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Article

Evaluating the Spectrum of Unlocked Injection Frequency Dividers in Pulling Mode

Antonio Buonomo * and Alessandro Lo Schiavo

Seconda Università di Napoli, Via Roma 29, Aversa (CE) 81031, Italy;
E-Mail: alessandro.loschiavo@unina2.it

* Author to whom correspondence should be addressed; E-Mail: antonio.buonomo@unina2.it;
Tel.: +39-081-5010222; Fax: +39-081-5037042.

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Abstract: We study the phenomenon of periodic pulling which occurs in certain integrated microcircuits of relevant interest in applications, namely the injection-locked frequency dividers (ILFDs). They are modelled as second-order driven oscillators working in the subharmonic (secondary) resonance regime, *i.e.*, when the self-oscillating frequency is close (resonant) to an integer submultiple n of the driving frequency. Under the assumption of weak injection, we find the spectrum of the system's oscillatory response in the unlocked mode through closed-form expressions, showing that such spectrum is double-sided and asymmetric, unlike the single-sided spectrum of systems with primary resonance (n=1). An analytical expression for the amplitude modulation of the oscillatory response is also presented. Numerical results are presented to support theoretical relations derived.

Keywords: Injection pulling; analog frequency dividers; injection-locked frequency dividers (ILFDs); nonlinear oscillators; synchronization; averaging method

1. Introduction

It is known that periodic pulling (or frequency pulling) is a general phenomenon that happens in any system involving the injection locking of self-sustained oscillations when the frequency of the periodic forcing is just outside the locking region (Arnold's tongue) [1–3]. The occurrence of the periodic pulling is easily recognized by the characteristic aspect of the pulled oscillations, usually called beats, which exhibit a simultaneous modulation of amplitude and frequency with a pulse-like envelope of the amplitude. A theoretical investigation of the oscillatory response in the pulling mode of driven oscillators is given in a number of papers (see [1–12] and references therein) starting from the pioneering investigation of Rjasin [6], who first performed a harmonic analysis of beats to establish the spectral composition. Later on, an approximate, but physically insightful, treatment of the pulling was given in a celebrated paper of Adler [7], who obtained an analytical expression for the phase difference between the forcing and the system response neglecting the amplitude modulation. Based on that approximation, valid for the regime of so-called weak injections, the spectrum of beats was derived analytically many years later by Armand [9], by using an appealing method as simple as effective.

The features of the spectrum of beats have therefore already been known for a long time and can be summarized as follows: unlike the single-line spectrum in a locked mode, or the two-lines spectrum in a quasi-periodic mode far from the locking region, in a pulling mode the spectrum has a single sideband and is spread over many frequencies, starting from the free-running frequency, in the opposite side to that of the injected frequency. This result [9] has been reported in literature to explain experimental observations of pulling in microwave solid-state oscillators [10], in a unijunction transistor based oscillator [11], and in many papers dealing with the study of plasma instabilities and with periodically driven oscillating plasma systems (see [11,12], and references therein). These systems are well modeled by the van der Pol equation and exhibit a variety of dynamical phenomena observed in forced oscillators of van der Pol type [13–15]. In particular, mode locking and periodic pulling, bifurcations between quasi-periodic and frequency entrained states have been observed, as well as period-doubling bifurcations as a route to deterministic chaos [12], for which the study of chaotic dynamics and the derivation of lower bounds on their topological entropy is yet an attractive problem [13–17].

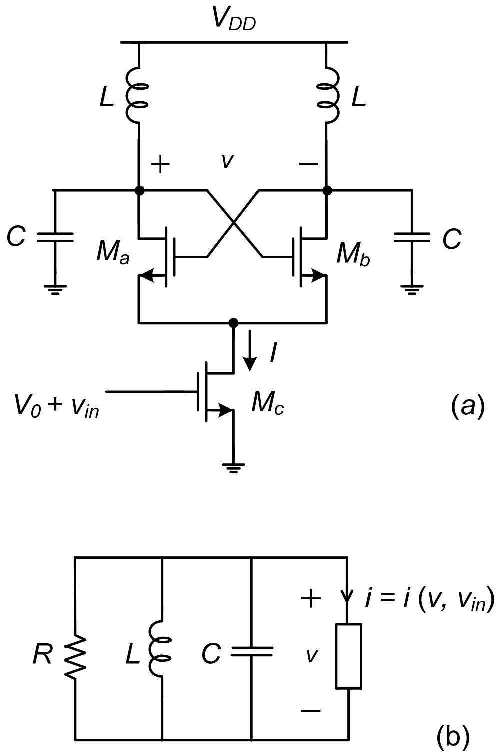
The pulling is observable in many electronic systems containing on-chip differential LC oscillators, and its occurrence is generally undesirable and harmful [18,19]. It is produced as a consequence of the unavoidable coupling of parts of the circuit, through the supply and the common substrate, or through parasitic paths [18,19]. It can therefore happen that an oscillator is subject to the action of an undesired periodic signal and, depending on its frequency, can operate in a locked-mode or in a pulling mode. Attempts to analytically calculate the simultaneous amplitude and frequency modulation in the pulling modes were recently made in [20,21] in the more simple case that the driving frequency is close to the self-oscillating frequency (primary resonance). The pulling phenomenon in injection-locked frequency dividers (ILFDs) is even more worrying, and its onset is to be avoided for a proper circuit operation as a divider. This imposes from one hand a reliable prediction of the locking range [22,23] and, on the other hand, a thorough understanding of the spectral properties of the oscillatory response during the pulling to avoid its effects. However, as far as is known to the authors, the pulling phenomenon in the frequency dividers, which operate in subharmonic resonance regime (secondary resonance), has never been investigated and some facets of the phenomenon yet are not known.

The present paper is devoted to the study of the pulling in subharmonic resonant systems, which is not only of theoretical but also of practical interest. By widening the analysis method in [9], we derive an analytical procedure for finding the spectral components of the unlocked oscillation in the pulling mode of injection-locked frequency dividers. The procedure is simple and straightforward, and allows us to calculate such components in the form of series taking into account both the amplitude and frequency modulation of the unlocked oscillation. We show that the power spectrum of the unlocked signal in the pulling mode is double-sided, and asymmetric, with respect to the natural frequency of the free-running oscillator, in contrast to the single-sided spectrum of systems with primary resonance [20,21]. Numerical results are presented to support theoretical relations derived.

2. Nonlinear Model of Injection-Locked *LC* Frequency Dividers

The circuit shown in Figure 1a is representative of the wide class of on-chip integrated circuits that perform the frequency division by exploiting the known phenomenon of injection-locking. It consists of a differential *LC* oscillator driven by a sinusoidal synchronization signal v_{in}, applied to the gate of the tail device M_c, , with a frequency close to an integer multiple *n* of the *LC*-tank resonance frequency ω_0, *i.e.*, $v_{in} = V_{in}\cos(\omega_{in}t)$, $\omega_{in} = n\omega$, $\omega_0 = \sqrt{1/LC}$, $\omega \approx \omega_0$.

Figure 1. (**a**) Circuit diagram of a conventional ILFD with injection via tail device; (**b**) its associated representation as a forced nonlinear *LC* oscillator.



Between the two output nodes, the circuit can be schematized by the simple equivalent circuit shown in Figure 1b, where *R* denotes the losses of the *LC*-tank. The active part of the circuit, made of two cross-coupled MOS devices biased by the tail device, is represented by a memoryless two-terminal whose constitutive relationship $i-v$ depends on the external signal v_{in}. To account for the frequency dependent behavior of the active part, due to the intrinsic capacitive effects of devices at high frequency operation, an equivalent capacitor can be used in the circuit of Figure 1b. In the present analysis, we investigate the behaviour of the equivalent circuit shown in Figure 1b, assuming that $i = i(v, v_{in})$ is a saturation function of the form $i = -I_0(1 + k\,v_{in})\mathrm{sign}(v)$ [22]. As a rule, the *LC*-tank is assumed to filter out all of the harmonics of the forcing current, so that the output voltage is purely sinusoidal, and the amplitude of the injection signal is assumed sufficiently small.

The forced *LC* oscillator shown in Figure 1b can operate as an injection-locked oscillator, if the external independent signal has a frequency close to the tank resonant frequency (primary resonance), while it can operate as an injection-locked frequency divider if the external signal has a frequency close to an integer multiple of the tank resonant frequency (secondary, or subharmonic resonance). In the following, we focus on the circuit operation as a divider with $n = 2$.

By making the substitution $\tau = \omega t$ and introducing the frequency detuning parameter $\sigma = 1 - \omega_0^2 / \omega^2 \approx 2(\omega - \omega_0)/\omega$, the describing equation of circuit in Figure 1b:

$$\left(D^2 + \frac{1}{RC} D + \frac{1}{LC} \right) v = -\frac{1}{C} D\, i(v, v_{in}) \tag{1}$$

can be written in the perturbation form:

$$D_\tau^2 v + v = \sigma v - \mu D_\tau \left(v + R\, i(v, v_{in}) \right) \tag{2}$$

where D_τ denotes the derivative operator with respect to τ, $\mu = \omega_0 / (\omega Q)$ is a small dimensionless parameter tending to zero as $1/Q$, where $Q = \omega_0 RC$ denotes the quality factor of the resonant circuit. Note that σ is a small parameter of the order of μ, that is, $\sigma = O(\mu)$.

The theory of the driven oscillator circuit in Figure 1b [22,23] predicts the existence of stable locked modes, if the ratio ω_{in} / ω is close to 2. In these states the ratio ω_{in} / ω remains constant, while the driving frequency is varied in a certain interval, called locking range. This interval widens by increasing the amplitude V_{in} of the driving signal and forms a tongue-shaped region, usually named Arnol'd tongue, in the parameter plane (V_{in}, ω_{in}). In the locked modes the phase relation between driving signal and locked oscillation is independent of time (phase-locking). The locking region has been studied in details in [22], and the bifurcation behavior which occurs at the transition point from entrainment to the loss of entrainment, in [24]. This does not exhaust the possible bifurcations. From the theory of dynamical systems we know that systems like the one in Figure 1b exhibit a multitude of dynamical regimes that occur in different parameter regions and, consequently, different bifurcations may occur [13–15]. In particular, period-doubling bifurcations, which generally happen in Arnold's tongues for strong amplitude of the driving signal, are noteworthy because in the limit of their sequence a chaotic behavior occurs. The structure of the bifurcation diagrams, the possible synchronization regimes, and the connection between desynchronization and chaos are reported elsewhere, together with the study of chaos in terms of the topological entropy [13–17].

In the following, we analyze the circuit operation in the region outside the locking region, called quasi-periodicity region, where the frequency entrainment is not possible as a consequence of the inherently different interaction between the driving signal and the oscillator. In particular, we focus on the system's response in the close proximity of the Arnold tongue, where the pulling phenomenon can be observed, just as in the case of the primary resonance. It is known that, near to the Arnold tongue, this interaction manifests as a periodically repeated and incomplete frequency entrainment process, known as periodic pulling. This process causes a simultaneous modulation of amplitude and phase of the system's oscillatory response, which has a complex time evolution and exhibits a power spectrum with very dense sidebands, which will be investigated in the next section.

As in the pulling modes neither the amplitude of oscillation, nor the instantaneous frequency remain constant, we seek a solution of Equation (2) in the form:

$$v(t) = V(t)\cos[\omega t + \theta(t)] \tag{3}$$

where $\theta(t)$ denotes the phase difference between the output voltage $v(t)$ and the driving signal $v_{in}(t)$. The formulation (2) of the circuit equation allows us to find its solution by the methods of asymptotic expansion [1–3,25], for small μ. By solving Equation (2) by the asymptotic method of Bogolyubov and Mitropol'skii [25] we obtain two coupled truncated equations by which we determine the amplitude $V(t)$ and the phase $\theta(t)$. Under the assumption that the amplitude of the injection signal is sufficiently small, it can be shown that the averaging equations associated to Equation (2) are [22]:

$$\dot{V}(t) = -\frac{\omega_0}{2Q}\left[V - V_{SS}\left(1 + \frac{k\,V_{in}}{3}\cos\left[2\theta(t)\right]\right)\right] \tag{4}$$

$$\dot{\theta}(t) = \Delta - \frac{V_{SS}}{V(t)}\frac{\omega_0}{Q}\frac{k\,V_{in}}{3}\sin\left[2\theta(t)\right] \tag{5}$$

where $\Delta = \omega_0 - \omega$ is the frequency detuning, and $V_{SS} = 4RI_0/\pi$ is the steady-state amplitude of the free-running oscillation, that is, for $v_{in} = 0$. Equations (4) and (5) allow us to calculate both the amplitude and the phase modulation of the output voltage, which are slowly-varying function on a time scale μt, and to study the nonlinear dynamics of the system in all of the operating modes. In the next section, we show that under a suitable approximation this function can be calculated in a closed form.

3. Analytical Treatment of Periodic Pulling

To get a comprehensive view of the pulling phenomenon in the circuit in Figure 1, we need to solve the nonlinear system of coupled differential Equations (4) and (5). However, Equations (4) and (5) cannot be solved by quadrature, in the general case, as generally it happens for the averaging equations [4]. This is possible in the more simple case $\dot{\theta}(t) = \dot{V}(t) = 0$ that defines the phase-locked operation mode, which has been analyzed earlier [22]. The problem of solving Equations (4) and (5) becomes analytically tractable in the weak injection regime when the amplitude V_{in} of the external signal is sufficiently small. This entails a substantial simplification since the assumption $V(t) = V_{SS} + \tilde{v}(t)$, with $\tilde{v}(t) << V_{SS}$ can be made, which is used in all the existing analytical treatments of the periodic pulling [7–9,20,21]. Under this assumption, by making the substitutions $\phi = 2\theta$, $\bar{\Delta} = 2\Delta$, and introducing the dimensionless pulling parameter $\alpha = \omega_0 m / Q\Delta$, system (4), (5) reduces to a system of decoupled equations:

$$\dot{\tilde{v}} = -\omega_p\tilde{v} + H\cos\phi \tag{6}$$

$$\dot{\phi} = \bar{\Delta} - \alpha\bar{\Delta}\sin\phi \tag{7}$$

where $m = kV_{in}/3$ is a parameter dependent on the amplitude of the driving signal, $H = \omega_p V_{SS} m$ and $\omega_p = \omega_0/2Q$.

System (6), (7) is derived assuming that the amplitude modulation does not significantly affect the phase variation in Equation (5), and thus considering the amplitude as a parameter, equal to V_{SS}. This allows us to analyze the phase dynamics of a driven oscillator independently from the amplitude, through a problem of reduced-order based on the single Equation (7). Note that this equation is formally similar to the celebrated Adler's equation who first introduced that approach [7], which was subsequently taken up in [26] using a nonlinear model and the perturbation theory [27]. The solution of Equation (7), valid for unlocked oscillation modes, is [7]:

$$\phi(t) = 2\left[\tan^{-1}\alpha + \sqrt{1-\alpha^2}\tan\left(\frac{\overline{\Delta}}{2}\sqrt{1-\alpha^2}\,t\right)\right]. \tag{8}$$

By exploiting the knowledge of the phase modulation (8), Armand [9] was able to analytically calculate the spectrum of the unlocked oscillation by using a simple and effective expedient, although little appreciated. In the present analysis, starting from the basic idea in [9], we show that, when $V(t) = V_{SS}$, the spectrum of Equation (3) can be obtained by the spectrum of the phase factor $\exp(i\,\theta(t))$ of the complex signal:

$$\hat{v} = V_{SS}e^{\omega t}e^{i\,\theta(t)} \; . \tag{9}$$

In the next two sections, we show how to calculate the phase factor $\exp[i\,\theta(t)]$ from (8), and how the calculation of the spectrum can be improved including the correction due to the amplitude modulation $\tilde{v}$, *i.e.*, finding the more accurate function:

$$\hat{v} = (V_{SS} + \tilde{v})e^{\omega t}e^{i\,\theta(t)} \tag{10}$$

obtained by solving Equation (6).

3.1. Phase Modulation and Spectrum

Firstly, we analyze the phase dynamics through the Adler's like Equation (7). We note that, as $\phi = 2\theta$, from Equation (8) the time evolution of the phase is:

$$\theta(t) = \tan^{-1}\alpha + \sqrt{1-\alpha^2}\tan\left(\Delta\sqrt{1-\alpha^2}\,t\right). \tag{11}$$

The periodic function $\theta(t)$ is essentially equivalent to Equation (8), except for the period that is equal to one-half, and thus the frequency:

$$\Omega = 2\Delta\sqrt{1-\alpha^2} = 2\Delta\sqrt{1-\left(\frac{\Delta_p}{\Delta}\right)^2} \tag{12}$$

is double. This frequency is usually termed beat frequency. We also note that the condition $\dot{\theta}(t) = 0$ corresponds to the mode-locking condition in which the circuit in Figure 1 operates in the synchronous mode, under frequency entrainment conditions, as a frequency divider. That condition can be satisfied when $|\alpha| > 1$, *i.e.* for $|\Delta| < m\omega_0 / Q$, which defines the critical detuning associated with the onset of pulling:

$$\Delta_p = \pm m\frac{\omega_0}{Q}. \tag{13}$$

In the present paper we are interested specifically in values $|\alpha| < 1$. When this inequality is fulfilled, the circuit ceases to behave like a frequency divider by 2, and beats take place in the circuit. The difference between the frequencies of the external signal and the output voltage becomes $\omega - \dot{\theta}(t)$.

By making the substitution $\alpha = \sin\gamma$, that defines a different pulling parameter lying in the interval $[-\pi/2, \pi/2]$, we can write $\Omega = \bar{\Delta}\cos\gamma$. Then, following the procedure in [9], from Equation (8) we can express $\cos\phi$ in terms of the beat frequency Ω. By using simple trigonometric relationships, we get:

$$e^{i\phi} = \frac{i\, e^{i(\Omega t+\gamma)} - \tan\gamma/2}{i + e^{i(\Omega t+\gamma)}\tan\gamma/2} \tag{14}$$

or, equivalently:

$$e^{i\phi} = \frac{e^{i\gamma} + e^{i(\Omega t+\gamma)} + e^{i(\Omega t+2\gamma)} - 1}{e^{i\gamma} + e^{i(\Omega t+\gamma)} - e^{i(\Omega t+2\gamma)} + 1} \tag{15}$$

allowing us to relate the phase factor $\exp[i\phi(t)]$ to the beat frequency Ω. By developing the function of the right hand side of Equation (14) in a power series of $y = \exp[i(\Omega t+\gamma)]$, we find:

$$e^{i\phi(t)} = iT + \left(1 - T^2\right)\sum_{n=1}^{\infty} (iT)^{n-1} y^n, \tag{16}$$

where we have put:

$$T = \tan(\gamma/2) = \tan\left(\frac{\arcsin\alpha}{2}\right). \tag{17}$$

Taking into account that $\gamma = \arcsin\alpha$, we deduce that the parameter T has the same sign as α and lies in the range $[-1, 1]$. From the above it results that the phase factor in Equation (9) can be developed into the Fourier series:

$$e^{i\phi(t)} = \sum_{n=-\infty}^{\infty} c_n e^{in(\Omega t+\gamma)}$$

$$c_n = \begin{cases} 0 & n < 0 \\ i\tan(\gamma/2) & n = 0 \\ \left(1 - \tan^2(\gamma/2)\right)\left(i\tan(\gamma/2)\right)^{n-1} & n > 0 \end{cases} \tag{18}$$

This is a well-known result [9] that provides some interesting insight into the spectral properties of the system's oscillatory response in the case of a strong periodic pulling for systems with a primary resonance, *i.e.*, when ϕ coincides with the phase angle between the driving signal and the system's response. The spectrum of $\exp[i\phi(t)]$ extends on only one side with respect to origin, *i.e.*, for $\omega_{in}/2 < \omega_0$ $(\Delta > 0)$ the spectrum components at a frequency less than zero are cancelled out, while for $\omega_{in}/2 > \omega_0$ $(\Delta < 0)$ this cancellation occurs for the spectrum components at a frequency greater than zero. In other words, the non vanishing sideband lies always on the side opposite to the frequency perturbation induced by the driving signal. The spectral density of the side band is thus given by a geometric series and has an unusual triangular shaped envelope in a semi-logarithmic plot. Note that, by increasing α the beat frequency Ω decreases and the time evolution of the phase becomes increasingly nonlinear. Consequently, more spectral lines are added making denser the spectrum.

The spectrum components in Equation (18) allow us to find the solution of Equation (6) in a closed-form, as we will show in the next section. However, to find the spectrum components of $V(t)$ under the approximation $\tilde{v}=0$, we need to find the spectrum of $\exp[i\,\theta(t)]$, according to Equation (9). To this ends, it is convenient to use the relationship (15) for $\exp[i\,\theta(t)]$, which we write in the form:

$$e^{j\theta}=\left(\frac{e^{i\gamma}+e^{i(\Omega t+\gamma)}+e^{i(\Omega t+2\gamma)}-1}{e^{i\gamma}+e^{i(\Omega t+\gamma)}-e^{i(\Omega t+2\gamma)}+1}\right)^{1/2}=f(z)\,g(y) \tag{19}$$

$$f(z)=\left(1+\frac{1/z-e^{-i\gamma}}{1+e^{-i\gamma}/z}\right)^{1/2} \tag{20}$$

$$g(y)=\left(1-\frac{y-e^{-i\gamma}}{1+y\,e^{-i\gamma}}\right)^{-1/2} \tag{21}$$

where $z=1/y$ and $y=\exp[i(\Omega t+\gamma)]$ as before. Taking into account that the function $\left(z^{-1}-B^{-1}\right)/\left(1+(z\,B)^{-1}\right)$, $B=\exp(i\gamma)$, can be developed in a power series in the neighborhood of $z=0$, *i.e.*:

$$x=\frac{z^{-1}-B^{-1}}{1+(z\,B)^{-1}}-B=(1+B^2)\sum_{n=1}^{\infty}(-1)^k B^{n-1}z^n \tag{22}$$

we deduce that the coefficients A_k of the power series for Equation (20), $f(y)=A_0+A_1/y+A_2/y^2+\cdots$, are obtained in a closed-form by substituting the right hand side of Equation (22) in the power series for the function $\sqrt{1+B+x}$, given by:

$$\sqrt{1+B+x}=\sqrt{1+B}\left[1+\frac{x}{2(1+B)}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!}{2^{2n-2}n!(n-2)!}\left(\frac{x}{1+B}\right)^n\right] \tag{23}$$

The leading coefficients A_k of the above series useful to evaluate the main output harmonics are obtained by the following formulas:

$$\begin{aligned}
A_0&=\sqrt{1+B}\\
A_1&=-\sqrt{1+B}\,\frac{(1+B^2)}{2(1+B)}\\
A_2&=\sqrt{1+B}\left(\frac{B(1+B^2)}{2(1+B)}-\frac{(1+B^2)^2}{8(1+B)^2}\right)\\
A_3&=\sqrt{1+B}\left(-\frac{B^2(1+B^2)}{2(1+B)}+\frac{2B(1+B^2)^2}{8(1+B)^2}-\frac{(1+B^2)^3}{16(1+B)^3}\right)\\
&\cdots\cdots
\end{aligned} \tag{24}$$

To obtain the coefficients of the power series for $g(y)$, *i.e.*, $g(y) = B_0 + B_1 y + B_2 y^2 + \cdots$, we observe that the function $y - B^{-1} / \left(1 + y B^{-1}\right)$ can be developed in a power series in a neighborhood of $y = 0$, and that:

$$x = \frac{y - B^{-1}}{1 + y B^{-1}} + \frac{1}{B} = (1 + B^2) \sum_{k=1}^{\infty} (-1)^{k-1} B^{-(k+1)} y^k . \tag{25}$$

Consequently, the coefficients B_k are obtained by substituting the right-hand side of Equation (25) into the power series of the function $1 / \sqrt{1 + 1/B - x}$, given by:

$$\frac{1}{\sqrt{1 + 1/B - x}} = \frac{1}{\sqrt{1 + B^{-1}}} \left[1 + \frac{x}{2\left(1 + B^{-1}\right)} + \sum_{n=2}^{\infty} \frac{(2n-1)!}{2^{2n-1} n!(n-1)!} \left(\frac{x}{1 + B^{-1}} \right)^n \right]. \tag{26}$$

With this substitution, it can be shown that the coefficients can be expressed by the following formulas:

$$\begin{aligned}
B_0 &= \frac{1}{\sqrt{1 + B^{-1}}} \\
B_1 &= \frac{1}{\sqrt{1 + B^{-1}}} \frac{1 + B^2}{2B(1 + B)} \\
B_2 &= \frac{1}{\sqrt{1 + B^{-1}}} \left(-\frac{1 + B^2}{2B^2(1 + B)} + \frac{3(1 + B^2)^2}{8B^2(1 + B)^2} \right) \\
B_3 &= \frac{1}{\sqrt{1 + B^{-1}}} \left(\frac{1 + B^2}{2B^3(1 + B)} - \frac{3(1 + B^2)^2}{4B^3(1 + B)^2} + \frac{5(1 + B^2)^3}{16B^3(1 + B)^3} \right) \\
&\cdots\cdots .
\end{aligned} \tag{27}$$

Finally, performing the product of the power series for $f(z)$ and $g(y)$, expressed by Equations (24) and (27), we find the coefficients of the power series of $\exp[i\,\theta(t)]$ in the following explicit form:

$$e^{i\,\theta(t)} = \sum_{m=-\infty}^{\infty} C_m e^{i\,m(\Omega t + \gamma)} , \tag{28}$$

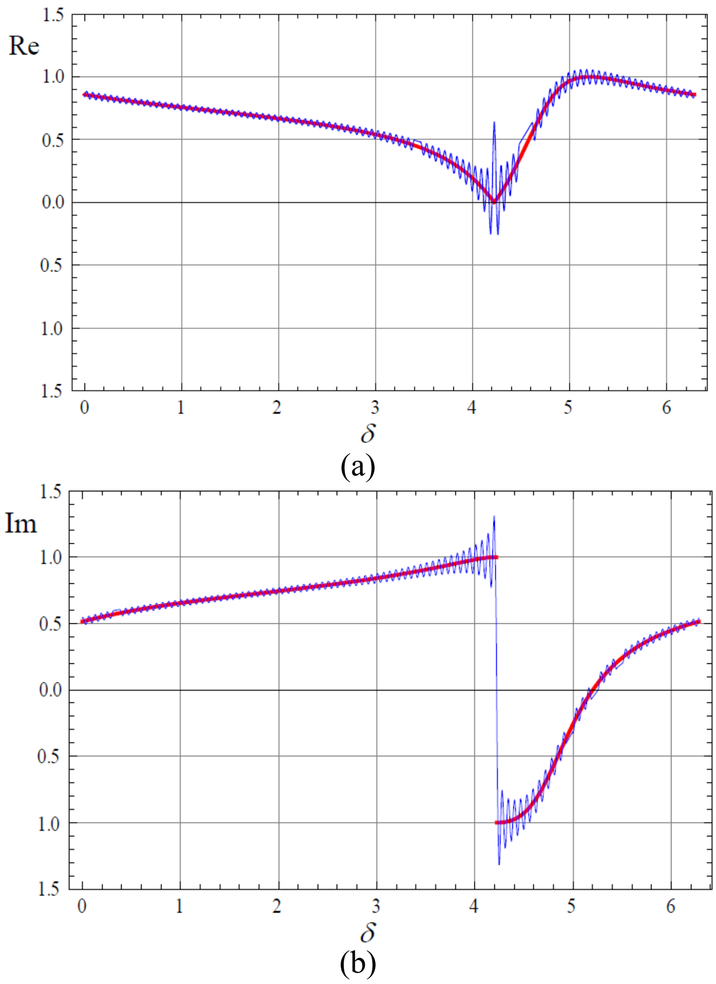
where the coefficients C_m are expressed in terms of coefficients A_k and B_k up to order N in the following explicit form:

$$C_m = \begin{cases} \displaystyle\sum_{k=0}^{N-m} A_k B_{k+m} & m \ge 0 \\ \displaystyle\sum_{k=0}^{N-|m|} A_{k-m} B_k & m < 0 \end{cases} . \tag{29}$$

By virtue of the frequency shift induced by Equation (9), the coefficient C_m gives the m-th component of the spectrum of the output voltage $v(t) = V_{SS}\cos(\omega t + \theta(t))$. From Equation (29) we deduce that sidebands, referenced to the frequency $\omega = \omega_{in}/2$, are generated at frequency $\omega_b = \omega + m\Omega$. Note that, according to Equation (12), the spacing Ω between sidebands can be smaller or greater than the frequency detuning $\Delta = \omega_0 - \omega$, differently from the case of primary resonance where it is always smaller than Δ.

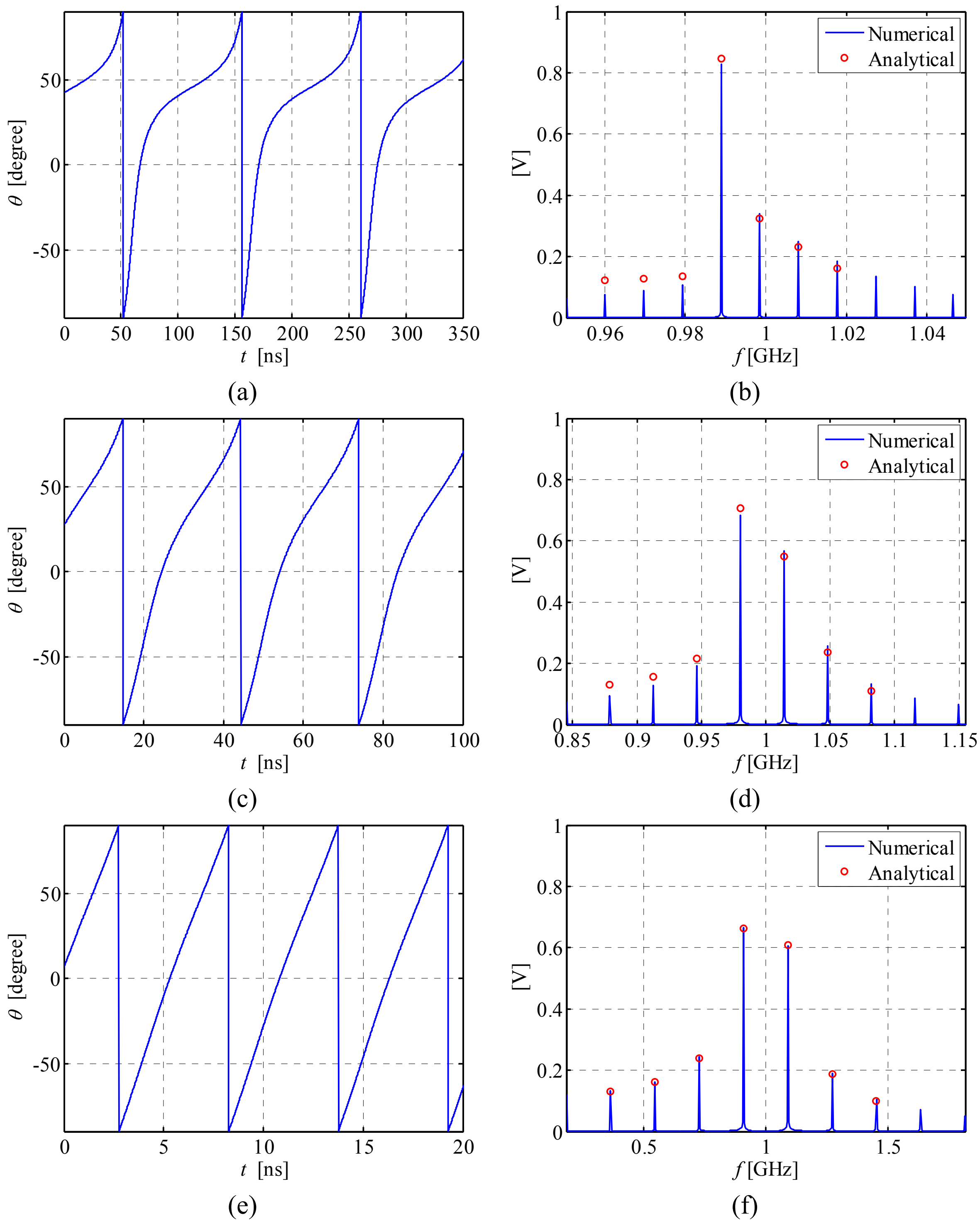
The numerical calculation of the sum of the truncated power series for $\exp[i\theta]$ showed that the error between the sum of the series (28) and the function (19), reduces increasing the number of terms taken into account. In Figure 2, we reported the real part and the imaginary part of the function in (19) and of the power series (28) to show its convergence.

Figure 2. (**a**) Real parts and (**b**) imaginary parts of the functions (19) and (28) evaluated for $\delta = \Omega t + \gamma$ ranging between 0 and 2π.



It is worth noting that, unlike what happens in the case of a driven oscillator (primary resonance), the oscillatory response of a divider in a pulling mode shows a double-sided asymmetric spectrum with respect to ω, as it follows from Equations (29) and (9). The time evolution of the phase $\theta(t)$ and the frequency spectrum of $V_{SS}\cos(\omega t + \theta(t))$ are depicted in Figure 3, which shows the asymmetric spectral broadening process for some values of the pulling parameter α. For small values of α, the time evolution of the phase is nearly linear, and it becomes linear for $\alpha = 0$, as expected for a conventional amplitude modulation. As α increases, the evolution of the phase becomes increasingly nonlinear, alternating a range in which varies slowly to one where it varies rapidly, which gives rise to the known phenomenon of beats.

Figure 3. Time evolution of $\theta(t)$ from (11) and frequency spectrum of $V_{SS}\cos(\omega t+\theta(t))$ calculated numerically starting from (11) via FFT and calculated by analytical formulas (9), (28). Parameters are: $f_0 = 1\ \text{GHz}$, $Q = 10$, $m = 0.1$, $V_{SS} = 1\ \text{V}$. In (**a**) and (**b**) $\alpha = 0.9$, in (**c**) and (**d**) $\alpha = 0.5$, in (**e**) and (**f**) $\alpha = 0.1$.



3.2. Amplitude Modulation and Spectrum

The previous analysis was carried out considering only the phase modulation, *i.e.*, by neglecting the slowly-varying modulation of the amplitude. However, the time evolution of the amplitude $V(t)$ is actually coupled to the time evolution of the phase $\theta(t)$, in our approximation through the term $\cos\phi$ in Equation (6). Hence, both amplitude and phase evolve synchronously in time (periodic pulling). To find the amplitude modulation we can solve Equation (6) in a closed form by virtue of (18).

For this purpose, we observe that the real part of the steady-state solution of the equation $\dot{\tilde{v}}(t) + \omega_p \tilde{v}(t) = i\,H\tan(\gamma/2)$ is equal to zero, and that the real part of the steady-state solution of the equation:

$$\dot{\tilde{v}}(t) + \omega_p \tilde{v}(t) = H\frac{1-T^2}{iT}(iT)^n e^{in(\Omega t+\gamma)} \tag{30}$$

is equal to:

$$d_n(-1)^{\frac{n-1}{2}}\left\{\cos\left[n(\Omega t+\gamma)\right] + \frac{n\Omega}{\omega_p}\sin\left[n(\Omega t+\gamma)\right]\right\} \tag{31}$$

$$d_n = -\frac{\omega_p H\left(T^2\right)^{\frac{n-1}{2}}\left(T^2-1\right)}{n^2\Omega^2+\omega_p^2} \tag{32}$$

for n odd, while for n even is equal to:

$$d_n\left\{(-1)^{\frac{n}{2}}\frac{n\Omega}{\omega_p}\cos[n(\Omega t+\gamma)] + (-1)^{\frac{n}{2}-1}\sin[n(\Omega t+\gamma)]\right\} \text{ for } T<0 \tag{33}$$

$$d_n\left\{\frac{n\Omega}{\omega_p}(-1)^{\frac{n}{2}-1}\cos[n(\Omega t+\gamma)] + (-1)^{\frac{n}{2}}\sin[n(\Omega t+\gamma)]\right\}\text{. for } T>0 \tag{34}$$

Consequently, we deduce that the harmonic components of $\tilde{v}(t)$ can be written in terms of the amplitude and phase in the following form:

$$\tilde{v}(t) = \sum_{n=1}^{\infty}\tilde{V}_n\sin\left[n(\Omega t+\gamma)+\varphi_n\right] \tag{35}$$

where:

$$\varphi_n = \tan^{-1}\left(\frac{n\Omega}{\omega_p}\right) \text{ for } n \text{ odd} \tag{36}$$

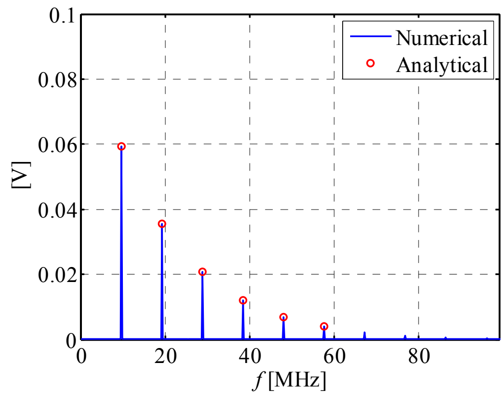
$$\varphi_n = -\tan^{-1}\left(\frac{n\Omega}{\omega_p}\right) \text{ for } n \text{ even} \tag{37}$$

and the amplitude $\tilde{V}_n$ is given by:

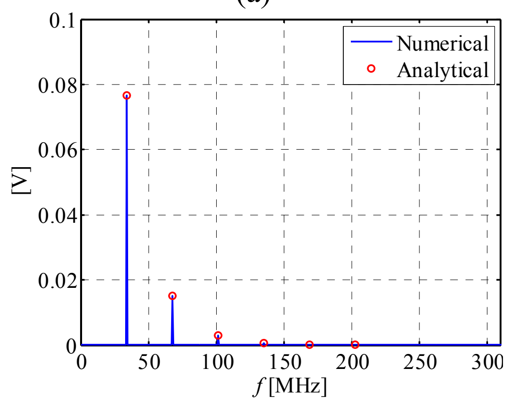
$$\tilde{V}_n = \frac{mV_{SS}\omega_p\left(1-T^2\right)\left|T\right|^{n-1}}{\sqrt{n^2\Omega^2+\omega_p^2}} \tag{38}$$

As expected, the harmonic components of $\tilde{v}(t)$ are separated by the beat frequency Ω and decrease progressively according to (38). The frequency spectrum of $\tilde{v}(t)$ is shown in Figure 4 for some values of the pulling parameter α. We highlight that the signal modulating the oscillation amplitude has a rich spectrum for large values of α, while reduces to a simple sinusoid for small values of α.

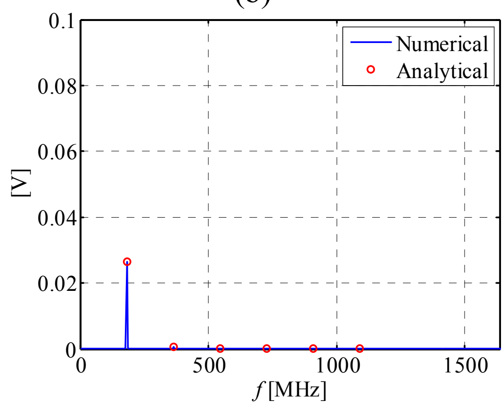
Figure 4. Frequency spectrum of $\tilde{v}$ calculated numerically from time expressions (6),(11) via FFT and calculated by analytical formula (38). Parameters are: $f_0 = 1$ GHz, $Q = 10$, $m = 0.1$, $V_{SS} = 1$ V. In (**a**) $\alpha = 0.9$, in (**b**) $\alpha = 0.5$, in (**c**) $\alpha = 0.1$.



(a)



(b)



(c)

Finally, we observe that the spectrum of $\tilde{v}(t)$ can be used to improve the calculation of the spectrum of $V(t)$ by simply making the product of two series, by virtue of Equation (10).

4. Conclusions

The presented investigation is the first attempt to develop an analytical procedure for analyzing the nonlinear dynamics of the periodic pulling in driven oscillators operating in a subharmonic resonance regime. The procedure has been developed by analyzing a driven oscillator of relevant practical interest, *i.e.*, a divide-by-two injection-locked frequency divider, and it allows us to evaluate the spectrum and the amplitude modulation of the unlocked system's response in the weak injection regime by closed-form expressions. It has proved a peculiar feature of the spectrum, which spreads asymmetrically on both sides of the driving signal frequency divided by two. Finally, we point out that the presented analysis procedure is general enough and it applies to any driven oscillator, irrespective of its nature, and to the more simple case of primary resonance. Moreover, the dynamical systems analyzed can be reduced to the classical forced van der Pol oscillator through a proper parameter setting. Consequently, results about the appearance of chaos and its investigation based on the topological entropy can be applied.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Hayashi, C. *Nonlinear Oscillations in Physical Systems*; McGraw-Hill: New York, NY, USA, 1964.
2. Minorsky, N. *Introduction to Non-Linear Mechanics*; Edwards Brothers Inc.: Ann Arbor, MI, USA, 1947.
3. Balanov, A.; Janson, N.; Postnov, D.; Sosnovtseva O. *Synchronization: From Simple to Complex*; Springer-Verlag: Berlin/Heidelberg, Gemany, 2009.
4. Khokhlov, R.V. A method of analysis in the theory of sinusoidal self-oscillations. *IRE Trans. Circuit Theory* **1960**, *7*, 398–413.
5. Schmackers, J.; Mathis, W. Entrainment of driven oscillators and the dynamic behavior of PLL's. In Proceedings of the 2005 International Symposium on Nonlinear Theory and its Applications (NOLTA2005), Bruges, Belgium, 18–21 October 2005; pp. 521–524.
6. Rjasin, P. Einstellungs-und Schwebungsprozesse bei der Mitnahme (Tuning or beat phenomena in entrainment). *J. Tech. Phys. USSR* **1935**, *5*, 195–214 (in Russian).
7. Adler, R. A study of locking phenomena in oscillators. *Proc. IRE Waves Electrons* **1946**, *34*, 351–357.
8. Stover, H.L. Theoretical explanation of the output spectra of unlocked driven oscillators. *Proc. IEEE* **1966**, *54*, 310–311.

9. Armand, M. On the output spectrum of unlocked driven oscillators. *Proc. IEEE*, **1969**, *59*, 798–799.
10. Kurokawa, K. Injection locking of microwave solid-state oscillators. *Proc. IEEE* **1973**, *61*, 1386–1410.
11. Koepke, M.E.; Hartley D.M. Experimental verification of periodic pulling in a nonlinear electronic oscillator. *Phys. Rev. A* **1991**, *44*, 6877–6887.
12. Rohde, A.; Piel, A.; Klostermann, H. Simulation of the nonlinear dynamics of grid sheath oscillations in double plasma devices. *Phys. Plasmas* **1997**, *4*, 3933–3942.
13. Mettin, R.; Parlitz, U.; Lauterborn, W. Bifurcation structure of the driven van der Pol oscillator. *Int. J. Bifurcat. Chaos* **1993**, *3*, 1529–1555.
14. Guckenheimer, J.; Hoffman, K.; Weckesser, W. The forced van der Pol equation I: The slow flow and its bifurcations. *SIAM J. Appl.Dyn. Syst.* **2003**, *l*, 1–35.
15. Shilnikov, A.; Shilnikov, L.; Turaev, D. On some mathematical topics in classical synchronization. A tutorial. *Int. J. of Bifurcat. Chaos* **2003**, *14*, 2143–2160.
16. Adler, R.L.; Konheim, A.G.; McAndrew, M.H. Topological entropy. *Trans. Amer. Math. Soc.* **1965**, *114*, 61–85.
17. Caneco, A.; Rocha, J.L.; Gracio, C. Topological entropy in the synchronization of piecewise linear and monotone maps. Coupled Duffing oscillators. *Int. J. of Bifurcat. Chaos* **2009**, *11*, 3855–3868.
18. Razavi, B. A study of injection pulling and locking in oscillators. *IEEE J. Solid-State Circuits* **2004**, *39*, 1415–1424.
19. Heidari, M.E.; Abidi, A.A. Behavioral models of frequency pulling in oscillator. In Proceedings of the IEEE International Behavioral Modeling and Simulation Workshop (BMAS 2007), San José, CA, USA, September, 2007; pp. 100–104.
20. Maffezzoni, P.; D'Amore, D. Evaluating pulling effects in oscillators due to small-signal injection. *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst*. **2009**, *28*, 22–31.
21. Ali, I.; Banerjee, A.; Mukherjee, A.; Biswas, B.N. Study of injection locking with amplitude perturbation and its effect on pulling of oscillator. *IEEE Trans. Circuits Syst., I. Reg. Papers* **2012**, *59*, 137–147.
22. Buonomo, A.; Lo Schiavo, A. Analytical approach to the study of injection-locked frequency dividers. *IEEE Trans. Circuits Syst., I*: *Reg. Papers* **2013**, *60*, 51–62.
23. Buonomo, A.; Lo Schiavo, A. A deep investigation of the synchronization mechanisms in LC-CMOS frequency dividers. *IEEE Trans. Circuits Syst., I*: *Reg. Papers* **2013**, doi: 10.1109/ TCSI.2013.2252452
24. Buonomo, A.; Lo Schiavo, A. Locking and pulling in injection-locked LC-CMOS dividers. *J. Low Power Electron.* **2013**, *9*, 221–228.
25. Bogolyubov, N.N.; Mitropol'skii, Y.A. *Asymptotic Methods in the Theory of Non-Linear Oscillations*; Gordon and Breach Publ., Inc.: New York, NY, USA, 1961.
26. Pikovsky, A.; Rosenblum, M.; Kurths, J. *Synchronization*; Cambridge University Press: Cambridge, UK, 2001.

27. Buonomo, A.; Kennedy, M.P.; Lo Schiavo, A. On the synchronization condition for superharmonic coupled QVCOs. *IEEE Trans. Circuits Syst., I*: *Reg. Papers* **2011**, *58*, 1637–1646.