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The Entropy of Co-Compact Open Covers

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Abstract: Co-compact entropy is introduced as an invariant of topological conjugation for perfect mappings defined on any Hausdorff space (compactness and metrizability are not necessarily required). This is achieved through the consideration of co-compact covers of the space. The advantages of co-compact entropy include: (1) it does not require the space to be compact and, thus, generalizes Adler, Konheim and McAndrew's topological entropy of continuous mappings on compact dynamical systems; and (2) it is an invariant of topological conjugation, compared to Bowen's entropy, which is metric-dependent. Other properties of co-compact entropy are investigated, e.g., the co-compact entropy of a subsystem does not exceed that of the whole system. For the linear system, (R, f), defined by $f(x) = 2x$, the co-compact entropy is zero, while Bowen's entropy for this system is at least $\log 2$. More generally, it is found that co-compact entropy is a lower bound of Bowen's entropies, and the proof of this result also generates the Lebesgue Covering Theorem to co-compact open covers of non-compact metric spaces.

Keywords: topological dynamical system; perfect mapping; co-compact open cover; topological entropy; topological conjugation; Lebesgue number

Classification: MSC 54H20; 37B40

1. Introduction

1.1. Measure-Theoretic Entropy

The concept of entropy per unit time was introduced by Shannon [1], by analogy with the standard Boltzmann entropy measuring a spatial disorder in a thermodynamic system. In the 1950s, Kolmogorov [2] and Sinai established a rigorous definition of K-S entropy per unit time for dynamical systems and other random processes [3]. Kolmogorov imported Shannon's probabilistic notion of entropy into the theory of dynamical systems, and the idea was vindicated later by Ornstein, who showed that metric entropy suffices to completely classify two-sided Bernoulli processes [4], a basic problem, which for many decades, appeared completely intractable. Kolmogorov's metric entropy is an invariant of measure theoretical dynamical systems and is closely related to Shannon's source entropy. The K-S entropy is a powerful concept, because it controls the top of the hierarchy of ergodic properties: K-S property ⇒ multiple mixing ⇒ mixing ⇒ weak mixing ⇒ ergodicity [3]. The K-S property holds if there exists a subalgebra of measurable sets in phase space, which generates the whole algebra by application of the flow [3]. The dynamical randomness of a deterministic system finds its origin in the dynamical instability and the sensitivity to initial conditions. In fact, the K-S entropy is related to the Lyapunov exponents, according to a generalization of Pesin's theorem [5,6]. A deterministic system with a finite number of degrees of freedom is chaotic if its K-S entropy per unit time is positive. More properties about K-S entropy can be found in papers [3,5,7]. The concept of space-time entropy or entropy per unit time and unit volume was later introduced by Sinai and Chernov [8]. A spatially extended system with a probability measure being invariant under space and time translations can be said to be chaotic if its space-time entropy is positive.

1.2. Topological Entropy and Its Relation to Measure-Theoretic Entropy

In 1965, Adler, Konheim and McAndrew introduced the concept of topological entropy for continuous mappings defined on compact spaces [9], which is an analogous invariant under conjugacy of topological dynamical systems and can be obtained by maximizing the metric entropy over a suitable class of measures defined on a dynamical system, implying that topological entropy and measure-theoretic entropy are closely related. Goodwyn in 1969 and 1971, motivated by a conjecture of Adler, Konheim and McAndrew [9], compared topological entropy and measure-theoretic entropy and concluded that topological entropy bounds measure-theoretic entropy [10,11]. In 1971, Bowen generalized the concept of topological entropy to continuous mappings defined on metric spaces and proved that the new definition coincides with that of Adler, Konheim and McAndrew's within the class of compact spaces [12]. However, the entropy according to Bowen's definition is metric-dependent [13] and can be positive even for a linear function (Example 5.1 or Walters' book, pp.176). In 1973, along with a study of measure-theoretic entropy, Bowen [12] gave another definition of topological entropy resembling Hausdorff dimension, which also equals to the topological entropy defined by Adler, Konheim and McAndrew when the space is compact. Recently, Cánovas and Rodríguez, and Malziri and Molaci proposed other definitions of topological entropy for continuous mappings defined on non-compact metric spaces [14,15].

1.3. The Importance of Entropy

The concepts of entropy are useful for studying topological and measure-theoretic structures of dynamical systems. For instance, two conjugated systems have the same entropy, and thus, entropy is a numerical invariant of the class of conjugated dynamical systems. Upper bounds on the topological entropy of expansive dynamical systems are given in terms of the ϵ−entropy, which was introduced by Kolmogorov-Tikhomirov [2]. The theory of expansive dynamical systems has been closely related to the theory of topological entropy [16–18]. Entropy and chaos are closely related, e.g., a continuous mapping, $f : I \to I$, is chaotic if and only if it has a positive topological entropy [19]. But this result may fail when the entropy is zero, because of the existence of minimum chaotic (transitive) systems [20,21]. A remarkable result is that a deterministic system together with an invariant probability measure defines a random process. As a consequence, a deterministic system can generate dynamical randomness, which is characterized by an entropy per unit time that measures the disorder of the trajectories along the time axis. Entropy has many applications, e.g., transport properties in escape-rate theory [22–26], where an escape of trajectories is introduced by absorbing conditions at the boundaries of a system. These absorbing boundary conditions select a set of phase-space trajectories, forming a chaotic and fractal repeller, which is related to an equation for K-S entropy. The escape-rate formalism has applications in diffusion [27], reaction-diffusion [28] and, recently, viscosity [29]. Another application is the classification of quantum dynamical systems, which is given by Ohya [30]. Symbolic dynamical systems $(\sum(p), \sigma)$ have various representative and complicated dynamical properties and characteristics, with an entropy $\log p$. When determining whether or not a given topological dynamical system has certain dynamical complexity, it is often compared with a symbolic dynamical system [21,31]. For the topological conjugation with symbolic dynamical systems, we refer to Ornstein [4], Sinai [32], Akashi [33] and Wang and Wei [34,35].

1.4. The Purpose, the Approach and the Outlines

The main purpose of this article is to introduce a topological entropy for perfect mappings defined on arbitrary Hausdorff spaces (compactness and metrizability are not necessarily required) and investigate fundamental properties of such an entropy.

Instead of using all open covers of the space to define entropy, we consider the open covers consisting of the co-compact open sets (open sets whose complements are compact).

Various definitions of entropy and historical notes are mentioned previously in this section. Section 2 investigates the topological properties of co-compact open covers of a space. Section 3 introduces the new topological entropy defined through co-compact covers of the space, which is called co-compact entropy in the paper, and further explores the properties of the co-compact entropy and compares it with Adler, Konheim and McAndrew's topological entropy for compact spaces. Sections 4 investigates the relation between the co-compact entropy and Bowen's entropy. More precisely, Section 4 compares the co-compact entropy with that given by Bowen for systems defined on metric spaces. Because the spaces under consideration include non-compact metric spaces, the traditional Lebesgue Covering Theorem does not apply. Thus, we generalize this theorem to co-compact open covers of non-compact metric spaces. Based on the generalized Lebesgue Covering Theorem, we show that the co-compact entropy

is a lower bound for Bowen's entropies. In Section 4.2, a linear dynamical system is studied. For this simple system, its co-compact entropy is zero, which is appropriate, but Bowen's entropy is positive.

2. Basic Concepts and Definitions

Let (X, f) be a topological dynamical system, where X is a Hausdorff and $f : X \to X$ is a continuous mapping. We introduce the concept of co-compact open covers as follows.

Definition 2.1 Let X be a Hausdorff space. For an open subset, U of X, if $X \backslash U$ is a compact subset of X, then U is called a co-compact open subset. If every element of an open cover $\mathcal{U}$ of X is co-compact, then $\mathcal{U}$ is called a co-compact open cover of X.

Theorem 2.1 *The intersection of finitely many co-compact open subsets is co-compact, and the union of any collection of co-compact open subsets is co-compact open.*

Proof. Suppose that $U_1, U_2, ..., U_n$ are co-compact open. Let $U = \bigcap_{i=1}^{n} U_i$. As $X \setminus U_i, i = 1, 2, ...n$ are compact, $X \setminus U = \bigcup_{i=1}^{n} (X \setminus U_i)$ is compact, and hence, U is co-compact open.

Suppose that $\{U_\lambda\}_{\lambda \in \Lambda}$ is a family of co-compact sets. Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. As any $\lambda \in \Lambda$ $X \backslash U_\lambda$ is compact, $X \backslash U = \bigcap_{\lambda \in \Lambda} (X \backslash U_\lambda)$ is compact. Hence, U is co-compact open. $\square$

Theorem 2.2 *Let X be Hausdorff. Then, any co-compact open cover has a finite subcover.*

Proof. Let $\mathcal{U}$ be a co-compact open cover. For any $U \in \mathcal{U}$, $X \backslash U$ is compact. Noting that $\mathcal{U}$ is also an open cover of $X \setminus U$, there exists a finite subcover, $\mathcal{V}$, of $X \setminus U$. Now, $\mathcal{V} \cup \{U\}$ is a finite subcover of $\mathcal{U}$. $\square$

Definition 2.2 Let X and Y be Hausdorff spaces and let $f : X \to Y$ be a continuous mapping. If f is a closed mapping and all fibers, $f^{-1}(x), x \in Y$, are compact, then f is called a perfect mapping.

In particular, if X is compact Hausdorff and Y is Hausdorff, every continuous mapping from X into Y is perfect. If $f : X \to Y$ is perfect, then $f^{-1}(F)$ is compact for each compact subset, $F \subseteq Y$ [36].

Theorem 2.3 *Let X and Y be two Hausdorff spaces and let $f : X \to Y$ be a perfect mapping. If U is co-compact open in Y, then $f^{-1}(U)$ is co-compact open in X. Moreover, if $\mathcal{U}$ is a co-compact open cover of Y, then $f^{-1}(\mathcal{U})$ is a co-compact Open Cover of X.*

Proof. It suffices to show that the pre-image of any co-compact set is co-compact. Let U be co-compact open in Y. Then, $F = Y \setminus U$ is compact in Y. As f is perfect, $f^{-1}(F)$ is compact in X. Hence, $f^{-1}(U) = X \setminus f^{-1}(F)$ is co-compact open in X. $\square$

3. The Entropy of Co-Compact Open Covers

For compact topological systems, Adler, Konheim and McAndrew introduced the concept of topological entropy and studied its properties [9]. Their definition is as follows: Let X be a compact topological space and $f : X \to X$ a continuous mapping. For any open cover, $\mathcal{U}$ of X, let $N_X(\mathcal{U})$ denote the smallest cardinality of all subcovers of $\mathcal{U}$, *i.e.*,

$$N_X(\mathcal{U}) = \min\{card(\mathcal{V}) : \mathcal{V} \text{ is a subcover of } \mathcal{U}\}$$

It is obvious that $N_X(\mathcal{U})$ is a positive integer. Let $H_X(\mathcal{U}) = \log N_X(\mathcal{U})$. Then, $ent(f, \mathcal{U}, X) = \lim\limits_{n\to\infty} \frac{1}{n} H_X(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U}))$ is called the topological entropy of f relative to $\mathcal{U}$, and $ent(f, X) = \sup\limits_{\mathcal{U}}\{ent(f, \mathcal{U}, X)\}$ is called the topological entropy of f.

Now, we will generalize Adler, Konheim and McAndrew's entropy to any Hausdorff space for perfect mappings. Therefore, in the remainder of the paper, a space is assumed to be Hausdorff and a mapping is assumed to be perfect.

Let X be Hausdorff. By Theorem 2.2, when $\mathcal{U}$ is a co-compact open cover of X, $\mathcal{U}$ has a finite subcover. Hence, $N_X(\mathcal{U})$, abbreviated as $N(\mathcal{U})$, is a positive integer. Let $H_X(\mathcal{U}) = \log N(\mathcal{U})$, abbreviated as $H(\mathcal{U})$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of X. Define

$$\mathcal{U}\bigvee\mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$$

If for any $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$, such that $U \subseteq V$, then $\mathcal{U}$ is said to be a refinement of $\mathcal{V}$ and is denoted by $\mathcal{V} \prec \mathcal{U}$.

The following are some obvious facts:

Fact 1: For any open covers, $\mathcal{U}$ and $\mathcal{V}$, of X, $\mathcal{U} \prec \mathcal{U}\bigvee\mathcal{V}$.
Fact 2: For any open covers, $\mathcal{U}$ and $\mathcal{V}$, of X, if $\mathcal{V}$ is a subcover of $\mathcal{U}$, then $\mathcal{U} \prec \mathcal{V}$.
Fact 3: For any co-compact open cover, $\mathcal{U}$, of X, $H(\mathcal{U}) = 0 \iff N(\mathcal{U}) = 1 \iff X \in \mathcal{U}$.
Fact 4: For any co-compact open covers, $\mathcal{U}$ and $\mathcal{V}$, of X, $\mathcal{V} \prec \mathcal{U} \Rightarrow H(\mathcal{V}) \leq H(\mathcal{U})$.
Fact 5: For any co-compact open covers, $\mathcal{U}$ and $\mathcal{V}$, $H(\mathcal{U}\bigvee\mathcal{V}) \leq H(\mathcal{U}) + H(\mathcal{V})$.

To prove Fact 5, let $\mathcal{U}_0$ be a finite subcover of $\mathcal{U}$, with the cardinality, $N(\mathcal{U})$. Let $\mathcal{V}_0$ be a finite subcover of $\mathcal{V}$ with the cardinality, $H(\mathcal{V})$. Then, $\mathcal{U}_0\bigvee\mathcal{V}_0$ is a subcover of $\mathcal{U}\bigvee\mathcal{V}$, and the cardinality of $\mathcal{U}_0\bigvee\mathcal{V}_0$ is at most $N(\mathcal{U}) \times N(\mathcal{V})$. Hence, $N(\mathcal{U}\bigvee\mathcal{V}) \leq N(\mathcal{U}) \times N(\mathcal{V})$, and therefore, $H(\mathcal{U}\bigvee\mathcal{V}) \leq H(\mathcal{U}) + H(\mathcal{V})$.

Fact 6: For any co-compact open cover, $\mathcal{U}$, of X, $H(f^{-1}(\mathcal{U})) \leq H(\mathcal{U})$, and if $f(X) = X$, the equality holds.

To prove Fact 6, let $\mathcal{U}_0$ be a finite subcover of $\mathcal{U}$, with the cardinality, $N(\mathcal{U})$. $f^{-1}(\mathcal{U}_0)$ is a subcover of $f^{-1}(\mathcal{U})$. Hence, we have $H(f^{-1}(\mathcal{U})) \leq H(\mathcal{U})$.

Now, assume $f(X) = X$. Let $\{f^{-1}(U_1), f^{-1}(U_2), ..., f^{-1}(U_n)\}$, $U_i \in \mathcal{U}$ be a finite subcover of $f^{-1}(\mathcal{U})$, with the cardinality, $N(f^{-1}(\mathcal{U}))$. As $X \subseteq \bigcup\limits_{i=1}^{n} f^{-1}(U_i)$, we have $X = f(X) \subseteq \bigcup\limits_{i=1}^{n} f(f^{-1}(U_i)) = \bigcup\limits_{i=1}^{n} U_i$. Hence, $U_1, U_2, ..., U_n$ is a finite subcover of $\mathcal{U}$. This shows $H(\mathcal{U}) \leq H(f^{-1}(\mathcal{U}))$. This inequality and the previous inequality together imply the required equality.

Lemma 3.1 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying $a_{n+p} \leq a_n + a_p, n \geq 1, p \geq 1$. Then, $\lim\limits_{n \to \infty} \frac{a_n}{n}$ exists and is equal to $\inf \frac{a_n}{n}$ (see [13]).* $\square$

Let $\mathcal{U}$ be a co-compact open cover of X. By Theorem 2.3, for any positive integer, n, and perfect mapping, $f : X \to X$, $f^{-n}(\mathcal{U})$ is a co-compact open cover of X. On the other hand, by Theorem 2.1, $\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})$ is a co-compact open cover of X. These two facts together lead to the following result:

Theorem 3.1 *Suppose that X is Hausdorff. Let $\mathcal{U}$ be a co-compact open cover of X, and $f : X \to X$, a perfect mapping. Then, $\lim\limits_{n \to \infty} \frac{1}{n} H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U}))$ exists.*

Proof. Let $a_n = H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U}))$. By Lemma 3.1, it suffices to show $a_{n+k} \leq a_n + a_k$. Now, Fact 6 gives $H(f^{-1}(\mathcal{U})) \leq H(\mathcal{U})$, and more generally, $H(f^{-j}(\mathcal{U})) \leq H(\mathcal{U}), j = 0, 1, 2, ...$ Hence, by applying Fact 5, we have $a_{n+k} = H(\bigvee\limits_{i=0}^{n+k-1} f^{-i}(\mathcal{U})) = H((\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})) \bigvee (\bigvee\limits_{j=n}^{n+k-1} f^{-j}(\mathcal{U}))) = H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U}) \bigvee (\bigvee\limits_{j=0}^{k-1} f^{-n}(f^{-j}(\mathcal{U})))) \leq H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})) + H(\bigvee\limits_{j=0}^{k-1} f^{-j}(\mathcal{U})) = a_n + a_k.$ $\square$

Next, we introduce the concept of entropy for co-compact open covers.

Definition 3.1 Let X be a Hausdorff space, $f : X \to X$ be a perfect mapping, and $\mathcal{U}$ be a co-compact open cover of X. The non-negative number, $h_c(f, \mathcal{U}) = \lim\limits_{n \to \infty} \frac{1}{n} H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U}))$, is said to be the co-compact entropy of f relative to $\mathcal{U}$, and the non-negative number, $h_c(f) = \sup\limits_{\mathcal{U}} \{h_c(f, \mathcal{U})\}$, is said to be the co-compact entropy of f.

In particular, when X is compact Hausdorff, any open set of X is co-compact, and any continuous mapping $f : X \to X$ is perfect. Hence, Adler, Konheim and McAndrew's topological entropy is a special case of our co-compact entropy. It should be made aware that the new entropy is well defined for perfect mappings on non-compact spaces, e.g., on R^n, but Adler, Konheim and McAndrew's topological entropy requires that the space be compact.

Co-compact entropy generalizes Adler, Konheim and McAndrew's topological entropy, and yet, it holds various similar properties, as well, as demonstrated by the fact that co-compact entropy is an invariant of topological conjugation (next theorem) and more explored in the next section.

Recall that ent denotes Adler, Konheim and McAndrew's topological entropy, and h_c denotes the co-compact entropy.

Theorem 3.2 *Let (X, f) and (Y, g) be two topological dynamical systems, where X and Y are Hausdorff, $f : X \to X$ and $g : Y \to Y$ are perfect mappings. If there exists a semi-topological conjugation, $h : X \to Y$, where h is also perfect, then $h_c(f) \geq h_c(g)$. Consequently, when h is a topological conjugation, we have $h_c(f) = h_c(g)$.*

Proof. Let $\mathcal{U}$ be any co-compact open cover of Y. As h is perfect and $\mathcal{U}$ is a co-compact open cover of Y, $h^{-1}(\mathcal{U})$ is co-compact open cover of X by applying Theorem 2.3. Hence, we have:

$$h_c(g,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{U})) = \lim_{n\to\infty} \frac{1}{n} H(h^{-1}(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{U})))$$

$$= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} h^{-1}(g^{-i}(\mathcal{U}))) = \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(h^{-1}(\mathcal{U})))$$

$$= h_c(f, h^{-1}(\mathcal{U})) \le h_c(f,\mathcal{U})$$

Therefore, $h_c(f) \ge h_c(g)$.

When h is a topological conjugation, it is, of course, perfect, too. Hence, we have both $h_c(f) \ge h_c(g)$ and $h_c(g) \ge h_c(f)$ from the above proof, implying $h_c(f) = h_c(g)$. $\square$

Remark: The condition that the conjugation map is perfect is crucial in this result. In the general case, the inequality given by conjugacy need not hold. Cánovas and Rodríguez [14] defined an entropy for non-compact spaces that has this property (Theorem 2.1 (a)), which can be applied for non-perfect maps. Notice that Cánovas and Rodríguez's definition does not depend on the metric that generates the given topology of X. This is due to the fact that for compact metric spaces, the definition of Bowen's entropy does not depend on the metric. Since Cánovas and Rodríguez's definition is based on invariant compact sets, and they are the same for equivalent metrics, that is, metrics that generate the same topology of X, Cánovas and Rodríguez's definition does not depend on the metric when topology is fixed [37].

We sum up some properties of the new definition of topological entropy in the following results. A minor adaptation of the proof of standard techniques on topological entropy (e.g., [13]) gives the proof of these results. These properties are comparable to that of Adler, Konheim and McAndrew's topological entropy.

Theorem 3.3 *Let X be Hausdorff and $id : X \to X$ be the identity mapping. Then $h_c(id) = 0$.*

When X is Hausdorff and $f : X \to X$ is perfect, $f^m : X \to X$ is also a perfect mapping [36].

Theorem 3.4 . *Let X be Hausdorff and $f : X \to X$ be perfect. Then, $h_c(f^m) = m \cdot h_c(f)$.*

Theorem 3.5 *Let X be Hausdorff and $f : X \to X$ be perfect. If Λ is a closed subset of X and invariant under f, i.e., $f(\Lambda) \subseteq \Lambda$, then $h_c(f|_\Lambda) \le h_c(f)$.*

4. Relations between Co-Compact Entropy and Bowen's Entropy

4.1. Co-Compact Entropy Less Than or Equal to Bowen's Entropy, $h_c(f) \le h_d(f)$

First let us recall the definition of Bowen's entropy [13,38]. Let (X, d) be a metric space and $f : X \to X$ a continuous mapping. A compact subset, E of X, is called a (n, ϵ)-separated set with respect to f if for any different $x, y \in E$, there exists an integer, j, with $0 \le j < n$, such that $d(f^j(x), f^j(y)) > \epsilon$. A subset, F, of X is called a (n, ϵ)-spanning set of a compact set, K, relative to f if for any $x \in K$, there exists $y \in F$, such that for all j satisfying $0 \le j < n$, $d(f^j(x), f^j(y)) \le \epsilon$.

Let K be a compact subset of X. Put

$$r_n(\epsilon, K, f) = \min\{card(F) : F \text{ is a } (n, \epsilon)-\text{spanning set for } K \text{ with respect to } f\}$$

$$s_n(\epsilon, K, f) = \max\{card(F) : F \subseteq K \text{ and } F \text{ is a } (n, \epsilon)-\text{separated set with respect to } f\}$$

$$r(\epsilon, K, f) = \lim_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K, f), \qquad s(\epsilon, K, f) = \lim_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K, f)$$

$$r(K, f) = \lim_{\epsilon \to 0} r(\epsilon, K, f), \qquad s(K, f) = \lim_{\epsilon \to 0} s(\epsilon, K, f)$$

Then, $\sup_K r(K, f) = \sup_K s(K, f)$, and this non-negative number, denoted by $h_d(f)$, is the Bowen entropy of f.

It should be pointed out that Bowen's entropy, $h_d(f)$, is metric-dependent, see e.g., [13,39]. For the topology of the metrizable space, X, the selection of different metrics may result in different entropies.

Next, recall the Lebesgue Covering Theorem and Lebesgue Number [36]. Let (X, d) be a metric space and $\mathcal{U}$ an open cover of X. $diam(\mathcal{U}) = \sup\{d(A) \mid A \in \mathcal{U}\}$ is called the diameter of $\mathcal{U}$, where $d(A) = \sup\{d(x, y) \mid x, y \in A\}$. A real number, δ, is said to be a Lebesgue number of $\mathcal{U}$ if every open subset, U, of X, satisfying $diam(U) < \delta$, is completely contained in an element of the cover, $\mathcal{U}$.

The Lebesgue Covering Theorem (see [36]): Every open cover of a compact metric space has a Lebesgue number. □

Our next theorem generalizes the Lebesgue Covering Theorem to all co-compact open covers of non-compact metric spaces.

Theorem 4.1 *Let (X, d) be a metric space, regardless of compactness. Then, every co-compact open cover of X has a Lebesgue number.*

Proof. Let $\mathcal{U}$ be any co-compact open cover of X. By Theorem 2.2, $\mathcal{U}$ has a finite subcover $\mathcal{V} = \{V_1, V_2, ..., V_m\}$. Put $Y = (X \setminus V_1) \cup (X \setminus V_2) \cup ... \cup (X \setminus V_m)$. Then, Y is compact as V_i's are co-compact.

We will prove that $\mathcal{V}$ has a Lebesgue number, so does $\mathcal{U}$. As it is obvious that the theorem holds when $Y = \emptyset$, thus in the following proof, we assume $Y \neq \emptyset$.

Assume that $\mathcal{V}$ does not have a Lebesgue number. Then, for any positive integer, n, $\frac{1}{n}$ is not a Lebesgue number of $\mathcal{V}$. Consequently, for each positive integer, n, there exists an open subset, O_n, of X, satisfying $diam(O_n) < \frac{1}{n}$, but O_n is not completely contained in any element of $\mathcal{V}$, *i.e.*, $O_n \cap (X \setminus V_j) \neq \emptyset, j = 1, 2, ..., m$. Hence, $O_n \cap Y \neq \emptyset$. Take an $x_n \in O_n \cap Y$. By the compactness of Y, the sequence x_n has a subsequence, x_{n_i}, that is convergent to some point, $y \in Y$, *i.e.*, $\lim_{i \to \infty} x_{n_i} = y \in Y \subseteq X$.

On the other hand, $\mathcal{V}$ is an open cover of X, thus there exists some $V \in \mathcal{V}$, such that $y \in V$. As V is open, there exists an open neighborhood, $S(y, \epsilon)$, of y, such that $y \in S(y, \epsilon) \subseteq V$. Since x_{n_i} converges to y, there exists a positive integer, M, such that $x_{n_i} \in S(y, \frac{\epsilon}{2})$ for $i > M$. Let k be any integer larger than $M + \frac{2}{\epsilon}$. Then, for any $z \in O_{n_k}$, we have $d(z, y) \leq d(z, x_{n_k}) + d(x_{n_k}, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, thus $O_{n_k} \subseteq S(y, \epsilon) \subseteq V \in \mathcal{V}$, which contradicts the selection of open sets, O_n's.

Therefore, $\mathcal{V}$ has a Lebesgue number. □

Theorem 4.2 *Let (X, d) be a metric space, $\mathcal{U}$ be any co-compact open cover of X, and $f : X \to X$ be a perfect mapping. Then, there exists $\delta > 0$ and a compact subset K of X, such that for all positive integers, n,*

$$N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq n \cdot r_n(\frac{\delta}{3}, K, f) + 1$$

Proof. Let $\mathcal{U}$ be any co-compact open cover of X. By Theorem 2.2, $\mathcal{U}$ has a finite subcover, $\mathcal{V} = \{V_1, V_2, ..., V_m\}$. By Theorem 4.1, $\mathcal{U}$ has a Lebesgue number, δ. Put $K = (X \setminus V_1) \cup (X \setminus V_2) \cup ... \cup (X \setminus V_m)$. If $K = \emptyset$, then $X = V_j$ for all $j = 1, 2, ..., m$, and in this case, the theorem clearly holds. Hence, we assume $K \neq \emptyset$; thus, the compact set, K, has a $(n, \frac{\delta}{3})$-spanning set, F, relative to f and satisfying $card(F) = r_n(\frac{\delta}{3}, K, f)$.

(a) For any $x \in K$, there exists, $y \in F$, such that $d(f^i(x), f^i(y)) \leq \frac{\delta}{3}, i = 0, 1, ..., n-1$; equivalently, $x \in f^{-i}(\overline{S(f^i(y), \frac{\delta}{3})}), i = 0, 1, ..., n-1$. Hence, $K \subseteq \bigcup_{y \in F} \bigcap_{i=0}^{n-1} f^{-i}(\overline{S(f^i(y), \frac{\delta}{3})})$. By the definition of the Lebesgue number, every $\overline{S(f^i(y), \frac{\delta}{3})}$ is a subset of an element of $\mathcal{V}$. Hence, $\bigcap_{i=0}^{n-1} f^{-i}(\overline{S(f^i(y), \frac{\delta}{3})})$ is a subset of an element of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$. Consequently, K can be covered by $r_n(\frac{\delta}{3}, K, f)$ elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$.

(b) For any $x, \in X \setminus K$, *i.e.*, $x \in V_1 \cap V_2 \cap ... \cap V_m$. In the following, we will consider points of $X \setminus K$, according to two further types of points.

First, consider those x for which there exists l with $1 \leq l \leq n-1$, such that $f^l(x) \in K$ and $x, f(x), f^2(x), ..., f^{l-1}(x) \in X \setminus K$ (l depends on x, but for convenience, we use l instead of l_x). Namely, we consider the set, $\{x \in X \setminus K : x \in X \setminus K, x, f(x), f^2(x), ..., f^{l-1}(x) \in X \setminus K, f^l(x) \in K\}$. For every such x, there exists $y \in F$, such that $d(f^{l+i}(x), f^i(y)) \leq \frac{\delta}{3}, i = 0, 1, ..., n-l-1$; equivalently, $x \in f^{-(l+i)}(\overline{S(f^i(y), \frac{\delta}{3})}), i = 0, 1, ..., n-l-1$. By the definition of the Lebesgue number, every $\overline{S(f^i(y), \frac{\delta}{3})}$ is a subset of an element of $\mathcal{V}$. Hence, $V_1 \cap f^{-1}(V_1) \cap ... \cap f^{-(l-1)}(V_1) \cap (\bigcap_{i=0}^{n-l-1} f^{-(l+i)}(\overline{S(f^i(y), \frac{\delta}{3})}))$ is a subset of an element of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$ and $x \in V_1 \cap f^{-1}(V_1) \cap ... \cap f^{-(l-1)}(V_1) \cap (\bigcap_{i=0}^{n-l-1} f^{-(l+i)}(\overline{S(f^i(y), \frac{\delta}{3})}))$. There are $r_n(\frac{\delta}{3}, K, f)$ such open sets, implying that $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$ has $r_n(\frac{\delta}{3}, K, f)$ elements that cover this type of point, x. As $1 \leq l \leq n-1$, $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$ has $(n-1) \cdot r_n(\frac{\delta}{3}, K, f)$ elements that actually cover this type of points, x.

Next, consider those x for which $f^i(x) \in X \setminus K$ for every $i = 0, 1, ..., n-1$. One (any) element of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$ covers all such points, x. Hence, $X \setminus K$ can be covered by no more than $(n-1) \cdot r_n(\frac{\delta}{3}, K, f) + 1$ elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})$.

By (a) and (b), for any $n > 0$, it holds $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})) \leq n \cdot r_n(\frac{\delta}{3}, K, f) + 1$. Now, it follows from $\mathcal{U} \prec \mathcal{V}$ and Fact 4, $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{V})) \leq n \cdot r_n(\frac{\delta}{3}, K, f) + 1$. $\square$

Theorem 4.3 *Let (X, d) be a metric space and $f : X \to X$ be a perfect mapping. Then $h_c(f) \leq h_d(f)$.*

Proof. For any co-compact open cover, $\mathcal{U}$ of X, if $X \in \mathcal{U}$, then $h_c(f, \mathcal{U}) = 0$. Hence, we can assume $X \notin \mathcal{U}$. By Theorem 4.2, there exists $\delta > 0$ and a non-empty compact subset, K, of X, such that for any $n > 0$, it holds $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq n \cdot r_n(\frac{\delta}{3}, K, f) + 1$. Hence, $h_c(f, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U}))$

$\leq \lim\limits_{n\to\infty} \frac{1}{n}\log(n \cdot r_n(\frac{\delta}{3},K,f)+1) = r(\frac{\delta}{3},K,f)$. Let $\delta \to 0$. It follows from the definition of Bowen's entropy (Walters' book [13], P.168, Definition 7.8 and Remark (2)) that $r(\frac{\delta}{3},K,f)$ is decreasing on δ and $r(K,f) = \lim\limits_{\delta\to0} r(\frac{\delta}{3},K,f)$. Therefore, $h_c(f,\mathcal{U}) \leq r(\frac{\delta}{3},K,f) \leq r(K,f)$. Moreover, $r(K,f) \leq h_d(f)$. Finally, because $\mathcal{U}$ is arbitrarily selected, $h_c(f) \leq h_d(f)$. $\square$

Bowen's entropy, $h_d(f)$, is metric-dependent. Theorem 4.3 indicates that the co-compact entropy, which is metric-independent, is always bounded by Bowen's entropy, *i.e.*, $h_c(f) \leq h_d(f)$, regardless of the choice of a metric for the calculation of Bowen's entropy. In the next section, we will give an example where co-compact entropy is strictly less than Bowen's entropy.

4.2. An Example

In this section, R denotes the one-dimensional Euclidean space equipped with the usual metric $d(x,y) = |x-y|, x,y \in R$. The mapping, $f : R \to R$, is defined by $f(x) = 2x, x \in R$. f is clearly a perfect mapping. It is known that $h_d(f) \geq \log 2$ [13]. We will show $h_c(f) = 0$.

Let $\mathcal{V}$ be any co-compact open cover of R. By Theorem 2.2, $\mathcal{V}$ has a finite co-compact subcover, $\mathcal{U}$. Let $m = card(\mathcal{U})$. As compact subsets of R are closed and bounded sets, there exist $U_r, U_l \in \mathcal{U}$, such that for any $U \in \mathcal{U}$, $\sup\{R \setminus U\} \leq \sup\{R \setminus U_r\}$ and $\inf\{R \setminus U\} \geq \inf\{R \setminus U_l\}$. Let $a_r = \sup\{R \setminus U_r\}$ and $b_l = \inf\{R \setminus U_l\}$. Observe that for any $n > 0$, $x \in \bigvee\limits_{i=0}^{n-1} f^{-i}(U_i) \iff x \in U_0, f(x) \in U_1,...,f^{n-1}(x) \in U_{n-1}$, where $U_i \in \mathcal{U}, i = 0,1,...,n-1$.

Case 1: $0 < b_l < a_r$. For any $n > 0$ and $x \in (a_r, +\infty)$, $x \in U_r, f(x) \in U_r,...,f^{n-1}(x) \in U_r$. So $(a_r, +\infty) \subseteq \bigcap\limits_{i=0}^{n-1} f^{-i}(U_r)$. For any $x \in (-\infty, 0]$, $x \in U_l, f(x) \in U_l,...,f^{n-1}(x) \in U_l$, thus $(-\infty, 0] \subseteq \bigcap\limits_{i=0}^{n-1} f^{-i}(U_l)$.

As f is a monotone increasing mapping, there exists $k > 0$, such that $f^k(b_l) > a_r$. We can assume $n > k > 0$. Consider the following two possibilities (1.1 and 1.2).

(1.1) $x \in [b_l, a_r]$.

This requires at most k iterations, so that $f^k(x) \in U_r$. Hence, $x \in U_{j_0}, f(x) \in U_{j_1},...,f^{k-1}(x) \in U_{j_{k-1}}, f^k(x) \in U_r,...,f^{n-1}(x) \in U_r$, where $U_{j_0}, U_{j_1},.,U_{j_{k-1}} \in \mathcal{U}$. Since $card(\mathcal{U}) = m$, $[b_l, a_r]$ can be covered by m^k elements of $\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})$.

(1.2) $x \in (0, b_l)$.

This is divided into three further possibilities as follows.

(1.2.1) $f^{n-1}(x) > a_r$.

Choose j with $0 < j < n$, such that $f^{j-1}(x) < b_l$, but $f^j(x) \geq b_l$. Then, $x \in U_l, f(x) \in U_l,...,f^{j-1}(x) \in U_l, f^j(x) \in U_{j_0},...,f^{j+k-1}(x) \in U_{j_{k-1}}, f^{j+k}(k) \in U_r,...,f^{n-1}(x) \in U_r$, where $U_{j_0}, U_{j_1},...,U_{j_{k-1}} \in \mathcal{U}$. Since $card(\mathcal{U}) = m$, $\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})$ has m^k elements that cover this kind of point, x.

(1.2.2) $b_l \leq f^{n-1}(x) \leq a_r$.

If $f^{n-2}(x) < b_l$, *i.e.*, for the last jump getting into $[b_l, a_r]$, it holds $x \in U_l, ..., f^{n-2}(x) \in U_l, f^{n-1}(x) \in U_{j_0}$, where $U_{j_0} \in \mathcal{U}$, while $card(\mathcal{U}) = m$; there are m elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$ that cover these kind of points, x.

If $f^{n-3}(x) < b_l$ and $f^{n-2}(x) \geq b_l$, *i.e.*, for the second jump to the last before getting into $[b_l, a_r]$, it holds $x \in U_l, ..., f^{n-3}(x) \in U_l, f^{n-2}(x) \in U_{j_2}, f^{n-1}(x) \in U_{j_1}$, where $U_{j_2}, U_{j_1} \in \mathcal{U}$, while $card(\mathcal{U}) = m$, $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$ has m^2 elements that cover this kind of point, x.

Continue in this fashion: if $f^{n-k}(x) < b_l$ and $f^{n-(k-1)}(x) \geq b_l$, *i.e.*, for the $(k-1)$th jump from the last before getting into $[b_l, a_r]$, it holds $x \in U_l, ..., f^{n-k}(x) \in U_l, f^{n-(k-1)}(x) \in U_{j_{k-1}}, ., f^{n-1}(x) \in U_{j_1}$, where $U_{j_1}, ..., U_{j_{k-1}} \in \mathcal{U}$, while $card(\mathcal{U}) = m$, $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$ has m^{k-1} elements that cover this kind of point, x.

If $f^{n-(k+1)}(x) < b_l$ and $f^{n-k}(x) \geq b_l$, *i.e.*, jump into $[b_l, a_r]$ on the kth, $f^{n-1}(x) > a_r$, and this is Case (1.2.1).

(1.2.3) $f^{n-1}(x) < b_l$.

Clearly, $x \in \bigcap_{i=0}^{n-1} f^{-i}(U_l) \in \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$.

Hence, in Case 1, where $0 < b_l < a_r$, for any $n > k > 0$, it holds $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq 2 + m^k + m^k + m + m^2 + ... + m^{k-1}$, and by the definition of co-compact entropy, $h_c(f, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq \lim_{n \to \infty} \frac{1}{n} \log(2 + m^k + m^k + m + m^2 + ... + m^{k-1}) = 0$.

Case 2: $b_l < a_r < 0$. This is similar to Case 1 above.

Case 3: $b_l < 0 < a_r$. For any $n > 0$ and $x \in (a_r, +\infty)$, $x \in U_r, f(x) \in U_r, ..., f^{n-1}(x) \in U_r$, thus $(a_r, +\infty) \subseteq \bigcap_{i=0}^{n-1} f^{-i}(U_r)$.

Similarly, for $x \in (-\infty, b_l)$, $x \in U_l, f(x) \in U_l, ..., f^{n-1}(x) \in U_l$, thus $(-\infty, b_l) \subseteq \bigcap_{i=0}^{n-1} f^{-i}(U_l)$. As $\mathcal{U}$ is an open cover of R, there exists $U_0 \in \mathcal{U}$, such that $0 \in U_0, f(0) = 0 \in U_0, ..., f^{n-1}(0) = 0 \in U_0$, and hence, $0 \in \bigcap_{i=0}^{n-1} f^{-i}(U_0)$.

For $x \in [b_l, a_r]$, U_0, as an open set of R, can be decomposed into a union of many countably open intervals. Noting that $0 \in U_0$, there are two further possibilities, as given in (3.1) and (3.2) below.

(3.1) The stated decomposition of U_0 has an interval, (b_0, a_0), that contains zero, *i.e.*, $0 \in (b_0, a_0)$. Since f is a monotone increasing mapping, there exists $k > 0$, such that $f^k(b_0) < b_l$ and $f^k(a_0) > a_r$. Here, we can assume $n > k > 0$. Similar to Case 1, $[b_l, b_0]$ can be covered by m^k elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$, $(b_0, 0)$ can be covered by $m^k + m + m^2 + ... + m^{k-1}$ elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$, $(0, a_0)$ can be covered by $m^k + m + m^2 + ... + m^{k-1}$ elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$ and $[a_0, a_r]$ can be covered by m^k elements of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})$. Hence, for any $n > k > 0$, $N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq 3 + m^k + m + m^2 + ... + m^{k-1} + m^k + m +$

$m^2 + ... + m^{k-1}$. Therefore, by the definition of co-compact entropy, $h_c(f, \mathcal{U}) = \lim\limits_{n \to \infty} \frac{1}{n} H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq \lim\limits_{n \to \infty} \frac{1}{n} \log(3 + m^k + m + m^2 + ... + m^{k-1} + m^k + m + m^2 + ... + m^{k-1}) = 0$.

(3.2) The only intervals covering zero are of the forms $(-\infty, a_0)$ or $(b_0, +\infty)$.

Consider the case, $0 \in (-\infty, a_0)$. As f is a monotone increasing mapping, there exists $k > 0$, such that $f^k(a_0) > a_r$. We can assume $n > k > 0$. Similar to Case 1, $(0, a_0)$ can be covered by $m^k + m + m^2 + ... + m^{k-1}$ elements of $\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})$ and $[a_0, a_r]$ can be covered by m^k elements of $\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})$, and it also holds $[b_l, 0) \subseteq \bigcap\limits_{i=0}^{n-1} f^{-i}(U_0)$. Hence, for any $n > k > 0$, $N(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq 3 + m^k + m + m^2 + ... + m^k$. By the definition of co-compact entropy, $h_c(f, \mathcal{U}) = \lim\limits_{n \to \infty} \frac{1}{n} H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\mathcal{U})) \leq \lim\limits_{n \to \infty} \frac{1}{n} \log(3 + m^k + m + m^2 + ... + m^k) = 0$. Therefore, when $b_l < 0 < a_r$, it holds $h_c(f, \mathcal{U}) = 0$.

The case, $0 \in (b_0, +\infty)$, is similar.

Now, by Cases 1, 2 and 3, it holds that $h_c(f, \mathcal{U}) = 0$. Noting that $\mathcal{V} \prec \mathcal{U}$, it holds that $h_c(f, \mathcal{V}) \leq h_c(f, \mathcal{U}) = 0$. Since $\mathcal{V}$ is arbitrary, $h_c(f) = 0$.

5. Concluding Remarks

The investigation of dynamical systems could be tracked back to Isaac Newton's era, when calculus and his laws of motion and universal gravitation were invented or discovered. Then, differential equations with time as a parameter played a dominant role. However, it was not realized until the end of the 19th century that the hope of solving all kinds of problems in celestial mechanics by following Newton's frame and methodology, e.g., the two body problem, becomes unrealistic when Jules Henri Poincaré's New Methods of Celestial Mechanics was publicized (shortly after this, in the early 20th century, fundamental changes in electrodynamics occurred when Albert Einstein's historical papers appeared: reconciling Newtonian mechanics with Maxwell's electrodynamics, separating Newtonian mechanics from quantum mechanics and extending the principle of relativity to non-uniform motion), in which the space of all potential values of the parameters of the system is included in the analysis, and the attention to the system was changed from individual solutions to dynamical properties of all solutions, as well as the relation among all solutions. Although this approach may not provide much information on individual solutions, it can obtain important information on most of the solutions. For example, by taking an approach similar to that in ergodic theory, Poincaré concluded that for all Hamiltonian systems, most solutions are stable [40].

The study of dynamical systems has become a central part of mathematics and its applications since the middle of the 20th century, when scientists from all related disciplines realized the power and beauty of the geometric and qualitative techniques developed during this period for nonlinear systems (see e.g., Robinson [31]).

Chaotic and random behavior of solutions of deterministic systems is now understood to be an inherent feature of many nonlinear systems (Devaney [41], 1989). Chaos and related concepts as main concerns in mathematics and physics were investigated through differentiable dynamical systems, differential equations, geometric structures, differential topology and ergodic theory, *etc.*, by

S. Smale, J. Moser, M. Peixoto, V.I. Arnol'd, Ya. Sinai, J.E. Littlewood, M.L. Cartwright, A.N. Kolmogorov and G.D. Birkhoff, among others, and even as early as H. Poincaré (global properties, nonperiodicity; 1900s) and J. Hadamard (stability of trajectories; 1890s).

Kolmogorov's metric entropy as an invariant of measure theoretical dynamical systems is a powerful concept, because it controls the top of the hierarchy of ergodic properties and plays a remarkable role in investigating the complexity and other properties of the systems. As an analogous invariant under conjugation of topological dynamical systems, topological entropy plays a prominent role for the study of dynamical systems and is often used as a measure in determining dynamical behavior (e.g., chaos) and the complexity of systems. In particular, topological entropy bounds measure-theoretic entropy (Goodwyn [10,11]). Other relations between various entropy characterizations were extensively studied, e.g., Dinaburg [42]. It is a common understanding that topological entropy, as a non-negative number and invariant of conjugation in describing dynamical systems, serves a unique and unsubstitutable role in dynamics. Consequently, an appropriate definition of topological entropy becomes important and difficult.

In the theory and applications of dynamical systems, locally compact systems appear commonly, e.g., R^n or other manifolds. The introduced concept of co-compact open covers is fundamental for describing the dynamical behaviors of systems as, for example, for locally compact systems, co-compact open sets are the neighborhoods of the infinity point in the Alexandroff compactification and, hence, admit the investigation of the dynamical properties near infinity.

The co-compact entropy introduced in this paper is defined based on the co-compact open covers. In the special case of compact systems, this new entropy coincides with the topological entropy introduced by Adler, Konheim and McAndrew (Sections 3 and 4). For non-compact systems, this new entropy retains various fundamental properties of Adler, Konheim and McAndrew's entropy (e.g., invariant under conjugation, entropy of a subsystem does not exceed that of the whole system).

Another noticeable property of the co-compact entropy is that it is metric-independent for dynamical systems defined on metric spaces, thus different from the entropy defined by Bowen. In particular, for the linear mapping given in Section 4.2 (locally compact system), its co-compact entropy is zero, which would be at least $\log 2$ according to Bowen's definition; as a positive entropy usually reflects certain dynamical complexity of a system, this new entropy is more appropriate.

For a dynamical system defined on a metric space, Bowen's definition may result in different entropies when different metrics are employed. As proven in Section 4, the co-compact entropy is a lower bound for Bowen's entropies, where the traditional Lebesgue Covering Theorem for open covers of compact metric spaces is generalized for co-compact open covers of non-compact metric spaces. As studied by Goodwyn in [10,11] and Goodman [43], when the space is compact, topological entropy bounds measure-theoretic entropy. The relation between co-compact entropy and measure-theoretic entropy (K-S entropy) remains an open question. Of course, when the space is compact, this relation degenerates to the variational principle [43]. Recently, M. Patrao (2010) [44] explored entropy and its variational principle for dynamical systems on locally compact metric spaces by utilizing one point compactification.

Acknowledgments

Conflict of Interest

The authors declare no conflict of interest.

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