Supplementary Material

Common equations used for the following derivations (cf. section (2.3) *Constrained maximization of a compound system*):

$$q_k = q_{i_1,i_2,\ldots,i_N} \tag{1}$$
$$g_k = g_{i_1,i_2,\ldots,i_N} = f_{i_1} + f_{i_2} + \ldots + f_{i_N} \tag{2}$$
$$\langle g \rangle = N \langle f \rangle \tag{3}$$

Used rule:

$$\sum_i \sum_j (x_i x_j) = \sum_i x_i \sum_j x_j = \sum_i x_i \cdot \sum_j x_j \tag{4}$$

The common set of summations splits into separate factors if the expression behind each summation only depends on the same index of its preceding summation symbol.

1 Derivation of eq. (31)

Initial expression (eq. (28)):

$$\langle g \rangle = \frac{\sum\limits_{i=1}^{m^N} g_i \; x_{\mathrm{c}}^{g_i}}{\sum\limits_{j=1}^{m^N} x_{\mathrm{c}}^{g_j}}$$

Replace the g_i by eq. (2):

$$\begin{aligned}
\langle g \rangle &= \frac{\sum\limits_{i_1=1}^{m} \sum\limits_{i_2=1}^{m} \ldots \sum\limits_{i_N=1}^{m} (f_{i_1} + f_{i_2} + \ldots + f_{i_N}) x_{\mathrm{c}}^{(f_{i_1}+f_{i_2}+\ldots+f_{i_N})}}{\sum\limits_{i_1=1}^{m} \sum\limits_{i_2=1}^{m} \ldots \sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{(f_{i_1}+f_{i_2}+\ldots+f_{i_N})}} \\
&= \frac{\sum\limits_{i_1=1}^{m} \sum\limits_{i_2=1}^{m} \ldots \sum\limits_{i_N=1}^{m} (f_{i_1} + f_{i_2} + \ldots + f_{i_N}) \, x_{\mathrm{c}}^{f_{i_1}} x_{\mathrm{c}}^{f_{i_2}} \ldots x_{\mathrm{c}}^{f_{i_N}}}{\sum\limits_{i_1=1}^{m} \sum\limits_{i_2=1}^{m} \ldots \sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_1}} x_{\mathrm{c}}^{f_{i_2}} \ldots x_{\mathrm{c}}^{f_{i_N}}}
\end{aligned}$$

Split the numerator into separate terms and place the factors in the numerator and the denominator behind the summations with the same indices. According to rule, eq. (4), this enables the introduction of factors consisting of the single summations in numerator and denominator:

$$\begin{aligned}
\langle g \rangle = &\frac{\sum\limits_{i_1=1}^{m} f_{i_1} x_{\mathrm{c}}^{f_{i_1}} \cdot \cancel{\sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \cancel{\sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}}{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \cancel{\sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \cancel{\sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}} \\
+ &\frac{\cancel{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \sum\limits_{i_2=1}^{m} f_{i_2} x_{\mathrm{c}}^{f_2} \cdot \ldots \cdot \cancel{\sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}}{\cancel{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \cancel{\sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}} \\
\vdots \\
+ &\frac{\cancel{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \cancel{\sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \sum\limits_{i_N=1}^{m} f_{i_N} x_{\mathrm{c}}^{f_{i_N}}}{\cancel{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \cancel{\sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}
\end{aligned}$$

Each of the terms can be reduced as follows:

$$\langle g\rangle = \frac{\sum\limits_{i_1=1}^{m} f_{i_1} x_{\mathrm{c}}^{f_{i_1}}}{\sum\limits_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} + \frac{\sum\limits_{i_2=1}^{m} f_{i_2} x_{\mathrm{c}}^{f_{i_2}}}{\sum\limits_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} + \ldots + \frac{\sum\limits_{i_N=1}^{m} f_{i_N} x_{\mathrm{c}}^{f_{i_N}}}{\sum\limits_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}$$

Each of the N terms represents the same expression with different indices $i_1 \ldots i_N$. Using the same index i for all terms results in the final expression (eq. (31)):

$$\langle g\rangle = N \frac{\sum\limits_{k=1}^{m} f_k x_{\mathrm{c}}^{f_k}}{\sum\limits_{k=1}^{m} x_{\mathrm{c}}^{f_k}}$$

2 Derivation of eq. (34) - first variant

Initial epression (eq. 30):

$$H_{\mathrm{c}} = -\frac{\sum\limits_{i=1}^{m^N} x_{\mathrm{c}}^{g_i} \ln x_{\mathrm{c}}^{g_i}}{\sum\limits_{j=1}^{m^N} x_{\mathrm{c}}^{g_j}} + \ln \sum_{i=1}^{m^N} x_{\mathrm{c}}^{g_i}$$

Abbreviations:

$$A = \sum_{i=1}^{m^N} x_{\mathrm{c}}^{g_i} \ln x_{\mathrm{c}}^{g_i} \tag{5}$$

$$B = \sum_{i=1}^{m^N} x_{\mathrm{c}}^{g_i} \tag{6}$$

$$H_{\mathrm{c}} = -\frac{A}{B} + \ln B \tag{7}$$

Replace the g_i by eq. (2).

Factor A:

$$\begin{aligned} A &= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_1}+f_{i_2}+\ldots+f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_1}+f_{i_2}+\ldots+f_{i_N}} \\ &= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_N=1}^{m} \left(x_{\mathrm{c}}^{f_{i_1}} \cdot x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot x_{\mathrm{c}}^{f_{i_N}} \right) \ln \left(x_{\mathrm{c}}^{f_{i_1}} \cdot x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot x_{\mathrm{c}}^{f_{i_N}} \right) \\ &= \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \left(\ln x_{\mathrm{c}}^{f_{i_1}} + \ln x_{\mathrm{c}}^{f_{i_2}} + \ldots + \ln x_{\mathrm{c}}^{f_{i_N}} \right) \end{aligned}$$

Split the numerator into separated terms and place the factors behind the summations with the same indices. According to the rule, eq. (4), this enables the introduction of factors consisting of the single summations:

$$
\begin{aligned}
A = {} & \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \\
& + \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \ln x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \\
& \vdots \\
& + \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_N}}
\end{aligned}
\tag{8}
$$

Factor B:

$$
\begin{aligned}
B &= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \ldots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{(f_{i_1}+f_{i_2}+\ldots+f_{i_N})} \\
&= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \ldots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_1}} x_{\mathrm{c}}^{f_{i_2}} \ldots x_{\mathrm{c}}^{f_{i_N}}
\end{aligned}
$$

According to rule, eq. (4), this is:

$$
B = \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \tag{9}
$$

Combine eqs. (8) and (9):

$$
\begin{aligned}
\frac{A}{B} = {} & \frac{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \cdot \cancel{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \cancel{\sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}}{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \cancel{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \cancel{\sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}} \\
& + \frac{\cancel{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \ln x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \cancel{\sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}}{\cancel{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \cancel{\sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}} \\
& \vdots \\
& + \frac{\cancel{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \cancel{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_N}}}{\cancel{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}} \cdot \cancel{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}
\end{aligned}
$$

This expression can be reduced as follows:

$$\frac{A}{B} = \frac{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}}}{\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}}}$$
$$+ \frac{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \ln x_{\mathrm{c}}^{f_{i_2}}}{\sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}}}$$
$$\vdots$$
$$+ \frac{\sum_{i_1=N}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_N}}}{\sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}}$$

All the N single terms have the same value, expressed only by different indices $i_1 \dots i_N$. By using the same index i this expression can by simplified to:

$$\frac{A}{B} = N \cdot \frac{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \ln x_{\mathrm{c}}^{f_i}}{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i}} \tag{10}$$

Factor B, eq. (9), can be simplified when taking into account that all the N terms give the same expression with different indices $i_1 \dots i_N$. Using the same index i for all factors yields:

$$B = \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^N \tag{11}$$

$$\ln B = N \cdot \ln \sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \tag{12}$$

Inserting eqs. (10) and (12) into eq. (7) gives the final expression (eq. (34)):

$$H_{\mathrm{c}} = N \left(- \frac{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \ln x_{\mathrm{c}}^{f_i}}{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i}} + \ln \sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)$$

3 Derivation of eq. (35)

Initial expression (eq. (29)):

$$q_k = \frac{x_c^{g_k}}{\sum_{j=1}^{m^N} x_c^{g_j}} \quad , k = 1 \ldots m^N \tag{13}$$

Insert eqs. (1) and (2) into eq. (13) and use the result for the denominator, eq. (11):

$$q_{i_1,i_2,\ldots,i_N} = \frac{x_c^{(f_{i_1}+f_{i_2}+\ldots+f_{i_N})}}{\left(\sum_{j=1}^{m} x_c^{f_j}\right)^N}$$

Splitting the numerator into separated factors immediately yields the final expression (eq. (35)):

$$q_{i_1,i_2,\ldots,i_N} = \frac{x_c^{f_{i_1}} \cdot x_c^{f_{i_2}} \cdot \ldots \cdot x_c^{f_{i_N}}}{\left(\sum_{i=j}^{m} x_c^{f_j}\right)^N} \tag{14}$$

4 Derivation of eq. (34) - second variant

Initial expression (eq. (27)):

$$H_\mathrm{c} = -\sum_{k=1}^{m^N} q_k \ln q_k \tag{15}$$

Use eqs. (1) and (14):

$$\begin{aligned} H_\mathrm{c} &= -\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\dots\sum_{i_N=1}^{m} q_{i_1,i_2,\dots,i_N} \ln q_{i_1,i_2,\dots,i_N} \\ &= -\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\dots\sum_{i_N=1}^{m} \frac{x_\mathrm{c}^{f_{i_1}} x_\mathrm{c}^{f_{i_2}} \dots x_\mathrm{c}^{f_{i_N}}}{\left(\sum_{i=1}^{m} x_\mathrm{c}^{f_i}\right)^N} \ln \frac{x_\mathrm{c}^{f_{i_1}} x_\mathrm{c}^{f_{i_2}} \dots x_\mathrm{c}^{f_{i_N}}}{\left(\sum_{i=1}^{m} x_\mathrm{c}^{f_i}\right)^N} \end{aligned}$$

Splitting the logarithm of the fraction into two logarithmic terms, this expression can be rewritten as

$$H_\mathrm{c} = -\frac{D}{\left(\sum_{i=1}^{m} x_\mathrm{c}^{f_i}\right)^N} + \frac{E}{\left(\sum_{i=1}^{m} x_\mathrm{c}^{f_i}\right)^N} \ , \tag{16}$$

with the abbreviations

$$D = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\dots\sum_{i_N=1}^{m} x_\mathrm{c}^{f_{i_1}} x_\mathrm{c}^{f_{i_2}} \dots x_\mathrm{c}^{f_{i_N}} \ln x_\mathrm{c}^{f_{i_1}} x_\mathrm{c}^{f_{i_2}} \dots x_\mathrm{c}^{f_{i_N}} \tag{17}$$

$$E = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\dots\sum_{i_N=1}^{m} x_\mathrm{c}^{f_{i_1}} x_\mathrm{c}^{f_{i_2}} \dots x_\mathrm{c}^{f_{i_N}} \ln \left(\sum_{i=1}^{m} x_\mathrm{c}^{f_i}\right)^N \tag{18}$$

Evaluation of D, eq. (17): split the logarithms of the products into logarithmic terms and position them according to the summation indices:

$$D = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \dots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_1}} x_{\mathrm{c}}^{f_{i_2}} \dots x_{\mathrm{c}}^{f_{i_N}} \left(\ln x_{\mathrm{c}}^{f_{i_1}} + \ln x_{\mathrm{c}}^{f_{i_2}} + \dots + \ln x_{\mathrm{c}}^{f_{i_N}} \right) \tag{19}$$

$$\begin{aligned} &= \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \dots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \\ &+ \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \ln x_{\mathrm{c}}^{f_{i_2}} \dots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \\ &\vdots \\ &+ \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \dots \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_2}} \end{aligned} \tag{20}$$

According to rule (4) the multiple summations in each of the terms can be written as multiple products. For example, the first term in eq. (20) can be written as:

$$D_{\text{first term}} = \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}}$$

Besides the first factor containing a logarithmic part, all the other $N-1$ factors represent the same expression with different indices $i_2 \dots i_N$. Using the same index i for all factors this line can be rewritten as:

$$D_{\text{first term}} = \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N-1}$$

The same holds for all of the N terms of D, eq. (20):

$$\begin{aligned} D = &\sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \ln x_{\mathrm{c}}^{f_{i_1}} \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N-1} \\ &+ \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \ln x_{\mathrm{c}}^{f_{i_2}} \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N-1} \\ &\vdots \\ &+ \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln x_{\mathrm{c}}^{f_{i_2}} \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N-1} \end{aligned}$$

Obviously all N terms represent the same expression by means of different indices $i_1 \ldots i_N$, and by using the same index i we get:

$$D = N \cdot \sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \ln x_{\mathrm{c}}^{f_i} \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N-1} \tag{21}$$

Evaluation of E, (18):Position the factors behind the according summations and apply rule (4):

$$E = \sum_{i_1=1}^{m} x_{\mathrm{c}}^{f_{i_1}} \cdot \sum_{i_2=1}^{m} x_{\mathrm{c}}^{f_{i_2}} \cdot \ldots \cdot \sum_{i_N=1}^{m} x_{\mathrm{c}}^{f_{i_N}} \ln \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N} \tag{22}$$

All N factors represent the same expression by means of different indices $i_1 \ldots i_N$, and by using the same index i we get:

$$E = \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N} \cdot \ln \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N} \tag{23}$$

$$= N \cdot \left(\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right)^{N} \cdot \ln \sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \tag{24}$$

Inserting expression (21) and (24) into eq. (16) yields the final expression (eq. (34)):

$$H_{\mathrm{c}} = N \left(- \frac{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \ln x_{\mathrm{c}}^{f_i}}{\sum_{i=1}^{m} x_{\mathrm{c}}^{f_i}} + \ln \sum_{i=1}^{m} x_{\mathrm{c}}^{f_i} \right) \tag{25}$$