponding confidential messages (no common message) to a legitimate receiver via a multiple-access channel (MAC), while a wiretapper wishes to obtain the messages via a physically degraded wiretap channel. The secrecy capacity region of this model is determined for both the discrete memoryless and Gaussian cases. For the Gaussian case, we find that this secrecy capacity region is exactly the same as the achievable secrecy rate region provided by Tekin and Yener, i.e., Tekin–Yener’s achievable region is exactly the secrecy capacity region of the degraded Gaussian MAC-WT. Second, we study a special Gaussian MAC-WT, and find the power control for two kinds of optimal points (max-min point and single user point) on the secrecy rate region of this special Gaussian model.

Keywords: multiple-access wiretap channel (MAC-WT); Gaussian multiple-access wiretap channel (GMAC-WT); secrecy capacity region; power control

1. Introduction

Transmission of confidential messages has been studied in the literature of several classes of channels. Wyner, in his well-known paper on the wiretap channel [1], studied the problem of how to transmit the confidential messages to a legitimate receiver via a degraded broadcast channel, while keeping the eavesdropper as ignorant of the messages as possible. Measuring the uncertainty of the eavesdropper by equivocation, the capacity-equivocation region was established. Furthermore, the secrecy capacity
was also established, which provided the maximum transmission rate with perfect secrecy. After the
publication of Wyner’s work, Csiszár and Körner [2] investigated a more general situation: broadcast
channels with confidential messages (BCC). In this model, a common message and a confidential
message were sent through a general broadcast channel. The common message was assumed to be
decoded correctly by the legitimate receiver and the eavesdropper, while the confidential message was
only allowed to be obtained by the legitimate receiver. This model is also a generalization of the model
in [3], where no confidentiality condition is imposed. The capacity-equivocation region and the secrecy
capacity region of BCC [2] were totally determined, and the results were also a generalization of those
in [1]. Based on Wyner’s work, Leung- Yan-Cheong and Hellman studied the Gaussian wiretap channel
(GWC) [4] and showed that its secrecy capacity was the difference between the main channel capacity
and the overall wiretap channel capacity. Some other related works on the wiretap channel (including
feedback, side information and secret key) can be found in [5–12].

Recently, by using the approach of [1,2], the information-theoretical security for other multi-user
communication systems has been widely studied.

For the relay channel, Lai-Gamal [13] and Xu et al. [14] studied the relay-eavesdropper channel,
where a source wishes to send messages to a destination while leveraging the help of a relay node to
hide those messages from the eavesdropper. Inner and outer bounds on the capacity-equivocation region
were provided in these papers. In addition, Oohama [15] studied the relay channel with confidential
messages, where a relay helps the transmission of messages from one sender to one receiver. The relay
is considered not only as a sender that helps the message transmission, but also as a wiretapper who
can obtain some knowledge about the transmitted messages. Measuring the uncertainty of the relay by
equivocation, the inner and outer bounds on the capacity-equivocation region were provided in [15].

For the interference channel, Liu et al. [16] studied the interference channel with two confidential
messages and provided inner and outer bounds on the secrecy capacity region. In addition, Liang et al. [17]
studied the cognitive interference channel with one common message and one
confidential message, and the capacity-equivocation region was totally determined for this model.

For the multiple-access channel (MAC), the security problems are split into two directions.

• The first is that two users wish to transmit their corresponding messages to a destination, and
  meanwhile, they also receive the channel output. Each user treats the other user as a wiretapper
  and wishes to keep its confidential message as secret as possible from the wiretapper. This
  model is usually called the MAC with confidential messages, and it was studied by Liang and
  Poor [18]. An inner bound on the capacity-equivocation region is provided for the model with
  two confidential messages, and the capacity-equivocation region is still not known. Furthermore,
  for the model of MAC with one confidential message [18], both inner and outer bounds on the
  capacity-equivocation region are derived. Moreover, for the degraded MAC with one confidential
  message, the capacity-equivocation region is totally determined.

• The second is that an additional wiretapper has access to the MAC output via a wiretap channel,
  and therefore, how to keep the confidential messages of the two users as secret as possible
  from the additional wiretapper is the main concern of the system designer. This model is
  usually called the multiple-access wiretap channel (MAC-WT). An inner bound on the secrecy
capacity region of the degraded Gaussian MAC-WT was provided in [19], and a \( n \)-letter form
of the secrecy capacity region of the degraded Gaussian MAC-WT was shown in (Theorem 6 in [20]). Moreover, an inner bound on the secrecy capacity region of the general Gaussian MAC-WT was provided in [21]. In [22,23], the MAC-WT with partially cooperating encoders (one encoder is allowed to confer and the other does not transmit any message) was studied, and inner and outer bounds on the capacity-equivocation region of this model were provided. The MAC-WT with two conference links between the encoders was investigated in [24], and inner and outer bounds on the secrecy capacity region were established for this model. Besides these works on the discrete memoryless and Gaussian cases of MAC-WT, He et al. [25] studied the MIMO MAC-WT, where the channel matrices of the legitimate users are fixed and revealed to all of the terminals, whereas the channel matrices of the eavesdropper are arbitrarily varying and only known to the eavesdropper. Recently, Zaidi et al. ([26,27]) investigated the secrecy problem of MIMO x-channels with output feedback and delayed CSI (an extension of the model of MAC-WT). The optimal sum secure degrees of freedom (SDoF) region was characterized in [26,27], and the artificial noise technique was used to construct the corresponding encoding-decoding scheme.

In this paper, first, we study the degraded MAC-WT, see Figure 1. The motivation of this work is to find the secrecy capacity region of the general (not degraded) MAC-WT. However, it is difficult to find a tight outer bound on the secrecy capacity region of the general MAC-WT, and thus, in this paper, we focus on the secrecy capacity region of the degraded MAC-WT. Compared with the capacity result of (Theorem 6 in [20]) (n-letter form), the main contribution of this paper is the single-letter characterization of the secrecy capacity region of the degraded MAC-WT.

In Figure 1, two users transmit their corresponding confidential messages (no common message) to a legitimate receiver via a multiple-access channel (MAC), while an eavesdropper wishes to obtain the messages via a physically degraded wiretap channel. The secrecy capacity region of the model of Figure 1 is determined for both the discrete memoryless and Gaussian cases. Furthermore, for the Gaussian case, we find that the secrecy capacity region provided in this paper is exactly the same as the achievable secrecy rate region provided by Tekin and Yener [21]. Then, we study the power control for two kinds of optimal points (max-min point and single user point) on the secrecy rate region of a special Gaussian MAC-WT and find that these optimum points tend to be constants when the power tends to infinity.

Figure 1. The degraded multiple-access wiretap channel (MAC-WT).

In this paper, random variables, sample values and alphabets are denoted by capital letters, lower case letters and calligraphic letters, respectively. A similar convention is applied to the random vectors and
their sample values. For example, \( U^N \) denotes a random \( N \)-vector \((U_1, ..., U_N)\), and \( u^N = (u_1, ..., u_N) \) is a specific vector value in \( U^N \) that is the \( N \)-th Cartesian power of \( U \). \( U_i^N \) denotes a random \( N-i+1 \)-vector \((U_i, ..., U_N)\), and \( u_i^N = (u_i, ..., u_N) \) is a specific vector value in \( U_i^N \). Let \( P_Y(v) \) denote the probability mass function \( Pr\{V = v\} \). Throughout the paper, the logarithmic function is to base two.

The organization of this paper is as follows. In Section 2, the secrecy capacity region of the degraded discrete memoryless MAC-WT is given by Theorem 1. In Section 3, the secrecy capacity region of the degraded Gaussian MAC-WT is given by Theorem 2. The power control for a special Gaussian MAC-WT is investigated in Section 4. Final conclusions are provided in Section 5.

## 2. Degraded Discrete Memoryless Multiple-Access Wiretap Channel

In this section, a description of the model of Figure 1 is given by Definition 1 to Definition 3. The secrecy capacity region \( \mathcal{R}^D \) composed of all achievable secrecy pairs \((R_1, R_2)\) in the model of Figure 1 is characterized in Theorem 1, where the achievable secrecy pair \((R_1, R_2)\) is defined in Definition 4.

**Definition 1.** (Channel encoder) The confidential messages \( W_1 \) and \( W_2 \) take values in \( \mathcal{W}_1 \), \( \mathcal{W}_2 \), respectively. \( W_1 \) and \( W_2 \) are independent and uniformly distributed over their ranges. The input of Encoder 1 (Encoder 2) is \( W_1 \) (\( W_2 \)), while the output of Encoder 1 (Encoder 2) is \( X_1^N \) (\( X_2^N \)). We assume that the encoders are stochastic encoders, i.e., the encoder \( g_i^N \) \((i = 1, 2)\) is a matrix of conditional probabilities \( g_i^N(x_i^n|w_i) \), where \( x_i^n \in \mathcal{X}_i^N \), \( w_i \in \mathcal{W}_i \), and \( g_i^N(x_i^n|w_i) \) is the probability that the message \( w_i \) is encoded as the channel input \( x_i^n \). Note that \( X_1^N \) is independent of \( X_2^N \). The transmission rates of the confidential messages are \( \frac{\log ||W_1||}{N} \) and \( \frac{\log ||W_2||}{N} \).

**Definition 2.** (Channels) The MAC is a discrete memoryless channel (DMC) with a finite input alphabet \( \mathcal{X}_1 \times \mathcal{X}_2 \), a finite output alphabet \( \mathcal{Y} \) and transition probability \( P_{Y|X_1, X_2}(y|x_1, x_2) \). Note that \( P_{Y|X_1^N, X_2^N}(y^N|x_1^n, x_2^n) = \prod_{n=1}^{N} P_{Y_n|x_1^n, x_2^n}(y_n|x_1^n, x_2^n) \). The inputs of the MAC are \( X_1^N \) and \( X_2^N \), while the output is \( Y^N \).

The wiretap channel is a DMC with finite input alphabet \( Y \), finite output alphabet \( Z \) and transition probability \( P_{Z|Y}(z|y) \). The wiretapper’s equivocation to the confidential messages \( W_1 \) and \( W_2 \) is defined as:

\[
\Delta = \frac{1}{N} H(W_1, W_2|Z^N). \tag{1}
\]

**Definition 3.** (Decoder) The decoder for the legitimate receiver is a mapping \( f_D : \mathcal{Y}^N \rightarrow \mathcal{W}_1 \times \mathcal{W}_2 \), with input \( Y^N \) and outputs \( \hat{W}_1 \), \( \hat{W}_2 \). Let \( P_e \) be the error probability of the receiver, and it is defined as \( Pr\{((W_1, W_2) \neq (\hat{W}_1, \hat{W}_2))\} \).

**Definition 4.** (Achievable secrecy pair \((R_1, R_2)\) in the model of Figure 1) A secrecy pair \((R_1, R_2)\) (where \( R_1, R_2 > 0 \)) is called achievable if, for any \( \epsilon > 0 \) (where \( \epsilon \) is an arbitrary small positive real number and \( \epsilon \rightarrow 0 \)), there exists a channel encoder-decoder \((N, \Delta, P_e)\), such that:

\[
\lim_{N \to \infty} \frac{\log ||W_1||}{N} = R_1, \quad \lim_{N \to \infty} \frac{\log ||W_2||}{N} = R_2, \quad \lim_{N \to \infty} \Delta \geq R_1 + R_2, \quad P_e \leq \epsilon. \tag{2}
\]
Theorem 1 gives a single-letter characterization of the secrecy capacity region $R^D$, which is composed of all achievable secrecy pairs $(R_1, R_2)$ in the model of Figure 1.

**Theorem 1.** A single-letter characterization of the secrecy capacity region $R^D$ is as follows,

$$\mathcal{R}^D = \{(R_1, R_2) :$$

$$R_1 \leq I(X_1; Y | X_2, U) - I(X_1; Z | U)$$

$$R_2 \leq I(X_2; Y | X_1, U) - I(X_2; Z | U)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y | U) - I(X_1, X_2; Z | U)\}$$

for some distribution:

$$P_{Z,Y,X_1,X_2,U}(z,y,x_1,x_2,u) = P_{Z|Y}(z|y)P_{Y|X_1,X_2}(y|x_1,x_2)P_{UX}(u,x_1,x_2).$$

**Proof.** The converse proof of Theorem 1 is given in Section 7, and it is from the standard technique used in [1,2]. Now, we focus on the direct (achievability) proof of Theorem 1, and it is considered into two cases.

- **Case 1:** the pair $(R_1 = I(X_1; Y | U) - I(X_1; Z | U), R_2 = I(X_2; Y | X_1, U) - I(X_2; Z | U))$ is achievable.

- **Case 2:** the pair $(R_1 = I(X_1; Y | X_2, U) - I(X_1; Z | U), R_2 = I(X_2; Y | U) - I(X_2; Z | U, X_1))$ is achievable.

The encoding schemes for Case 1 and Case 2 are roughly illustrated in Figures 2 and 3, respectively. The proposed achievable encoding schemes combine the random binning, superposition coding and artificial noise techniques.

In Figure 2, the dummy message $w^*$ is encoded as $u^N$, and the channel input $x_1^N$ represents the superposition code in which the confidential message $w_1$ is superimposed on $w^*$. In addition, the channel input $x_2^N$ represents the random binning codeword encoded by the confidential message $w_2$.

Analogously, in Figure 3, the dummy message $w^*$ is encoded as $u^N$, and the channel input $x_2^N$ represents the superposition code in which the confidential message $w_2$ is superimposed on $w^*$. In addition, the channel input $x_1^N$ represents the random binning codeword encoded by the confidential message $w_1$.

The details of the complete proof will be provided in Section 6.

**Remark 1.** There are some notes on Theorem 1; see the following.

- The MAC-WT was first investigated by Tekin and Yener [19,21]. In [21], an achievable secrecy rate region (inner bound on the secrecy capacity region) is given by:

$$\mathcal{R}^{Di} = \{(R_1, R_2) :$$

$$R_1 \leq I(X_1; Y | X_2) - I(X_1; Z)$$

$$R_2 \leq I(X_2; Y | X_1) - I(X_2; Z)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y) - I(X_1, X_2; Z)\}$$
Letting $U$ be a constant, it is easy to see that the region $\mathcal{R}^D$ of Theorem 1 reduces to $\mathcal{R}^{D_i}$, i.e., $\mathcal{R}^{D_i} \subseteq \mathcal{R}^D$.

- Note that the above $\mathcal{R}^{D_i}$ is constructed according to the random binning technique. In this paper, we combine the artificial noise technique (the dummy message $w^\ast$ can be also viewed as an artificial noise) with the classical random binning technique to construct the encoding scheme of the model of Figure 1. To be more specific, first, we randomly choose a dummy message (artificial noise) $w^\ast$. Second, the transmitted codeword is constructed by using the double binning technique, where the index of the bin is related to $w^\ast$ and the index of the sub-bin is related to the transmitted message $w_1$ or $w_2$. Finally, we randomly choose a codeword in sub-bin $w_1$ or $w_2$ to transmit. By using this double binning technique, we prove that $\mathcal{R}^D$ is achievable. Here, note that the double binning technique (combination of artificial noise and binning) is also used in [22,23].

- The region $\mathcal{R}^D$ is convex. The proof is directly obtained by introducing a time sharing random variable into Theorem 1, and thus, it is omitted here.

**Figure 2.** The encoding scheme for Case 1.

![Figure 2](image1.png)

**Figure 3.** The encoding scheme for Case 2.

![Figure 3](image2.png)

3. Degraded Gaussian Multiple-Access Wiretap Channel

3.1. Secrecy Capacity Region of the Degraded Gaussian Multiple-Access Wiretap Channel

In this subsection, we investigate the Gaussian case of the model of Figure 1, where the channel input-output relationships at each time instant $i$ ($1 \leq i \leq N$) are given by:
respectively, i.e.

\[ X = X_1 + X_2 + \eta_1, \]  

and:

\[ Z = X_1 + X_2 + \eta_1 + \eta_2, \]  

where \( \eta_1 \sim \mathcal{N}(0, N_1) \) and \( \eta_2 \sim \mathcal{N}(0, N_2) \). The random vectors \( \eta_1^N \) and \( \eta_2^N \) are independent with i.i.d. components. The channel inputs \( X_1^N \) and \( X_2^N \) are subject to the average power constraints \( P_1 \) and \( P_2 \), respectively, i.e.,

\[
\frac{1}{N} \sum_{i=1}^{N} E[X_{1,i}^2] = p_1 \leq P_1, \quad \frac{1}{N} \sum_{i=1}^{N} E[X_{2,i}^2] = p_2 \leq P_2.
\]  

Note that \( X_1^N \) is independent of \( X_2^N \).

**Theorem 2.** The secrecy capacity region \( \mathcal{R}^G \) of the Gaussian model of Figure 1 is given by:

\[
\mathcal{R}^G = \mathcal{A} \cup \mathcal{B},
\]

where:

\[
\mathcal{A} = \bigcup_{0 \leq \alpha \leq 1} \bigg\{ (R_1, R_2) : 
\begin{align*}
R_1 &\leq \frac{1}{2} \log(1 + \frac{(1-\alpha)p_1}{N_1}) - \frac{1}{2} \log(1 + \frac{(1-\alpha)p_1}{N_1+N_2+p_2}) \\
R_2 &\leq \frac{1}{2} \log(1 + \frac{p_2}{N_1}) - \frac{1}{2} \log(1 + \frac{p_2}{N_1+N_2+\alpha p_1}) \\
R_1 + R_2 &\leq \frac{1}{2} \log(1 + \frac{(1-\alpha)p_1+p_2}{N_1}) - \frac{1}{2} \log(1 + \frac{(1-\alpha)p_1+p_2}{N_1+N_2})
\end{align*}
\bigg\},
\]

and:

\[
\mathcal{B} = \bigcup_{0 \leq \alpha \leq 1} \bigg\{ (R_1, R_2) : 
\begin{align*}
R_1 &\leq \frac{1}{2} \log(1 + \frac{p_1}{N_1}) - \frac{1}{2} \log(1 + \frac{p_1}{N_1+N_2+\alpha p_2}) \\
R_2 &\leq \frac{1}{2} \log(1 + \frac{(1-\alpha)p_2}{N_1}) - \frac{1}{2} \log(1 + \frac{(1-\alpha)p_2}{N_1+N_2+p_1}) \\
R_1 + R_2 &\leq \frac{1}{2} \log(1 + \frac{(1-\alpha)p_2+p_1}{N_1}) - \frac{1}{2} \log(1 + \frac{(1-\alpha)p_2+p_1}{N_1+N_2})
\end{align*}
\bigg\}.
\]

**Proof.** The proof of Theorem 2 is considered in the following two parts:

- (Proof of \( \mathcal{A} \)): The direct proof follows by computing the mutual information terms in Theorem 1 with the following distributions: \( X_1 = U + V, U \sim \mathcal{N}(0, \alpha p_1), V \sim \mathcal{N}(0, (1 - \alpha)p_1) \) and \( X_2 \sim \mathcal{N}(0, p_2) \). \( U \), \( V \) and \( X_2 \) are independent. The details are omitted here. The converse proof follows from Section 7, and it is omitted here, too. Thus, the proof of \( \mathcal{A} \) is completed.

- (Proof of \( \mathcal{B} \)): The direct proof follows by computing the mutual information terms in Theorem 1 with the following distributions: \( X_2 = U + V, U \sim \mathcal{N}(0, \alpha p_2), V \sim \mathcal{N}(0, (1 - \alpha)p_2) \) and \( X_1 \sim \mathcal{N}(0, p_1) \). \( U \), \( V \) and \( X_1 \) are independent. The details are omitted here. The converse proof follows from Section 7, and it is omitted here, too. Thus, the proof of \( \mathcal{B} \) is completed.

The proof of Theorem 2 is completed.
3.2. Discussions

First, note that an achievable secrecy rate region of the degraded Gaussian MAC-WT is provided in [21], and it is given by:

\[
R_{Gi} = \bigcup_{0 \leq p_1 \leq P_1} \bigcup_{0 \leq p_2 \leq P_2} \left\{ (R_1, R_2) : 
\begin{align*}
R_1 &\leq \frac{1}{2} \log(1 + \frac{p_2 + N_2}{N_1}) - \frac{1}{2} \log(1 + \frac{p_2 + N_2}{N_1 + N_2 + p_2}) \\
R_2 &\leq \frac{1}{2} \log(1 + \frac{p_1 + N_1}{N_2}) - \frac{1}{2} \log(1 + \frac{p_1 + N_1}{N_1 + N_2 + p_1}) \\
R_1 + R_2 &\leq \frac{1}{2} \log(1 + \frac{p_1 + p_2}{N_1}) - \frac{1}{2} \log(1 + \frac{p_1 + p_2}{N_1 + N_2})
\end{align*}
\right\}.
\]

The secrecy capacity region \( R_G \) is achieved when \( \alpha = 0 \), and it coincides with Tekin–Yener’s inner bound \( R_{Gi} \), i.e., Tekin–Yener’s inner bound \( R_{Gi} \) is, in fact, the secrecy capacity region of the degraded Gaussian MAC-WT. The rigorous proof is as follows.

**Proof.** Observing that the region \( A \) of Theorem 2 can be rewritten as:

\[
A = \bigcup_{0 \leq \alpha \leq 1} \bigcup_{0 \leq p_1 \leq P_1} \bigcup_{0 \leq p_2 \leq P_2} \left\{ (R_1, R_2) : 
\begin{align*}
R_1 &\leq \frac{1}{2} \log(\frac{p_2 + N_2}{N_1} (1 - \frac{p_2 + N_2}{p_2 + N_2 + N_1 + (1-\alpha)p_1})) \\
R_2 &\leq \frac{1}{2} \log(\frac{p_1 + N_1}{N_2} (1 - \frac{p_1 + N_1}{p_2 + N_2 + N_1 + (1-\alpha)p_2})) \\
R_1 + R_2 &\leq \frac{1}{2} \log(\frac{N_1 + N_2}{N_1} (1 - \frac{N_1 + N_2}{p_2 + N_2 + N_1 + (1-\alpha)p_2}))
\end{align*}
\right\}.
\]

It is easy to see that the region \( A \) achieves its maximum when \( \alpha = 0 \). Analogously, the region \( B \) achieves its maximum when \( \alpha = 0 \). Note that the regions \( A \) and \( B \) are exactly the same as the region \( R_{Gi} \) if \( \alpha = 0 \). Thus, the proof is completed.

\( \square \)

4. Power Control for Two Kinds of Optimal Points on the Secrecy Rate Region of a Special Gaussian Multiple-Access Wiretap Channel

In this section, we investigate a special Gaussian MAC-WT; see Figure 4. The model of Figure 4 is characterized by:

\[
Y^N = X_1^N + X_2^N + N_M, \quad Z^N = \alpha X_1^N + (1 - \alpha) X_2^N + N_W,
\]

where \( N_M, N_W \sim \mathcal{N}(0, 1) \) and \( 0 < \alpha \leq \frac{1}{2} \).

**Figure 4.** A special Gaussian multiple-access wiretap channel.
An achievable secrecy rate region $\mathcal{R}$ of the model of Figure 4 is given by (2), where $p_1$ and $p_2$ are transmission powers for the codewords $x_1^N$ and $x_2^N$, respectively, and $0 \leq p_1, p_2 \leq P$. Note that the region $\mathcal{R}$ is directly from [21].

$$\mathcal{R} = \bigcup_{0 \leq p_1 \leq P_1}^{P_1} \bigcup_{0 \leq p_2 \leq P_2}^{P_2} \left\{ (R_1, R_2) : \begin{array}{ll} R_1 \leq \frac{1}{2} \log(1 + p_1) - \frac{1}{2} \log\left(1 + \frac{\alpha p_1}{1 + (1 - \alpha) p_2}\right) \\ R_2 \leq \frac{1}{2} \log(1 + p_2) - \frac{1}{2} \log\left(1 + \frac{(1 - \alpha) p_2}{1 + \alpha p_1}\right) \\ R_1 + R_2 \leq \frac{1}{2} \log(1 + p_1 + p_2) - \frac{1}{2} \log(1 + \alpha p_1 + (1 - \alpha) p_2) \end{array} \right\}. \quad (2)$$

In addition, the optimum power control for the maximization of the total secrecy sum rate is given by:

$$(p_1^*, p_2^*) = \begin{cases} (P, P), & \text{if } 0 \leq P \leq \frac{\alpha}{1 - 2\alpha}, \\ (P, 0), & P > \frac{\alpha}{1 - 2\alpha}. \end{cases} \quad (3)$$

and the corresponding maximum secrecy sum rate $R_{\text{sum}}^*$ is given by:

$$R_{\text{sum}}^* = \max(R_1 + R_2) = \begin{cases} \frac{1}{2} \log \frac{1 + 2P}{1 + \alpha P}, & \text{if } 0 \leq P \leq \frac{\alpha}{1 - 2\alpha}, \\ \frac{1}{2} \log \frac{1 + P}{1 + \alpha p_1}, & P > \frac{\alpha}{1 - 2\alpha}. \end{cases} \quad (4)$$

In the remainder of this section, the power control for two kinds of optimum points (max-min point and single user point) on the secrecy rate region of Figure 4 is provided in Sections 4.1 and 4.2. Numerical examples and discussions are in Section 4.3.

### 4.1. Max-Min Point

We first define an optimal point in the following sense:

$$R_{\text{min}}^* \triangleq \max_{p_1, p_2} \min \{ R_1, R_2 \}. \quad (5)$$

**Theorem 3.** For the model of Figure 4, the optimum point $R_{\text{min}}^*$ satisfies:

$$R_{\text{min}}^* = \begin{cases} \frac{1}{2} \log(1 + \alpha P), & \text{if } 0 \leq P \leq \frac{\sqrt{(a - 2)^2 + 4 - (a + 2)}}{2a}, \\ \frac{1}{2} \log \frac{1 + 2P}{1 + \alpha P}, & P > \frac{\sqrt{(a - 2)^2 + 4 - (a + 2)}}{2a}. \end{cases}$$

$R_{\text{min}}^*$ is achieved if $(p_1^*, p_2^*) = (P, P)$.

**Proof.** First, for convenience, define:

$$a = \frac{1}{2} \log \frac{(1 + p_1)(1 + (1 - \alpha)p_2)}{1 + (1 - \alpha)p_2 + \alpha p_1}, \quad (6)$$

$$b = \frac{1}{2} \log \frac{(1 + p_2)(1 + \alpha p_1)}{1 + (1 - \alpha)p_2 + \alpha p_1}, \quad (7)$$

$$c = \frac{1}{2} \log \frac{1 + p_1 + p_2}{1 + (1 - \alpha)p_2 + \alpha p_1}. \quad (8)$$
Then, (2) can be rewritten as:

\[ R = \bigcup_{0 \leq p_1 \leq P_1, 0 \leq p_2 \leq P_2} \begin{cases} (R_1, R_2) : \\ R_1 \leq a \\ R_2 \leq b \\ R_1 + R_2 \leq c \end{cases} \]  

(9)

The calculation of \( R_{\text{min}}^* \) depends on the following three cases; see Figure 5. The regions \( A \) and \( B \) of these three figures imply that \( R_1 \leq R_2 \) and \( R_1 \geq R_2 \), respectively. In region \( A \), \( R_{\text{min}}^* = \max \min \{ R_1, R_2 \} = \max R_1 \), and in region \( B \), \( R_{\text{min}}^* = \max \min \{ R_1, R_2 \} = \max R_2 \).

Therefore, from Figure 5a, it is easy to see that:

\[ R_{\text{min}}^* = \max_{p_1, p_2} b, \text{ s.t. } b \leq \frac{1}{2} c \leq a. \]  

(10)

Similarly, from Figure 5b, we see that:

\[ R_{\text{min}}^* = \max_{p_1, p_2} \frac{1}{2} c, \text{ s.t. } \frac{1}{2} c \leq \min \{ a, b \}. \]  

(11)

From Figure 5c, we see that:

\[ R_{\text{min}}^* = \max_{p_1, p_2} a, \text{ s.t. } a \leq \frac{1}{2} c \leq b. \]  

(12)

By using the well-known method of Lagrange multipliers on (10), (11) and (12), Theorem 3 is proven.

\[ \Box \]

**Figure 5.** All cases for the calculation of \( R_{\text{min}}^* \).

4.2. Single User Point

We now investigate another point, called the single user point, on which the legitimate receiver tries to maximize the secrecy rate \( R_1 \) (or \( R_2 \)) with the help of the senders, i.e., \( R_{\text{su}, i}^* = \max R_i \) (i = 1, 2).

**Theorem 4.** For the model of Figure 4, the optimum point \( R_{\text{su}, i}^* \) satisfies:

\[ R_{\text{su}, 1}^* = \begin{cases} \frac{1}{2} \log (1 + (1 - \alpha)P), & \text{if } 0 \leq P \leq \frac{\alpha}{1-\alpha}, \\
\frac{1}{2} \log \frac{1+2P}{1+P}, & \text{if } \frac{\alpha}{1-\alpha} < P \leq \frac{\alpha}{1-2\alpha}, \\
\frac{1}{2} \log \frac{1+P}{1+\alpha P}, & \text{if } P > \frac{\alpha}{1-2\alpha}. \end{cases} \]
The optimum power control achieving $R_{su,1}^*$ is given by:

$$(p_1^*, p_2^*) = \begin{cases} 
(P, P), & \text{if } 0 \leq P \leq \frac{\alpha}{1-\alpha}, \\
(P, P), & \text{if } \frac{\alpha}{1-\alpha} \leq P \leq \frac{\alpha}{1-2\alpha}, \\
(P, 0), & P > \frac{\alpha}{1-2\alpha}.
\end{cases}$$

The optimum point $R_{su,2}^*$ satisfies:

- If $0 \leq \alpha \leq \frac{3-\sqrt{5}}{\alpha}$,

$$R_{su,2}^* = \begin{cases} 
\frac{1}{2} \log(1 + \alpha P), & \text{if } 0 \leq P \leq \frac{\alpha}{1-\alpha}, \\
\frac{1}{2} \log(1 + \alpha P), & \text{if } \frac{\alpha}{1-\alpha} \leq P \leq \frac{1-2\alpha}{\alpha^2}, \\
\frac{1}{2} \log \frac{1+P}{1+\alpha P}, & P > \frac{1-2\alpha}{\alpha^2}.
\end{cases}$$

The optimum power control achieving $R_{su,2}^*$ is given by:

$$(p_1^*, p_2^*) = \begin{cases} 
(P, P), & \text{if } 0 \leq P \leq \frac{\alpha}{1-\alpha}, \\
(P, 0), & \text{if } \frac{\alpha}{1-\alpha} \leq P \leq \frac{1-2\alpha}{\alpha^2}, \\
(P, 0), & P > \frac{1-2\alpha}{\alpha^2}.
\end{cases}$$

- If $\frac{3-\sqrt{5}}{\alpha} \leq \alpha \leq \frac{1}{2}$,

$$R_{su,2}^* = \begin{cases} 
\frac{1}{2} \log(1 + \alpha P), & \text{if } 0 \leq P \leq \frac{1-\alpha}{\alpha}, \\
\frac{1}{2} \log \frac{1+2P}{1+\alpha P}, & \text{if } \frac{1-\alpha}{\alpha} \leq P \leq \frac{\alpha}{1-2\alpha}, \\
\frac{1}{2} \log \frac{1+P}{1+\alpha P}, & P > \frac{\alpha}{1-2\alpha}.
\end{cases}$$

The optimum power control achieving $R_{su,2}^*$ is given by:

$$(p_1^*, p_2^*) = \begin{cases} 
(P, P), & \text{if } 0 \leq P \leq \frac{1-\alpha}{\alpha}, \\
(P, P), & \text{if } \frac{1-\alpha}{\alpha} \leq P \leq \frac{\alpha}{1-2\alpha}, \\
(P, 0), & P > \frac{\alpha}{1-2\alpha}.
\end{cases}$$

**Proof.** By using (2), $R_{su,1}^*$ and $R_{su,2}^*$ can be rewritten as $R_{su,1}^* = \max R_1 = \max\{a, c\}$ and $R_{su,2}^* = \max R_2 = \max\{b, c\}$, respectively. Here, $a$, $b$ and $c$ are defined in (6), (7) and (8), respectively.

By using the method of Lagrange multipliers, Theorem 4 is proven.

□

4.3. Numerical Examples and Discussions

Figure 6 shows the max-min point $R_{min}^*$ and the maximum secrecy sum rate $R_{sum}^*$ for $\alpha = 0.2$ and $\alpha = 0.4$. It is easy to see that $R_{sum}^*$ increases while $\alpha$ decreases and that $R_{min}^*$ increases while $\alpha$ increases. Furthermore, $R_{min}^*$ tends to be a constant (0.5) while $P$ tends to infinity. $R_{sum}^*$ tends to be $\frac{1}{2} \log \frac{1}{\alpha}$ while $P$ tends to infinity.
Figure 7 shows the single user points $R_{su,1}^*$ and $R_{su,2}^*$ for $\alpha = 0.2$ and $\alpha = 0.4$. It is easy to see that the curve for $R_{su,1}^*$ is always better than that for $R_{su,2}^*$. Furthermore, $R_{su,1}^*$ and $R_{su,2}^*$ tend to be the same constant $\frac{1}{2} \log \frac{1}{\alpha}$, while $P$ tends to infinity. In addition, for a fixed $\alpha$, when $P$ tends to infinity, $R_{sum}^*$, $R_{su,1}^*$ and $R_{su,2}^*$ are the same.

The above results show that the secrecy rate region of Gaussian MAC-WT behaves significantly different from the classical capacity of Gaussian MAC. When classical capacity is concerned, the max-min point is always attained when the sum rate $R_1 + R_2$ is also maximized. However, for secrecy capacity, the point $\max(R_1 + R_2)$ does not necessarily coincide with $R_{min}^*$ all the time.

**Figure 6.** The $R_{min}^*$ and $R_{sum}^*$ for $\alpha = 0.2$ and $\alpha = 0.4$.

5. Conclusions

In this paper, first, we study the degraded multiple-access wiretap channel (MAC-WT). The secrecy capacity region is determined for both the discrete memoryless and Gaussian cases. Furthermore, for the Gaussian case, we find that the secrecy capacity region provided in this paper is exactly the same as the achievable secrecy rate region provided by Tekin and Yener. Then, we study the power control for two kinds of optimal points (max-min point and single user point) on the secrecy rate region of a special Gaussian MAC-WT and find that these optimum points tend to be constants when the power tends to infinity.
6. Direct Proof of Theorem 1

We consider the achievability proof of Theorem 1 for the case that the pair \( (R_1 = I(X_1; Y | X_2, U) - I(X_1; Z | U), R_2 = I(X_2; Y | U) - I(X_2; Z | U, X_1) ) \) is achievable, and the achievability proof for the pair \( (R_1 = I(X_1; Y | U) - I(X_1; Z | U, X_2), R_2 = I(X_2; Y | X_1, U) - I(X_2; Z | U) ) \) follows by symmetry.

The coding scheme combines the random binning, superposition coding and artificial noise techniques; see Figure 3. Define the messages \( W_1, W_2 \) and \( W^* \) (dummy message) taking values in the alphabets \( \mathcal{W}_1, \mathcal{W}_2 \) and \( \mathcal{W}^* \), respectively, where:

\[
\mathcal{W}_1 = \{1, 2, ..., 2^{NR_1}\}, \quad \mathcal{W}_2 = \{1, 2, ..., 2^{NR_2}\}, \quad \mathcal{W}^* = \{1, 2, ..., 2^{NR^*}\}.
\]

Fix the joint probability mass function \( P_{Z,Y,X_1,X_2,U}(z, y, x_1, x_2, u) \). For arbitrary \( \epsilon > 0 \), define:

\[
R_1 = I(X_1; Y | X_2, U) - I(X_1; Z | U), \quad (1)
\]

\[
R_2 = I(X_2; Y | U) - I(X_2; Z | U, X_1), \quad (2)
\]

\[
R^* = \min\{I(U; Y), I(U; Z)\} - \epsilon_1 \overset{(a)}{=} I(U; Z) - \epsilon_1, \quad (3)
\]

\[
R^{**} = I(X_2; Z | U, X_1) - \epsilon_1, \quad (4)
\]

where (a) is from the Markov chain \( U \rightarrow Y \rightarrow Z \) and \( \epsilon_1 \rightarrow 0 \) as \( N \rightarrow \infty \).

Here, note that:

\[
R_2 + R^* + R^{**} = I(X_2; Y | U) + I(U; Z) - 2\epsilon_1 \leq I(X_2; Y | U) + I(U; Y) - 2\epsilon_1 \overset{(b)}{=} I(X_2; Y) - 2\epsilon_1, \quad (5)
\]

where (b) is from the Markov chain \( U \rightarrow X_2 \rightarrow Y \).
Now, the remainder of this section is organized as follows. The code construction is introduced in Section 6.1. For any $\epsilon > 0$, the proofs of $\lim_{N \to \infty} \frac{\log ||W_1||}{N} = R_1$, $\lim_{N \to \infty} \frac{\log ||W_2||}{N} = R_2$, $\lim_{N \to \infty} \Delta \geq R_1 + R_2$ and $P_e \leq \epsilon$ are given in Section 6.2.

6.1. Coding Construction

Construction of $X_1^N$: Generate $2^{N(I(X_1;Y|X_2,U) - \epsilon_2)}$ i.i.d. codewords $x_1^n$ ($\epsilon_2 \to 0$ as $N \to \infty$) according to $\prod_{i=1}^{N} P_{X_1}(x_1,i)$, and divide them into $2^{NR_1}$ bins. Each bin contains $2^{N(I(X_1;Y|X_2,U) - \epsilon_2 - R_1)}$ codewords. Here, note that:

$$I(X_1;Y|X_2,U) - \epsilon_2 - R_1 \overset{(c)}{=} I(X_1;Z|U) - \epsilon_2,$$

where (c) is from (1). For a given confidential message $w_1$, randomly choose a codeword in bin $w_1$ to transmit.

Construction of $U^N$ (dummy message): Generate $2^{NR^*}$ i.i.d. codewords $u^N$ according to $\prod_{i=1}^{N} P_{U}(u_i)$. Randomly choose a $u^N(w^*)$ to transmit. Note that here, $U^N$ is independent of $X_1^N$.

Construction of $X_2^N$: Generate $2^{N(R_2 + R^* + R^{**})}$ i.i.d. codewords $x_2^N$ according to $\prod_{i=1}^{N} P_{X_2|U}(x_2,i|u_i)$, and divide them into $2^{NR^*}$ bins. Each bin contains $2^{N(R_2 + R^{**})}$ codewords. Divide the codewords in each bin into $2^{NR_2}$ sub-bins, and each sub-bin contains $2^{NR^{**}}$ codewords.

For a transmitted dummy message $w^*$ and a given message $w_2$, first choose the index of the bin according to $w^*$, and then, choose the index of the sub-bin in bin $w^*$ according to $w_2$. Finally, randomly choose a codeword in sub-bin $w_2$ to transmit.

Decoding scheme for the legitimate receiver: for a given $y^N$, try to find a sequence $u^N(\hat{w}^*)$, such that $(\hat{u}^N(\hat{w}^*), y^N)$ are jointly typical. If there exists a unique sequence with the index $\hat{w}^*$, put out the corresponding $\hat{w}^*$, else declare a decoding error. Based on the AEP and (3), the probability $Pr\{\hat{w}^* = w^*\}$ goes to one.

After decoding $\hat{w}^*$, the legitimate receiver tries to find a sequence $x_2^N(\hat{w}_2, \hat{w}^*)$, such that $(\hat{u}^N(\hat{w}^*), x_2^N(\hat{w}_2, \hat{w}^*), y^N)$ are jointly typical. If there exists a unique sequence with the index $\hat{w}_2$, put out the corresponding $\hat{w}_2$; else declare a decoding error. Based on the AEP, (2), (3), (4), (5) and the construction of $x_2^N$, the probability $Pr\{\hat{w}_2 = w_2\}$ goes to one.

Finally, after decoding $\hat{w}_2$ and $\hat{w}^*$, the legitimate receiver tries to find a sequence $x_1^N(\hat{w}_1)$, such that $(\hat{u}^N(\hat{w}^*), x_1^N(\hat{w}_1), x_2^N(\hat{w}_2, \hat{w}^*), y^N)$ are jointly typical. There exists a unique sequence with the index $\hat{w}_1$; put out the corresponding $\hat{w}_1$; else declare a decoding error. Based on the AEP, (1) and the construction of $x_1^N$, the probability $Pr\{\hat{w}_1 = w_1\}$ goes to one.

6.2. Proof of the Achievability

By using the above definitions, it is easy to verify that $\lim_{N \to \infty} \frac{\log ||W_1||}{N} = R_1$ and $\lim_{N \to \infty} \frac{\log ||W_2||}{N} = R_2$. Then, by using the above encoding–decoding scheme, $P_e \leq \epsilon$ is easy to be checked. It remains to be shown that $\lim_{N \to \infty} \Delta \geq R_1 + R_2$; see the following.

$$\lim_{N \to \infty} \Delta = \lim_{N \to \infty} \frac{1}{N} H(W_1, W_2 | Z^N)$$

$$= \lim_{N \to \infty} \frac{1}{N} (H(W_1 | Z^N) + H(W_2 | W_1, Z^N)).$$

(7)
The first term in (7) is bounded as follows.

\[
\lim_{N \to \infty} \frac{1}{N} H(W_1|Z^N) \geq \lim_{N \to \infty} \frac{1}{N} H(W_1|Z^N, U^N)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(W_1, Z^N, U^N) - H(Z^N, U^N))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(W_1, Z^N, U^N, X^N) - H(X_1^N|W_1, Z^N, U^N) - H(Z^N, U^N))
\]

\[
\equiv \lim_{N \to \infty} \frac{1}{N} (H(Z^N|U^N, X_1^N) + H(U^N) + H(X_1^N) - H(X_1^N|W_1, Z^N, U^N)
\]

\[
- H(Z^N, U^N))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(X_1^N) - I(X_1^N; Z^N|U^N) - H(X_1^N|W_1, Z^N, U^N)),
\]

(8)

where (a) is from \(H(W_1|X_1^N) = 0\) and \(U^N\) is independent of \(X_1^N\).

Consider the first term in (8); the codeword generation and [18, Lemma 3] ensure that:

\[
\lim_{N \to \infty} \frac{1}{N} H(X_1^N) \geq I(X_1^N; Y|X_2, U).
\]

(9)

For the second term in (8), using the same approach as that in [2, Lemma 3], we get:

\[
\lim_{N \to \infty} \frac{1}{N} I(X_1^N; Z^N|U^N) \geq I(X_1; Z|U).
\]

(10)

Now, we consider the last term of (8). From (6), given \(U^N, Z^N\) and \(W_1\), the total number of possible codewords of \(X_1^N\) is \(2^{N(I(X_1|Z; U) - \epsilon_2)}\). By using Fano’s inequality and the fact that \(\epsilon_2 \to 0\) as \(N \to \infty\), we have:

\[
\lim_{N \to \infty} \frac{1}{N} H(X_1^N|W_1, Z^N, U^N) = 0.
\]

(11)

Substituting (9), (10) and (11) into (8), we have:

\[
\lim_{N \to \infty} \frac{1}{N} H(W_1|Z^N) \geq I(X_1; Y|X_2, U) - I(X_1; Z|U) = R_1.
\]

(12)

The second term in (7) is bounded as follows.

\[
\lim_{N \to \infty} \frac{1}{N} H(W_2|W_1, Z^N) \geq \lim_{N \to \infty} \frac{1}{N} H(W_2|W_1, Z^N, U^N)
\]

\[
\geq \lim_{N \to \infty} \frac{1}{N} H(W_2|W_1, Z^N, U^N, X_1^N)
\]

\[
\equiv \lim_{N \to \infty} \frac{1}{N} H(W_2|Z^N, U^N, X_1^N)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(W_2, Z^N, U^N, X_1^N) - H(Z^N, U^N, X_1^N))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(W_2, Z^N, U^N, X_1^N, X_2^N) - H(X_1^N|W_2, Z^N, U^N, X_1^N)
\]

\[
- H(Z^N, U^N, X_1^N))
\]

\[
\equiv \lim_{N \to \infty} \frac{1}{N} (H(Z^N|U^N, X_1^N, X_2^N) + H(X_1^N) + H(U^N, X_2^N)
\]

\[
- H(X_2^N|W_2, Z^N, U^N, X_1^N) - H(Z^N|U^N, X_1^N) - H(U^N) - H(X_1^N))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} (H(X_2^N|U^N) - I(X_2^N; Z^N|U^N, X_1^N) - H(X_2^N|W_2, Z^N, U^N, X_1^N)),
\]

(13)
where (1) is from $H(W_1|X_1^N) = 0$ and (2) is from $X_1^N$ independent of $U^N$ and $X_2^N$.

Consider the first term in (13); the codeword generation and ([18] Lemma 3) LP1 ensure that:

$$\lim_{N \to \infty} \frac{1}{N} H(X_2^N|U^N) \geq I(X_2;Y|U).$$

(14)

For the second term in (13), using the same approach as that in ([2] Lemma 3), we get:

$$\lim_{N \to \infty} \frac{1}{N} I(X_2^N;Z^N|U^N, X_1^N) \geq I(X_2;Z|U, X_1).$$

(15)

Now, we consider the last term of (13). Given $U^N$, $Z^N$, $X_1^N$ and $W_2$, the total number of possible codewords of $X_1^N$ is $2^{NR^*}$. By using Fano’s inequality and (4), we have:

$$\lim_{N \to \infty} \frac{1}{N} H(W_2|Z_2^N, U^N, X_1^N) = 0.$$  

(16)

Substituting (14), (15) and (16) into (13), we have:

$$\lim_{N \to \infty} \frac{1}{N} H(W_2|W_1, Z^N) \geq I(X_2;Y|U) - I(X_2;Z|U, X_1) = R_2.$$  

(17)

Substituting (12) and (17) into (7), $\lim_{N \to \infty} \Delta \geq R_1 + R_2$ is proven.

The achievability proof of Theorem 1 is completed.

7. Converse Proof of Theorem 1

In this section, we prove the converse part of Theorem 1: all the achievable secrecy pairs $(R_1, R_2)$ are contained in the set $\mathcal{R}^P$. We will prove the inequalities of Theorem 1 in the remainder of this section.

(Proof of $R_1 \leq I(X_1;Y|X_2,U) - I(X_1;Z|U)$):

$$\frac{1}{N} H(W_1) \overset{(1)}{=} \frac{1}{N} H(W_1|Z^N)$$

$$= \frac{1}{N} (H(W_1|Z^N) - H(W_1|Z^N, W_2, Y^N) + H(W_1|Z^N, W_2, Y^N))$$

$$\overset{(2)}{=} \frac{1}{N} (H(W_1|W_2, Y^N|Z^N) + \delta(P_e))$$

$$\leq \frac{1}{N} (H(W_1|Z^N) - H(W_1|Z^N, W_2, Y^N, X_2^N) + \delta(P_e))$$

$$\overset{(3)}{=} \frac{1}{N} (H(W_1|Z^N) - H(W_1|Z^N, Y^N, X_2^N) + \delta(P_e))$$

$$= \frac{1}{N} (I(W_1;Y^N, X_2^N|Z^N) + \delta(P_e))$$

$$\leq \frac{1}{N} (H(Y^N, X_2^N|Z^N) - H(Y^N, X_2^N|Z^N, W_1, X_1^N) + \delta(P_e))$$

$$\overset{(4)}{=} \frac{1}{N} (H(Y^N, X_2^N|Z^N) - H(Y^N, X_2^N|Z^N, X_1^N) + \delta(P_e))$$

$$= \frac{1}{N} (I(Y^N, X_2^N; X_1^N|Z^N) + \delta(P_e))$$

$$\overset{(5)}{=} \frac{1}{N} (H(X_1^N|Z^N) - H(X_1^N|Z^N, Y^N, X_2^N) - H(X_1^N) + H(X_1^N|X_2^N) + \delta(P_e))$$

$$= \frac{1}{N} (I(X_1^N; Y^N|X_2^N) - I(X_1^N; Z^N) + \delta(P_e))$$
= \frac{1}{N} \sum_{i=1}^{N} (H(Y_i|Y^{i-1}, X_2^N) - H(Y_i|X_1,i, X_2,i) - H(Z_i|Z^{i-1}) + H(Z_i|Z^{i-1}, X_1^N)) + \frac{\delta(P_e)}{N}

\leq \frac{1}{N} \sum_{i=1}^{N} (H(Y_i|X_2,i, Z^{i-1}) - H(Y_i|X_1,i, X_2,i, Z^{i-1}) - H(Z_i|Z^{i-1}) + H(Z_i|Z^{i-1}, X_1,i)) + \frac{\delta(P_e)}{N}

\leq \frac{1}{N} \sum_{i=1}^{N} (H(Y_i|X_2,i, Z^{i-1}, J = i) - H(Y_i|X_1,i, X_2,i, Z^{i-1}, J = i) - H(Z_i|Z^{i-1}, J = i) + \frac{\delta(P_e)}{N}

\leq \frac{1}{N} \sum_{i=1}^{N} [H(Y_j|X_2,j, Z^{j-1}, J) - H(Y_j|X_1,j, X_2,j, Z^{j-1}, J) - H(Z_j|Z^{j-1}, J) + H(Z_j|Z^{j-1}, X_1,j, J) + \frac{\delta(P_e)}{N}

= \lim_{N \to \infty} \frac{1}{N} H(W_1, W_2|Z^N)

\leq \lim_{N \to \infty} \frac{1}{N} [H(Y|Y^{i-1}) - H(Y_i|X_1,i, X_2,i, Z^{i-1}) - H(Z_i|Z^{i-1}) + H(Z_i|X_1,i, X_2,i, Z^{i-1})] + \frac{\delta(P_e)}{N}

\leq \lim_{N \to \infty} \frac{1}{N} [H(Y|Y^{i-1}) - H(Y_i|X_1,i, X_2,i, Z^{i-1}) - H(Z_i|Z^{i-1}) + H(Z_i|X_1,i, X_2,i, Z^{i-1})] + \frac{\delta(P_e)}{N}

where (1) is from the definition of the perfect secrecy; (2) is from Fano’s inequality; (3) is from $H(W_2|X_2^N) = 0$; (4) is from $H(W_1|X_1^N) = 0$; (5) is from the Markov chain $X_1^N \to (X_2^N, Y^N) \to Z^N$ and the fact that $X_1^N$ is independent of $X_2^N$; (6) is from the Markov chains $Y_i \to (Y^{i-1}, X_2^i) \to Z^{i-1}$ and $Y_i \to (X_1,i, X_2,i) \to Z^{i-1}$; (7) is from $J$ is a random variable (uniformly distributed over $\{1, 2, ..., N\}$), and it is independent of $X_1^N, X_2^N, Y^N$ and $Z^N$; (8) is from $J$ is uniformly distributed over $\{1, 2, ..., N\}$; and (9) is from the definitions that $X_1 \triangleq X_1,j, X_2 \triangleq X_2,j, Y \triangleq Y_j, Z \triangleq Z_j$ and $U \triangleq (Z^{j-1}, J)$. 

By using $P_e \leq \epsilon, \epsilon \to 0$ as $N \to \infty$, $\lim_{N \to \infty} \frac{H(W_1)}{N} = R_1$ and (1), it is easy to see that $R_1 \leq I(X_1; Y|X_2, U) - I(X_1; Z|U)$.

(Proof of $R_2 \leq I(X_2; Y|X_1, U) - I(X_2; Z|U)$):

The proof is analogous to the proof of $R_1 \leq I(X_1; Y|X_2, U) - I(X_1; Z|U)$, and it is omitted here.

Proof of $R_1 + R_2 \leq I(X_1, X_2; Y|U) - I(X_1, X_2; Z|U)$:
\[
\lim_{N \to \infty} \left( \frac{1}{N} \sum_{i=1}^{N} (H(Y_i | Z_i^{i-1}, J = i) - H(Y_i | X_{1,i}, X_{2,i}, Z_i^{i-1}, J = i)) - H(Z_i | Z_i^{i-1}, J = i) + H(Z_i | X_{1,i}, X_{2,i}, Z_i^{i-1}, J = i)) + \frac{\delta(P_e)}{N} \right)
\]

\[
\lim_{N \to \infty} \left( H(Y_J | Z_J^{J-1}, J) - H(Y_J | X_{1,J}, X_{2,J}, Z_J^{J-1}, J) - H(Z_J | Z_J^{J-1}, J) + H(Z_J | X_{1,J}, X_{2,J}, Z_J^{J-1}, J) + \frac{\delta(P_e)}{N} \right)
\]

\[
I(X_1, X_2; Y | U) - I(X_1, X_2; Z | U),
\]

where (1) is from Fano’s inequality; (2) is from \((W_1, W_2) \to (X_1^N, X_2^N, Z^N) \to Y^N\); (3) is from \(Y_i \to (X_{1,i}, X_{2,i}) \to Z_i^{i-1}\) and \(Z_i \to (X_{1,i}, X_{2,i}) \to Z_i^{i-1}\); (4) is from \(Y_i \to Y_i^{i-1} \to Z_i^{i-1}\); (5) is from \(J\) is a random variable (uniformly distributed over \(\{1, 2, ..., N\}\)), and it is independent of \(X_1^N, X_2^N, Y^N\) and \(Z^N\); (6) is from \(J\) is uniformly distributed over \(\{1, 2, ..., N\}\); and (7) is from the definitions that \(X_1 \triangleq X_{1,J}, X_2 \triangleq X_{2,J}, Y \triangleq Y_J, Z \triangleq Z_J\) and \(U \triangleq (Z_J^{J-1}, J)\) and the fact that \(P_e \to 0\) as \(N \to \infty\).

By using \(\lim_{N \to \infty} \Delta \geq R_1 + R_2\) and (2), it is easy to see that \(R_1 + R_2 \leq I(X_1, X_2; Y | U) - I(X_1, X_2; Z | U)\).

The converse proof of Theorem 1 is completed.

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Author Contributions

Bin Dai and Zheng Ma did the theoretical work and wrote this paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References


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