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Self-Energy Closure for Inhomogeneous Turbulent Flows and Subgrid Modeling

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Abstract: A new statistical dynamical closure theory for general inhomogeneous turbulent flows and subgrid modeling is presented. This Self-Energy (SE) closure represents all eddy interactions through nonlinear dissipation or forcing 'self-energy' terms in the mean-field, covariance and response function equations. This makes the renormalization of the bare dissipation and forcing, and the subgrid modeling problem, transparent. The SE closure generalizes the quasi-diagonal direct interaction closure to allow for more complex interactions. The SE closure is applicable to flows in different geometries, is exact near maximum entropy states corresponding to canonical equilibrium, and provides a framework for deriving simpler realizable closures.

Keywords: inhomogeneous closures; turbulence; subgrid modeling; statistical dynamics

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1. Introduction

Statistical dynamical closure theory describes the direct evolution of the mean and transient statistics of generally turbulent fluid flows. It is also the natural formalism for deriving subgrid models of the unresolved scales of motion needed in large eddy simulations (LES) of fluid systems with very large numbers of degrees of freedom. Much of statistical closure theory has focused on homogeneous turbulence [1] since this somewhat idealized problem nevertheless captures many of the essential issues of this complex field. Kraichnan's [2] pioneering work in developing the direct interaction approximation (DIA) closure theory for homogeneous turbulence paved the way for advances in fluid turbulence. Herring [3] and McComb [1,4] independently developed the related non-Markovian self

consistent field theory (SCFT) and local energy transfer theory (LET) closures that subsequently have been shown to differ from the DIA only in how a fluctuation dissipation theorem is invoked [5–7]. The SCFT and LET closures have very similar performance to the DIA for homogeneous turbulence at finite Reynolds number [5,7] while the LET closure has the property of being consistent with the Kolmogorov 1941 [8] scaling laws at very high Reynolds numbers [1,9].

More recently there has been increasing attention focused on the development and performance of closure theories for inhomogeneous turbulent flows. A computationally tractable non-Markovian closure theory, termed the quasi-diagonal direct interaction approximation (QDIA), was formulated by Frederiksen [10] and applied to the subgrid modeling problem for two-dimensional turbulent flow over topography. In the QDIA closure the off-diagonal elements of the covariance and response functions are expressed in terms of the diagonal elements and the mean-field and topography. The theory was generalized to inhomogeneous turbulent flows on a β -plane and applied to Rossby wave dispersion and predictability by Frederiksen and O'Kane [11]. The theory has also recently been generalized to classical field theories with quadratic nonlinearity, including quasigeostrophic (QG) baroclinic and three-dimensional (3D) turbulence, by Frederiksen [12].

The QDIA statistical closure has been implemented numerically [11,13] and extensively tested and applied to problems in dynamics [11,13], predictability [11,14], data assimilation [15] and subgrid modeling [16,17]. O'Kane and Frederiksen [13] found that the *f*-plane QDIA closure has similar performance to the DIA for homogeneous turbulence and is only a few times more computationally intensive. Frederiksen and O'Kane [11] found that the β -plane QDIA closure [11] was as successful as the *f*-plane QDIA. Indeed, in a study of Rossby wave dispersion when eastward zonal flows impinge on an isolated topography in a turbulent environment, they found pattern correlations as high as 0.9999 between the closure and the statistics of 1800 DNSs for the mean Rossby wave trains in 10 day simulations. This is quite a remarkable result since Rossby wave dispersion in a turbulent environment is a far from equilibrium process and a severe test of the closure. As in earlier homogeneous DIA, SCFT and LET closure [18] to enhance the performance of the QDIA [11,13]. A regularization procedure, similar to that employed by Frederiksen and Davies [14] for homogeneous turbulence, which corresponds to an empirical vertex renormalization, has also been implemented in the QDIA to ensure it has the right power law behavior [13].

The QDIA closure theory also provides a framework for developing subgrid parameterizations of unresolved eddy-eddy, eddy-mean-field, eddy-topographic and mean-field-mean-field interactions. This is because the QDIA allows unambiguous identification of the terms that renormalize the generalized dissipation and forcing terms in the mean-field equation, as well as in the equations for the covariance and response functions. In the language of field theory [19–21] these interactions are represented as integrals over the product of nonlinear dissipation and forcing 'self-energy' terms and the mean-field, topography, covariance and response functions. Because of this factorization of the interactions it is transparent that the 'self-energy' terms renormalize the generalized dissipation and forcing for the mean-fields and the transients.

In the inhomogeneous direct interaction approximation (IDIA) of Kraichnan (1972) [22] the three-point functions in the equations for the covariance and response functions are also represented by integrals over products of 'self-energy' (nonlinear damping and noise) terms and the covariance and

response functions to close the system. This again means that the renormalization of the bare dissipation and random forcing in the equations for the transients is transparent in the IDIA (and in the homogeneous DIA). However, the covariance of the statistics of the transients in the mean field equation, unlike in the QDIA, is not represented in terms of integrals over products of 'self-energy' terms and the mean-field or topography. This different structure of the mean-field equation in the IDIA makes it more difficult to determine how to renormalize the dissipation and forcing terms in the mean-field equation. In the IDIA the covariance and response function equations are second order in perturbation theory and in the interaction coefficient. When the second order covariance is inserted in the mean-field equation it in turn becomes third order in perturbation theory and in the interaction coefficient. When the second order covariance is inserted in the mean-field equation it in turn becomes third order in perturbation theory and in the interaction coefficients. The IDIA closes the statistical dynamical equations at the level of the propagators, the covariance and response functions, and the mean-field. If the system is closed at the next level then the explicit three-point function appears in the equations for the propagators, rather than through a representation as integrals over 'self-energy' terms and propagators. In that case the renormalization in the propagator equations is again less transparent. This argument can clearly be generalized to closure at any level of n-point functions.

The primary purpose of this article is to develop a closure theory for inhomogeneous turbulent flows which generalizes the QDIA closure to more complex interactions. The SE closure, like the QDIA closure, is second order in perturbation theory, and the nonlinear interactions are represented through 'self-energy' terms, in the mean-field equation, as well as in the propagator equations. It is however, more general in that it does not employ the quasi-diagonal approximation. This 'Self-Energy' closure theory allows interactions that are essentially as complex as those in the IDIA but importantly makes the general inhomogeneous subgrid modeling problem much more transparent. The general SE closure is also useful for deriving simpler realizable closures for different geometries and boundary conditions as will also be demonstrated in this paper. The SE closure, like the IDIA [22] and QDIA [10,12], has the important property of being exact near maximum entropy states corresponding to canonical equilibrium [10].

The subgrid modeling problem has had a long history in the field of fluid turbulence with most studies and applications focusing on homogeneous turbulence. In Kraichnan's [2] homogeneous DIA, like in his inhomogeneous IDIA [22], the tendency of the covariance, or two-point function, is driven by the three-point function which is represented by nonlinear damping and nonlinear noise terms. As noted above, this makes the subgrid modeling problem transparent with eddy-eddy interactions consisting of an eddy damping term and a stochastic backscatter term. A theory of eddy viscosity was also developed by Kraichnan [23] and he found that the eddy viscosity increases to a cusp near the cut-off wavenumber. Rose [24] and Leith [25] noted the importance of eddy noise in subgrid modeling. Leith [25] combined a heuristic stochastic backscatter parameterization with the Smagorinsky [26] empirical eddy viscosity model in numerical studies of a turbulent shear mixing layer. Further subgrid modeling studies for three-dimensional homogeneous turbulence were carried out by Leslie and Quarini [27], Chollet and Lesieur [28], Chasnov [29], Domaradzki *et al.* [30], McComb *et al.* [31–33].

There has also been considerable progress in developing subgrid-scale parameterizations for large eddy simulations (LES) of quasigeostrophic flows. Frederiksen and Davies [34] developed self consistent representations of eddy viscosity and stochastic backscatter based on eddy damped

quasi-normal Markovian (EDQNM) and DIA closures for two-dimensional turbulence. A direct stochastic modeling approach to subgrid processes, based on the statistics of DNS, was employed by Frederiksen and Kepert [35] and shown to produce very similar results. Zidikheri and Frederiksen [36–38] also successfully applied this stochastic modeling methodology to two-level QG model studies with typical atmospheric flows and oceanic flows. More recently, Kitsios, Frederiksen and Zidikheri [39] derived universal scaling laws for subgrid models of eddy-eddy interactions applicable to baroclinic atmospheric flows. The more complex problem of determining subgrid-scale parameterizations in the presence of strong inhomogeneities has also been examined for quasigeostrophic flows. General expressions for the eddy-topographic force, eddy viscosity and stochastic backscatter were derived by Frederiksen [10] based on the QDIA closure. O'Kane and Frederiksen [16] and Frederiksen and O'Kane [17] analysed the numerical implementation of these subgrid terms.

In the last decade there has been increasing interest in exploring how parameterizations of stochastic backscatter may improve simulations and predictions of weather and climate [34,40–42]. Stochastic backscatter may be particularly important in predictability studies where it increases ensemble spread [34,43,44] in a similar manner to ensemble methods [45]. It may also result in reductions in systematic errors in seasonal forecasts [46] and affect the climate sensitivity to increased greenhouse gas forcing [47].

The derivation of the Self-Energy closure and its application to the subgrid modeling problem will be carried out with generic prognostic equations that take the form [12]:

$$\frac{\partial}{\partial t}\zeta_{\mathbf{k}}^{a}(t) + D_{0}^{a\beta}(\mathbf{k})\zeta_{\mathbf{k}}^{\beta}(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}} \left[\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)\zeta_{-\mathbf{q}}^{c}(t) + \mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c} \right] + f_{0}^{a}(\mathbf{k},t)$$
(1)

Here $\zeta_{\mathbf{k}}^{a}(t)$ is typically a field variable in spectral space depending on time t, the level or field type a and the wave vector \mathbf{k} . The equations are quadratic in the fields and also contain forcing $f_{0}^{a}(\mathbf{k},t)$, a linear term with coefficients $D_{0}^{a\beta}(\mathbf{k})$ and a term bilinear in the dynamical fields $\zeta_{\mathbf{k}}^{a}(t)$ and in constant fields $h_{\mathbf{k}}^{a}$ that typically specify topography. Throughout this paper we assume summation over repeated superscripts.

We suppose that the interaction coefficients $\mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ and $\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ satisfy the relationships:

$$\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \mathcal{K}^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})$$
(2)

and:

$$\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2} \left[\mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) + \mathcal{A}^{acb}(\mathbf{k},\mathbf{q},\mathbf{p}) \right]$$
(3)

The plan of this paper is as follows. In Section 2 we summarize the spectral equations for two-level QG turbulence, as well as for 3D turbulence, as specific examples of classical field theories to which the Self-Energy closure formalism applies. The inhomogeneous IDIA closure equations, as well as the generalized Langevin equation that guarantees realizability of the closure, are presented in Section 3. In Section 4, the SE closure and the corresponding generalized Langevin equation are detailed. In Sections 5 and 6 subgrid-scale parameterizations based on the SE closure are formulated for application to large eddy simulations. The QDIA closure for flow in planar geometry is recovered from the SE closure in Section 7 by assuming that the covariance and response functions are diagonal in

spectral space to lowest order. In Section 8, the SE closure is considered for QG flow on a sphere and we discuss how to derive from it simplified closures that are nevertheless realizable. The conclusions are presented in Section 9. Appendix A details the interaction coefficients for QG flows on the plane and on the sphere and for 3D turbulent flows. Appendix B contains a derivation relating the first order inhomogeneous elements of the covariance [Equation (B.9)] and response functions [Equation (B.12)] to the mean-field and topography.

2. Equations for Inhomogeneous Fluid Turbulence

The statistical closure equations formulated in this paper apply to classical field theories with quadratic nonlinearity. In this section and in Appendix A we outline some typical systems of equations for quasigeostrophic and three-dimensional turbulence to which the theory can be applied. Further examples have been given by Frederiksen [12] and include the primitive equations [48].

2.1. Quasigeostrophic Equations for Turbulent Flow

Taking suitable length and time scales, the nondimensional equation for 2-level baroclinic quasigeostrophic flow over topography on an *f*-plane may be written in the form:

$$\frac{\partial q^a}{\partial t} = -J(\psi^a, q^a + h^a) - D_0^{ab} q^b + f_0^a$$
(4)

Here, a = 1 or 2, ψ^a is the streamfunction and $q^a = \nabla^2 \psi^a + (-1)^a F_L(\psi^1 - \psi^2)$ is the reduced potential vorticity, $\omega^a = \nabla^2 \psi^a$ is the relative vorticity, $h^2 = h$, $h^1 = 0$ where h is the scaled topography, D_0^{ab} are dissipation operators to be specified below and f_0^a are forcing functions. Also, F_L is the layer coupling parameter [36], which is inversely proportional to the static stability. In planar geometry:

$$J(\psi,\zeta) = \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x}$$
(5)

As shown in Appendix A, with the identification $q_k^a(t) \Rightarrow \zeta_k^a(t)$, where **k** is a two-dimensional wave vector, the spectral equations for quasigeostrophic flow on the periodic domain $0 \le x \le 2\pi, 0 \le y \le 2\pi$ can then be written in the standard form:

$$\frac{\partial}{\partial t}\zeta_{\mathbf{k}}^{a}(t) + D_{0}^{a\beta}(\mathbf{k})\zeta_{\mathbf{k}}^{\beta}(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q}) \Big[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)\zeta_{-\mathbf{q}}^{c}(t) + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}\Big] + f_{0}^{a}(\mathbf{k},t)$$
(6)

Here, we suppose that the dissipation may be related to the viscosity through:

$$D_0^{a\beta}(\mathbf{k}) = v_0^{a\beta}(\mathbf{k})k^2 \tag{7}$$

where we refer to $v_0^{a\beta}(\mathbf{k})$ as the bare viscosity, although more general forms can be considered including wave frequencies; we shall also refer to $f_0^a(\mathbf{k},t)$ as the bare forcing.

In the above spectral equations:

$$k = |\mathbf{k}| \tag{7a}$$

$$\zeta_{-\mathbf{k}}^{a} = \zeta_{\mathbf{k}}^{a^{*}} \tag{7b}$$

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2} \Big[A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) + A^{acb}(\mathbf{k},\mathbf{q},\mathbf{p}) \Big]$$
(7c)

$$\delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{k} + \mathbf{p} + \mathbf{q} = 0 \\ 0 & \text{otherwise} \end{cases}$$
(7d)

The interaction coefficients are given in Appendix A. We note that the interaction coefficients $K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q})$ satisfy the relationship:

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = K^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})$$
(8)

2.2. Navier Stokes Equations for Three-Dimensional Turbulent Flow

The Navies Stokes equations for three-dimensional inhomogeneous turbulent flow may be written in the form [1]:

$$\frac{\partial}{\partial t}u^{a} + u^{\beta}\frac{\partial u^{a}}{\partial x^{\beta}} = -\frac{1}{\rho}\frac{\partial p}{\partial x^{a}} + v_{0}\nabla^{2}u^{a} + f_{0}^{a}$$
(9)

Here, $u^{a}(\mathbf{x},t)$, a = 1,2,3 is the fluid velocity at position \mathbf{x} and time t. Also, ρ is the density and $p(\mathbf{x},t)$ is the pressure. The bare viscosity is specified by v_0 , and $f_0^{a}(\mathbf{x},t)$ are forcing functions.

As shown in Appendix A, with the identification $u_k^a(t) \Rightarrow \zeta_k^a(t)$, the spectral equations for threedimensional flow on the periodic domain $0 \le x = x^1 \le 2\pi$, $0 \le y = x^2 \le 2\pi$, $0 \le z = x^3 \le 2\pi$ can then be written in the standard form given in Equations (1) and (6). Here $h_k^a = 0$ and the following analysis applies equally to the Navier Stokes equations where **k** is a three-dimensional wave vector as it does to the baroclinic quasigeostrophic equations where **k** is a two-dimensional wave vector.

3. Inhomogeneous DIA Closure Equations — IDIA

We consider an ensemble of flows satisfying Equation (1) or (6) where the ensemble mean is denoted by $\langle \zeta_k^a \rangle$ and angle brackets denote expectation value. We express the field component for a given realization by:

$$\zeta_{\mathbf{k}}^{a} = \langle \zeta_{\mathbf{k}}^{a} \rangle + \hat{\zeta}_{\mathbf{k}}^{a} \tag{10}$$

where $\hat{\zeta}_{\mathbf{k}}^{a}$ denotes the deviation from the ensemble mean. The spectral equation can then be expressed in terms of $\langle \zeta_{\mathbf{k}}^{a} \rangle$ and $\hat{\zeta}_{\mathbf{k}}^{a}$ as follows:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > +\sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}') < \zeta_{\mathbf{k}'}^{\beta}(t) > =$$

$$\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \left\{ < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > +C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t) \right\}$$

$$+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}] + \overline{f}_{0}^{a}(\mathbf{k},t)$$
(11)

$$\frac{\partial}{\partial t} \hat{\zeta}^{a}_{\mathbf{k}}(t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k}, \mathbf{k}') \hat{\zeta}^{\beta}_{\mathbf{k}'}(t) =$$

$$\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) [K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \{ \langle \zeta^{b}_{-\mathbf{p}}(t) \rangle \hat{\zeta}^{c}_{-\mathbf{q}}(t) + \hat{\zeta}^{b}_{-\mathbf{p}}(t) \langle \zeta^{c}_{-\mathbf{q}}(t) \rangle + \hat{\zeta}^{b}_{-\mathbf{p}}(t) \langle \zeta^{c}_{-\mathbf{q}}(t) \rangle + \hat{\zeta}^{b}_{-\mathbf{p}}(t) \langle \zeta^{c}_{-\mathbf{q}}(t) \rangle + A^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \hat{\zeta}^{b}_{-\mathbf{p}}(t) h^{c}_{-\mathbf{q}}] + \hat{f}^{a}_{0}(\mathbf{k}, t)$$
(12)

Here:

$$f_0^a(\mathbf{k}) = \overline{f}_0^a(\mathbf{k}) + \hat{f}_0^a(\mathbf{k})$$
(13)

$$\overline{f}_0^a(\mathbf{k}) = \langle f_0^a(\mathbf{k}) \rangle \tag{14}$$

$$C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,s) = \langle \hat{\zeta}_{-\mathbf{p}}^{b}(t) \hat{\zeta}_{-\mathbf{q}}^{c}(s) \rangle$$
(15)

are the mean and fluctuating forcing functions and two-time covariance matrix elements. Also,

$$D_0^{a\beta}(\mathbf{k},\mathbf{k}') = D_0^{a\beta}(\mathbf{k})\delta_{\mathbf{k},\mathbf{k}'}$$
(16)

where $\delta_{\mathbf{k},\mathbf{k}'}$ is the Kronecker delta function. Thus $D_0^{a\beta}(\mathbf{k},\mathbf{k}') = D_0^{a\beta}(\mathbf{k})$ when $\mathbf{k}' = \mathbf{k}$ and is zero otherwise. We have introduced the more general form in Equation (12) for later convenience.

3.1. Statistical Closure Equations

Direct interaction approximation closure equations may be derived for general inhomogeneous flows following Kraichnan [22] or Martin *et al.* [20]. The equation for the mean-field is just Equation (11) which requires an expression for the single-time two-point cumulant $C^{bc}_{-p,-q}(t,t)$ in order to close it.

From Equation (12) we can obtain an equation for the two-time cumulant by multiplying by $\hat{\zeta}_{-1}^{\alpha}(t')$ and taking the statistical average:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{a\alpha}(t,t') + \sum_{\mathbf{k}'} D_0^{a\beta}(\mathbf{k},\mathbf{k}') C_{\mathbf{k}',-\mathbf{l}}^{\beta\alpha}(t,t') =
\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) C_{-\mathbf{p},-\mathbf{l}}^{b\alpha}(t,t') h_{-\mathbf{q}}^{c} + \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) [<\zeta_{-\mathbf{p}}^{b}(t) > C_{-\mathbf{q},-\mathbf{l}}^{c\alpha}(t,t')
+ C_{-\mathbf{p},-\mathbf{l}}^{b\alpha}(t,t') < \zeta_{-\mathbf{q}}^{c}(t) > + <\hat{\zeta}_{-\mathbf{p}}^{b}(t) \hat{\zeta}_{-\mathbf{q}}^{c}(t) \hat{\zeta}_{-\mathbf{q}}^{a}(t') >] + \int_{t_0}^{t'} ds \sum_{\mathbf{k}'} F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) R_{-\mathbf{l},-\mathbf{k}'}^{\alpha\beta}(t',s)$$
(17)

This equation in turn involves the three-point function. The method of representing three-point function in terms of the response function and covariance follows Kraichnan [2] (see also Martin *et al.* [20] and the review of Frederiksen [49]) and results in the expression:

$$\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) < \hat{\zeta}^{b}_{-\mathbf{p}}(t) \hat{\zeta}^{c}_{-\mathbf{q}}(t) \hat{\zeta}^{a}_{-1}(t') >$$

$$= \int_{t_{o}}^{t'} ds \sum_{\mathbf{k}'} S^{a\beta}_{\mathbf{k}, -\mathbf{k}'}(t, s) R^{\alpha\beta}_{-1, -\mathbf{k}'}(t', s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta^{a\beta}_{\mathbf{k}, \mathbf{k}'}(t, s) C^{\alpha\beta}_{-1, \mathbf{k}'}(t', s)$$
(18)

for Gaussian initial conditions. Non-Gaussian initial conditions can also be considered as described by Rose [18] and Frederiksen *et al.* [5]; for the sake of brevity we do not detail the non-Gaussian initial conditions here but refer the interested reader to these papers and O'Kane and Frederiksen [13].

Thus, we obtain the closure:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{a\alpha}(t,t') + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}') C_{\mathbf{k}',-\mathbf{l}}^{\beta\alpha}(t,t') = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) C_{-\mathbf{p},-\mathbf{l}}^{b\alpha}(t,t') h_{-\mathbf{q}}^{c} \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^{b}(t) > C_{-\mathbf{q},-\mathbf{l}}^{c\alpha}(t,t') + C_{-\mathbf{p},-\mathbf{l}}^{b\alpha}(t,t') < \zeta_{-\mathbf{q}}^{c}(t) >] \\
+ \int_{t_{o}}^{t'} ds \sum_{\mathbf{k}'} S_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) R_{-\mathbf{l},-\mathbf{k}'}^{\alpha\beta}(t',s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{l},\mathbf{k}'}^{\alpha\beta}(t',s) \\
+ \int_{t_{o}}^{t'} ds \sum_{\mathbf{k}'} F_{0}^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) R_{-\mathbf{l},-\mathbf{k}'}^{\alpha\beta}(t',s) \qquad (19)$$

Here, the nonlinear damping (self-energy):

$$\eta_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) = -4\sum_{\mathbf{p}}\sum_{\mathbf{q}'}\sum_{\mathbf{p}'}\sum_{\mathbf{q}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q})\delta(\mathbf{k}',\mathbf{p}',\mathbf{q}')K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\beta\gamma\alpha}(-\mathbf{p}',-\mathbf{q}',-\mathbf{k}')R^{b\beta}_{-\mathbf{p},-\mathbf{p}'}(t,s)C^{c\gamma}_{-\mathbf{q},\mathbf{q}'}(t,s)$$
(20)

the nonlinear noise (self-energy):

$$S_{\mathbf{k},-\mathbf{k}'}^{a\alpha}(t,s) = 2\sum_{\mathbf{p}}\sum_{\mathbf{q}}\sum_{\mathbf{p}'}\sum_{\mathbf{q}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q})\delta(\mathbf{k}',\mathbf{p}',\mathbf{q}')K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\alpha\beta\gamma}(-\mathbf{k}',-\mathbf{p}',-\mathbf{q}')C_{-\mathbf{p},\mathbf{p}'}^{b\beta}(t,s)C_{-\mathbf{q},\mathbf{q}'}^{c\gamma}(t,s)$$
(21)

and covariance of the bare random forcing:

$$F_0^{a\alpha}(\mathbf{k}, -\mathbf{k}', t, s) = \langle \hat{f}_0^a(\mathbf{k}, t) \hat{f}_0^\alpha(-\mathbf{k}', s) \rangle$$
(22)

We assume homogeneous random forcing so that:

$$F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) = F_0^{a\beta}(\mathbf{k},-\mathbf{k},t,s)\delta_{\mathbf{k},\mathbf{k}'}$$
(23)

where $\delta_{\mathbf{k},\mathbf{k}'}$ is the Kronecker delta function. Thus $F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) = F_0^{a\beta}(\mathbf{k},-\mathbf{k},t,s)$ when $\mathbf{k}' = \mathbf{k}$ and is zero otherwise. We have introduced the more general form for later convenience.

The equation for the response function is derived in a similar way. We find:

$$\frac{\partial}{\partial t}R_{\mathbf{k},\mathbf{l}}^{a\alpha}(t,t') + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}')R_{\mathbf{k}',\mathbf{l}}^{\beta\alpha}(t,t') = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})R_{-\mathbf{p},\mathbf{l}}^{b\alpha}(t,t')h_{-\mathbf{q}}^{c}$$

$$+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})[<\zeta_{-\mathbf{p}}^{b}(t)>R_{-\mathbf{q},\mathbf{l}}^{c\alpha}(t,t')+R_{-\mathbf{p},\mathbf{l}}^{b\alpha}(t,t')<\zeta_{-\mathbf{q}}^{c}(t)>]$$

$$-\int_{t'}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s)R_{\mathbf{k}',\mathbf{l}}^{\beta\alpha}(s,t') \qquad (24)$$

with $R_{k,l}^{ab}(t,t) = \delta_{k,l}\delta^{ab}$ and δ is the Kronecker delta function.

The singe-time cumulant equation may be obtained from the expression:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t) = \lim_{t' \to t} \left\{ \frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') + \frac{\partial}{\partial t'} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') \right\}$$

$$= \lim_{t' \to t} \left\{ \frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') + \frac{\partial}{\partial t'} C_{-\mathbf{l},\mathbf{k}}^{ba}(t',t) \right\}$$
(25)

This yields:

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$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{aa}(t,t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}') C_{\mathbf{k}',-\mathbf{l}}^{\betaa}(t,t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(-\mathbf{l},-\mathbf{k}') C_{-\mathbf{k}',\mathbf{k}}^{\betaa}(t,t) \\
= \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) C_{-\mathbf{p},-\mathbf{l}}^{ba}(t,t') h_{-\mathbf{q}}^{c} \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^{b}(t) > C_{-\mathbf{q},-\mathbf{l}}^{ca}(t,t') + C_{-\mathbf{p},-\mathbf{l}}^{ba}(t,t') < \zeta_{-\mathbf{q}}^{c}(t) >] \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(-\mathbf{l},\mathbf{p},\mathbf{q}) A^{abc}(-\mathbf{l},\mathbf{p},\mathbf{q}) C_{-\mathbf{p},\mathbf{k}}^{ba}(t,t') h_{-\mathbf{q}}^{c} \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(-\mathbf{l},\mathbf{p},\mathbf{q}) A^{abc}(-\mathbf{l},\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^{b}(t) > C_{-\mathbf{q},\mathbf{k}}^{ca}(t,t') + C_{-\mathbf{p},\mathbf{k}}^{ba}(t,t') < \zeta_{-\mathbf{q}}^{c}(t) >] \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(-\mathbf{l},\mathbf{p},\mathbf{q}) K^{abc}(-\mathbf{l},\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^{b}(t) > C_{-\mathbf{q},\mathbf{k}}^{ca}(t,t') + C_{-\mathbf{p},\mathbf{k}}^{ba}(t,t') < \zeta_{-\mathbf{q}}^{c}(t) >] \\
+ \int_{t_{o}} ds \sum_{\mathbf{k}'} \{S_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) + F_{0}^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{l},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{k},\mathbf{k}}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{k},\mathbf{k}}^{a\beta}(t,s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{-\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{k},\mathbf{k}}^{a\beta}(t,s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{-\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{-\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) C_{-\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}$$

The system of statistical dynamical equations (11) for the mean-field, (19) for the two-time covariance, (24) for the response function and (26) for the single-time covariance constitute the IDIA closure in conjunction with the expressions (20) and (21) for the nonlinear damping and nonlinear noise.

3.2. Langevin Equation for IDIA Closure

As in the case of the homogeneous DIA closure equations [50], the general inhomogeneous IDIA closure equations (11), (19) (24) and (26) have an exact stochastic model representation [22]. The generalized Langevin equation which exactly reproduces the IDIA closure equations is as follows:

$$\frac{\partial}{\partial t}\widetilde{\zeta}_{\mathbf{k}}^{a}(t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}')\widetilde{\zeta}_{\mathbf{k}'}^{\beta}(t)
= \sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{\langle \zeta_{-\mathbf{p}}^{b}(t) \rangle \widetilde{\zeta}_{-\mathbf{q}}^{c}(t) + \widetilde{\zeta}_{-\mathbf{p}}^{b}(t) \langle \zeta_{-\mathbf{q}}^{c}(t) \rangle - C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\}
+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\widetilde{\zeta}_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}]
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s)\widetilde{\zeta}_{\mathbf{k}'}^{\beta}(s) + \widehat{f}_{0}^{a}(\mathbf{k},t) + f_{s}^{a}(\mathbf{k},t)$$
(27)

where:

$$f_{S}^{a}(\mathbf{k},t) = \sqrt{2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \rho_{-\mathbf{p}}^{(1)b}(t) \rho_{-\mathbf{q}}^{(2)c}(t)$$
(28)

Here, $\rho_{\mathbf{k}}^{(j)a}(t)$, where j = 1 or 2, are statistically independent random variables such that:

$$<\rho_{\mathbf{k}}^{(j)a}(t)\rho_{-\mathbf{l}}^{(j')b}(t') >= \delta^{jj'}C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t')$$
⁽²⁹⁾

and:

$$\langle \tilde{\zeta}^{a}_{\mathbf{k}}(t) \tilde{\zeta}^{b}_{-\mathbf{l}}(t') \rangle = C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t')$$
(30)

In Equation (29), $\delta^{jj'}$ is the Kronecker delta function.

The Langevin equation (27) guarantees realizability for the IDIA closure equations. The closure equations also preserve conservation of the quadratic invariants of the original equations (in the absence of forcing and dissipation).

4. Self-Energy Closure Equations — SE

In the IDIA equations of Kraichnan [22] the covariance and response function equations are second order in perturbation theory, or in the interaction coefficient. This implies that the mean-field equation becomes third order in the interaction coefficient. In contrast, in the QDIA [10,12] closure both the mean-field equation and the covariance and response functions are second order in perturbation theory. Importantly, in the QDIA closure the eddy-eddy interaction is expressed in terms of self-energies in the mean-field equation as well as in the covariance and response function equations. Thus the bare dissipation is renormalized by the nonlinear damping (self-energy) and the bare force is renormalized by the eddy-topographic interaction (self-energy) in the mean-field equation.

In this section we develop the Self-Energy closure equations which are structurally similar to the QDIA closure but allow for more complex interaction terms since they do not employ the quasidiagonal approximation.

4.1. Statistical Closure Equations

The derivation of the SE closure equations differs from that for the IDIA equations in the following ways. Firstly, in the mean-field equation (11), the first order expression for the two-point cumulant $C_{-p,-q}^{bc}(t,t)$, derived in Equation (B.9) is used. The mean-field equation then becomes second order in the interaction coefficients and perturbation theory. Thus, using equation (B.9) in Equation (11) yields:

$$\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) C^{bc}_{-\mathbf{p}, -\mathbf{q}}(t, t)$$

$$= 2 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{k}'} \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\mathbf{k}', \mathbf{p}', \mathbf{q}') K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \int_{t_o}^{t} ds \, R^{b\beta}_{-\mathbf{p}, -\mathbf{p}'}(t, s) C^{c\gamma}_{-\mathbf{q}, \mathbf{q}'}(t, s).$$

$$\cdot \left[A^{\beta\gamma\alpha} (-\mathbf{p}', -\mathbf{q}', -\mathbf{k}') h^{\alpha}_{\mathbf{k}'} + 2K^{\beta\gamma\alpha} (-\mathbf{p}', -\mathbf{q}', -\mathbf{k}') < \zeta^{\alpha}_{\mathbf{k}'}(s) > \right]$$

$$= -\int_{t_o}^{t} ds \sum_{\mathbf{k}'} \eta^{a\beta}_{\mathbf{k}, \mathbf{k}'}(t, s) < \zeta^{\beta}_{\mathbf{k}'}(s) > + \sum_{\mathbf{k}'} h^{\alpha}_{\mathbf{k}'} \int_{t_o}^{t} ds \, \chi^{a\alpha}_{\mathbf{k}, \mathbf{k}'}(t, s)$$
(31)

where the eddy-topographic interaction (self-energy):

$$\chi_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) = 2\sum_{\mathbf{p}} \sum_{\mathbf{q}'} \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) A^{\beta\gamma\alpha}(-\mathbf{p}',-\mathbf{q}',-\mathbf{k}') R^{b\beta}_{-\mathbf{p},-\mathbf{p}'}(t,s) C^{c\gamma}_{-\mathbf{q},\mathbf{q}'}(t,s)$$
(32)

with the eddy-topographic force is given by:

$$f_{H}^{a}(\mathbf{k},t) = \sum_{\mathbf{k}'} h_{\mathbf{k}'}^{\alpha} \int_{t_{o}}^{t} ds \, \chi_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) \equiv \sum_{\mathbf{q}} h_{-\mathbf{q}}^{c} \int_{t_{0}}^{t} ds \, \chi_{\mathbf{k},-\mathbf{q}}^{ac}(t,s)$$
(33)

and the nonlinear damping $\eta_{k,k'}^{a\alpha}(t,s)$ is given in Equation (20). That is, the nonlinear damping due to eddy-eddy interactions that occurs in the mean-field equation is *identical* with that which occurs in the covariance equation (19). Substituting Equation (31) into Equation (11) yields the SE mean-field equation:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > +\sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}') < \zeta_{\mathbf{k}'}^{\beta}(t) > = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c} \\
+\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > \\
-\int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) < \zeta_{\mathbf{k}'}^{\beta}(s) > +\overline{f}_{0}^{a}(\mathbf{k},t) + f_{H}^{a}(\mathbf{k},t)$$
(34)

It is also clear from the second form on the right hand side of Equation (33) that the eddy-topographic force has a similar structural form to the Jacobian term involving the topography on the right side of Equation (34) (after using the delta function (7d) to eliminate the summation over \mathbf{p}).

Secondly, the SE equation for the two-time cumulant is obtained by using the first order expression for the two-point cumulant $C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t)$, derived in Equation (B.9), on the right hand side of Equation (17). Then, using as well Equation (19) we have the SE two-time cumulant equation:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') + \sum_{\mathbf{k}'} D_0^{a\beta}(\mathbf{k},\mathbf{k}') C_{\mathbf{k}',-\mathbf{l}}^{\beta b}(t,t')
= \int_{t_0}^{t'} ds \sum_{\mathbf{k}'} \{S_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) + P_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) + F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)\} R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t',s)
- \int_{t_0}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s)\} C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t',s)$$
(35)

Again this equation is for Gaussian initial conditions and non-Gaussian initial conditions can also be added by following the methods of Rose [18], Frederiksen *et al.* [5] and O'Kane and Frederiksen [13]. In Equation (35):

$$P_{\mathbf{k},-\mathbf{k}'}^{a\alpha}(t,s) = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') C_{-\mathbf{p},\mathbf{p}'}^{b\beta}(t,s) \cdot \left[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c} \right] \cdot \left[2K^{\alpha\beta\gamma}(-\mathbf{k}',-\mathbf{p}',-\mathbf{q}') < \zeta_{\mathbf{q}'}^{\gamma}(s) > +A^{\alpha\beta\gamma}(-\mathbf{k}',-\mathbf{p}',-\mathbf{q}')h_{\mathbf{q}'}^{\gamma} \right]$$
(36)

$$\pi_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) = -\sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') R_{-\mathbf{p},-\mathbf{p}'}^{b\beta}(t,s) \cdot \left[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c} \right] \cdot \left[2K^{\beta\alpha\gamma}(-\mathbf{p}',-\mathbf{k}',-\mathbf{q}') < \zeta_{\mathbf{q}'}^{\gamma}(s) > +A^{\beta\alpha\gamma}(-\mathbf{p}',-\mathbf{k}',-\mathbf{q}')h_{\mathbf{q}'}^{\gamma} \right]$$
(37)

are nonlinear noise (self-energy) and nonlinear damping (self-energy) terms associated with eddy-mean-field and eddy-topographic interactions. Again, $\eta_{k,k'}^{a\alpha}(t,s)$ and $S_{k,-k'}^{a\alpha}(t,s)$ are given in Equations (20) and (21). Both $S_{k,-k'}^{a\alpha}(t,s)$ and $P_{k,-k'}^{a\alpha}(t,s)$ are positive semi-definite in the sense of equation (19) of Bowman *et al.* [51].

Thirdly, the equation for the response function in the SE closure is derived in a similar way by using the first order expression for the response function in Equation (B.12) in the Jacobian terms on the right hand side of Equation (24). We find:

$$\frac{\partial}{\partial t}R^{ab}_{\mathbf{k},\mathbf{l}}(t,t') + \sum_{\mathbf{k}'} D^{a\beta}_{0}(\mathbf{k},\mathbf{k}')R^{\beta b}_{\mathbf{k}',\mathbf{l}}(t,t') = -\int_{t'}^{t} ds \sum_{\mathbf{k}'} \{\eta^{a\beta}_{\mathbf{k},\mathbf{k}'}(t,s) + \pi^{a\beta}_{\mathbf{k},\mathbf{k}}(t,s)\}R^{\beta b}_{\mathbf{k}',\mathbf{l}}(s,t')$$
(38)

with $R_{\mathbf{k},\mathbf{l}}^{ab}(t,t) = \delta_{\mathbf{k},\mathbf{l}}\delta^{ab}$ and δ is the Kronecker delta function.

The singe-time cumulant equation for the SE closure then follows on using Equations (25) and (35):

$$\frac{\partial}{\partial t} C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t) + \sum_{\mathbf{k}'} D^{a\beta}_{0}(\mathbf{k},\mathbf{k}') C^{\beta b}_{\mathbf{k}',-\mathbf{l}}(t,t) + \sum_{\mathbf{k}'} D^{b\beta}_{0}(-\mathbf{l},-\mathbf{k}') C^{\beta a}_{-\mathbf{k}',\mathbf{k}}(t,t)
= \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S^{a\beta}_{\mathbf{k},-\mathbf{k}'}(t,s) + P^{a\beta}_{\mathbf{k},-\mathbf{k}'}(t,s) + F^{a\beta}_{0}(\mathbf{k},-\mathbf{k}',t,s)\} R^{b\beta}_{-\mathbf{l},-\mathbf{k}'}(t,s)
+ \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S^{b\beta}_{-\mathbf{l},\mathbf{k}'}(t,s) + P^{b\beta}_{-\mathbf{l},\mathbf{k}'}(t,s) + F^{b\beta}_{0}(-\mathbf{l},\mathbf{k}',t,s)\} R^{a\beta}_{\mathbf{k},\mathbf{k}'}(t,s)
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta^{a\beta}_{\mathbf{k},\mathbf{k}'}(t,s) + \pi^{a\beta}_{\mathbf{k},\mathbf{k}'}(t,s)\} C^{b\beta}_{-\mathbf{l},\mathbf{k}'}(t,s)
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta^{b\beta}_{-\mathbf{l},-\mathbf{k}'}(t,s) + \pi^{b\beta}_{-\mathbf{l},-\mathbf{k}'}(t,s)\} C^{a\beta}_{\mathbf{k},-\mathbf{k}'}(t,s)$$
(39)

In summary, the Self-Energy closure consists of Equation (34) for the mean-field, (35) for the two-time covariance, (38) for the response function and (39) for the single-time covariance together with the expressions (20), (21), (32), (36) and (37) for the self energy terms.

4.2. Langevin Equation for SE Closure

The SE closure, like the IDIA of Section 3 has an exact stochastic model representation. The generalized Langevin equation which exactly reproduces the inhomogeneous SE closure equations is as follows:

$$\frac{\partial}{\partial t}\widetilde{\zeta}_{\mathbf{k}}^{a}(t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}')\widetilde{\zeta}_{\mathbf{k}'}^{\beta}(t)
= -\int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s)\}\widetilde{\zeta}_{\mathbf{k}'}^{\beta}(s) + \hat{f}_{0}^{a}(\mathbf{k},t) + f_{S}^{a}(\mathbf{k},t) + f_{P}^{a}(\mathbf{k},t)$$
(40)

where:

$$f_{S}^{a}(\mathbf{k},t) = \sqrt{2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \rho_{-\mathbf{p}}^{(1)b}(t) \rho_{-\mathbf{q}}^{(2)c}(t)$$
(41a)

$$f_P^a(\mathbf{k},t) = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^c(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^c]\rho_{-\mathbf{p}}^{(3)b}(t)$$
(41b)

Here, $\rho_{k}^{(j)a}(t)$, where j = 1, 2 or 3, are statistically independent random variables such that

$$<\rho_{\mathbf{k}}^{(j)a}(t)\rho_{-\mathbf{l}}^{(j')b}(t') >= \delta^{jj'}C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t')$$
(42)

and:

$$\langle \tilde{\zeta}^{a}_{\mathbf{k}}(t)\tilde{\zeta}^{b}_{-\mathbf{l}}(t')\rangle = C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t')$$

$$\tag{43}$$

In Equation (42), $\delta^{jj'}$ is the Kronecker delta function.

The Langevin Equation (40) guarantees realizability for the SE closure equations. The closure equations also preserve conservation of the quadratic invariants of the original equations (in the absence of forcing and dissipation).

5. Subgrid-Scale Parameterizations

We can now derive expressions for subgrid scale terms when the resolution is reduced from C_T to $C_R < C_T$, where C_R is the resolution of the resolved scales. In the previous sections, the summations over **p** and **q** are such that $p \le C_T$, $q \le C_T$ or $(\mathbf{p}, \mathbf{q}) \in \mathbf{7}$ where the set:

$$\mathbf{7} = \left\{ \mathbf{p}, \mathbf{q} \,\middle|\, p \le C_T, q \le C_T \right\} \tag{44}$$

We also define the set \mathcal{R} of resolved scales by:

$$\mathcal{R} = \left\{ \mathbf{p}, \mathbf{q} \,\middle|\, p \le C_R, q \le C_R \right\} \tag{45}$$

and the set \boldsymbol{S} of subgrid scales by:

$$S = 7 - \mathcal{R} \tag{46}$$

The functions defined in the previous sections, which involve summations over **p** and **q**, can now be split into resolved scale terms for which $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}$ and subgrid scale terms for which $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}$. For $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}$ we define $\eta_{\mathbf{k},\mathbf{k}'}^{sab}(t,s)$, $f_{H}^{sa}(\mathbf{k},t)$, $\chi_{\mathbf{k},\mathbf{k}'}^{sab}(t,s)$ by right hand side of Equations (20), (33), (32), $S_{\mathbf{k},-\mathbf{k}'}^{sab}(t,s)$, $\pi_{\mathbf{k},\mathbf{k}'}^{sab}(t,s)$, $p_{\mathbf{k},-\mathbf{k}'}^{sab}(t,s)$ by (21), (36), (37), and $f_{S}^{sa}(\mathbf{k},t)$ and $f_{P}^{sa}(\mathbf{k},t)$ by Equations (41a) and (41b) respectively. Similar expressions with superscript \mathcal{R} may be defined for $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$.

5.1. Large Eddy Simulations with SE Subgrid Model

First we consider the modifications to the equations for direct numerical simulation when the SE closure based subgrid model is applied to form equations for large eddy simulation.

5.1.1. Mean-Field Equation for Large Eddy Simulations with SE Subgrid Model

The dynamical equation for the mean resolved scale vorticity, including subgrid scale terms, may then be derived as follows. For $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}$, we use the original Equation (11) for $\langle \zeta_k^a \rangle$ and the subgrid scale contributions are taken from the closure based Equation (34). Thus, for $k \leq C_R$ we have:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > +\sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}') < \zeta_{\mathbf{k}'}^{\beta}(t) > + \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \eta_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s) < \zeta_{\mathbf{k}}^{\beta}(s) >$$

$$= \sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \left\{ < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > + C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t) \right\}$$

$$+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}] + \bar{f}_{0}^{a}(\mathbf{k},t) + f_{H}^{sa}(\mathbf{k},t)$$

$$+ \sum_{(\mathbf{p},\mathbf{q})\in\mathcal{S}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}]$$
(47a)

This can also be written in the form:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > + \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \overline{d}_{r}^{a\beta}(\mathbf{k}, \mathbf{k}', t, s) < \zeta_{\mathbf{k}'}^{\beta}(s) > \\
= \sum_{(\mathbf{p}, \mathbf{q}) \in \mathcal{R}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) [K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \left\{ < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > + C_{-\mathbf{p}, -\mathbf{q}}^{bc}(t, t) \right\} \\
+ A^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}] + \overline{f}_{r}^{a}(\mathbf{k}, t)$$
(47b)

where the two-time renormalized dissipation elements:

$$\overline{d}_{r}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) = D_{0}^{\alpha\beta}(\mathbf{k},\mathbf{k}')\delta(t-s) + \overline{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s)$$
(48a)

and $\delta(t-s)$ is the Dirac delta function. Here we have defined the two-time drain dissipation elements by:

$$\overline{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) = \eta_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s)$$
(48b)

As well, the renormalized mean force is defined by:

$$\overline{f}_{r}^{a}(\mathbf{k},t) = \overline{f}_{0}^{a}(\mathbf{k},t) + \overline{f}_{h}^{a}(\mathbf{k},t) + j^{a}(\mathbf{k},t)$$
(48c)

where the 'residual Jacobian' term is given by:

$$j^{a}(\mathbf{k},t) = \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{S}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta^{b}_{-\mathbf{p}}(t) > \zeta^{c}_{-\mathbf{q}}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta^{b}_{-\mathbf{p}}(t) > h^{c}_{-\mathbf{q}}]$$
(48d)

Here, we have also denoted the subgrid eddy-topographic force by:

$$\overline{f}_{h}^{a}(\mathbf{k},t) \equiv f_{H}^{sa}(\mathbf{k},t)$$
(48e)

5.1.2. Fluctuating Field Equation for Large Eddy Simulations with SE Subgrid Model

Again, for $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}$ the equation for the resolved scale vorticity fluctuations is taken from Equation (12) and for the subgrid scale terms the Langevin Equation (40) is used with $\tilde{\zeta}_{\mathbf{k}}^{a} \rightarrow \hat{\zeta}_{\mathbf{k}}^{a}$ to give:

$$\frac{\partial}{\partial t}\hat{\zeta}_{\mathbf{k}}^{a}(t) + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}')\hat{\zeta}_{\mathbf{k}'}^{\beta}(t) + \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s)\}\hat{\zeta}_{\mathbf{k}'}^{\beta}(s)$$

$$= \sum_{(\mathbf{p},\mathbf{q})} \sum_{\varepsilon \in \mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{\langle \zeta_{-\mathbf{p}}^{b}(t) \rangle \hat{\zeta}_{-\mathbf{q}}^{c}(t) + \hat{\zeta}_{-\mathbf{p}}^{b}(t) \langle \zeta_{-\mathbf{q}}^{c}(t) \rangle + \hat{\zeta}_{-\mathbf{q}}^{c}(t) \rangle + \hat{\zeta}_{-\mathbf{q}}^{b}(t)\hat{\zeta}_{-\mathbf{q}}^{c}(t) - C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\} + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\zeta}_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}]$$

$$+ \hat{f}_{0}^{a}(\mathbf{k},t) + f_{S}^{sa}(\mathbf{k},t) + f_{p}^{sa}(\mathbf{k},t)$$
(49a)

This can also be written as:

$$\frac{\partial}{\partial t} \hat{\zeta}^{a}_{\mathbf{k}}(t) + \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \hat{d}^{a\beta}_{r}(\mathbf{k}, \mathbf{k}', t, s) \hat{\zeta}^{\beta}_{\mathbf{k}'}(s)$$

$$= \sum_{(\mathbf{p}, \mathbf{q}) \in \mathcal{R}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) [K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \{ \langle \zeta^{b}_{-\mathbf{p}}(t) \rangle \hat{\zeta}^{c}_{-\mathbf{q}}(t) + \hat{\zeta}^{b}_{-\mathbf{p}}(t) \langle \zeta^{c}_{-\mathbf{q}}(t) \rangle + \hat{\zeta}^{c}_{-\mathbf{q}}(t) + \hat{\zeta}^{b}_{-\mathbf{p}}(t) \langle \zeta^{c}_{-\mathbf{q}}(t) \rangle + \hat{\zeta}^{b}_{-\mathbf{q}}(t) + \hat{\zeta}^{b}_{-\mathbf{q}}(t) \rangle + \hat{\zeta}^{b}_{-\mathbf{q}}(t) - C^{bc}_{-\mathbf{p},-\mathbf{q}}(t, t) \} + A^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \hat{\zeta}^{b}_{-\mathbf{p}}(t) h^{c}_{-\mathbf{q}}] + \hat{f}^{a}_{r}(\mathbf{k}, t)$$
(49b)

where the two-time renormalized dissipation elements:

$$\hat{d}_{r}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) = D_{0}^{\alpha\beta}(\mathbf{k},\mathbf{k}')\delta(t-s) + \hat{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s)$$
(50a)

Here, the two-time drain dissipation elements are given by:

$$\hat{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) = \eta_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{sa\beta}(t,s)$$
(50b)

As well, the renormalized random force is defined by:

$$\hat{f}_r^a(\mathbf{k},t) = \hat{f}_0^a(\mathbf{k},t) + \hat{f}_b^a(\mathbf{k},t)$$
(50c)

where:

$$\hat{f}_b^a(\mathbf{k},t) = f_S^{sa}(\mathbf{k},t) + f_P^{sa}(\mathbf{k},t)$$
(50d)

These mean-field and fluctuation equations are generalizations of the original Equations (11) and (12) with additional forcing contributions and linear terms modifying the bare viscous dissipation; these additional terms are due to the subgrid scale eddies. Note also that the linear terms now have an integral representation. That is, our parameterization of the subgrid scale eddies changes the original coupled ordinary differential Equations (11) and (12) to coupled integro-differential equations for the resolved scales. This is due to the subgrid scales having memory effects.

5.2. Self-Energy Closure with SE Subgrid Model

Mean-Field Equation for Self-Energy Closure with SE Subgrid Model

The mean-field equation for the Self-Energy closure with the SE subgrid model is again given by Equation (47) but with the replacement:

$$\sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{R}} \sum_{\boldsymbol{\kappa}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t)$$

$$\Rightarrow -\int_{t_o}^{t} ds \sum_{\mathbf{k}'} \eta^{\mathcal{R}a\beta}_{\mathbf{k},\mathbf{k}'}(t,s) < \zeta^{\beta}_{\mathbf{k}'}(s) > + f^{\mathcal{R}a}_{H}(\mathbf{k},t)$$
(51)

5.3. Response Function and Covariance for Self-Energy Closure with SE Subgrid Model

The equation for the response function, including subgrid terms, follows from Equation (38):

$$\frac{\partial}{\partial t} R_{\mathbf{k},\mathbf{l}}^{ab}(t,t') + \int_{t'}^{t} ds \sum_{\mathbf{k}'} \hat{d}_{r}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) R_{\mathbf{k},\mathbf{l}}^{\beta b}(s,t')$$

$$= -\int_{t'}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s)\} R_{\mathbf{k},\mathbf{l}}^{\beta b}(s,t')$$
(52)

Similarly, from Equation (35), the two-time cumulant equation including subgrid terms is:

$$\frac{\partial}{\partial t} C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t') + \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \hat{d}^{a\beta}_r(\mathbf{k},\mathbf{k}',t,s) C^{b\beta}_{-\mathbf{l},\mathbf{k}'}(t',s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} F^{a\beta}_r(\mathbf{k},-\mathbf{k}',t,s) R^{b\beta}_{-\mathbf{l},-\mathbf{k}'}(t',s) \\
= \int_{t_o}^{t'} ds \sum_{\mathbf{k}'} \{S^{\mathcal{R}a\beta}_{\mathbf{k},-\mathbf{k}'}(t,s) + P^{\mathcal{R}a\beta}_{\mathbf{k},-\mathbf{k}'}(t,s)\} R^{b\beta}_{-\mathbf{l},-\mathbf{k}'}(t',s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta^{\mathcal{R}a\beta}_{\mathbf{k},\mathbf{k}'}(t,s) + \pi^{\mathcal{R}a\beta}_{\mathbf{k},\mathbf{k}'}(t,s)\} C^{b\beta}_{-\mathbf{l},\mathbf{k}'}(t',s)$$
(53)

Finally, from Equation (39), the single-time cumulant equation including subgrid terms is:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t) + \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \hat{d}_r^{a\beta}(\mathbf{k},\mathbf{k}',t,s) C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} F_r^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \hat{d}_r^{b\beta}(-\mathbf{l},-\mathbf{k}',t,s) C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} F_r^{b\beta}(-\mathbf{l},\mathbf{k}',t,s) R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) \\
= \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{\mathbf{k},-\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + P_{\mathbf{k},-\mathbf{k}'}^{\mathbf{z}a\beta}(t,s)\} R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s)\} C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \pi_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \pi_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \eta_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{k},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \eta_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s) - \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{k},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \eta_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \eta_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \eta_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) \\
+ \int_{t_o}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{k},\mathbf{k}'}^{\mathbf{z}$$

Here, we have defined renormalized noise covariance matrix elements by:

$$F_r^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) = \langle \hat{f}_r^a(\mathbf{k},t)\hat{f}_r^\beta(-\mathbf{k}',s) \rangle$$

= $F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) + F_b^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)$ (55a)

The renormalized random force $\hat{f}_r^a(\mathbf{k},t)$ is given by Equation (50c), $F_0^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)$ is given in Equation (22) and the backscatter covariance matrix elements by:

$$F_{b}^{a\beta}(\mathbf{k},-\mathbf{k}',t,s) = S_{\mathbf{k},-\mathbf{k}'}^{s\alpha\beta}(t,s) + P_{\mathbf{k},-\mathbf{k}'}^{s\alpha\beta}(t,s) \equiv \langle \hat{f}_{b}^{a}(\mathbf{k},t)\hat{f}_{b}^{\beta}(-\mathbf{k}',s) \rangle$$
(55b)

As well, the stochastic backscatter noise $\hat{f}_b^a(\mathbf{k},t)$ is given by Equation (50d).

5.4. Langevin Equation for Self-Energy Closure with Subgrid-Scale Parameterizations

The generalized Langevin equation which exactly reproduces the Self-Energy closure equations with subgrid-scale parameterizations is as follows:

$$\frac{\partial}{\partial t}\widetilde{\zeta}_{\mathbf{k}}^{a}(t) + \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \hat{d}_{r}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) \widetilde{\zeta}_{\mathbf{k}'}^{\beta}(s)
= -\int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s)\} \widetilde{\zeta}_{\mathbf{k}'}^{\beta}(s) + \hat{f}_{r}^{a}(\mathbf{k},t) + f_{s}^{\mathbf{z}a}(\mathbf{k},t) + f_{p}^{\mathbf{z}a}(\mathbf{k},t) + f_{p}^{\mathbf{z}a}(\mathbf{k},t)$$

where:

$$f_{S}^{\mathcal{R}a}(\mathbf{k},t) = \sqrt{2} \sum_{(\mathbf{p},\mathbf{q})} \sum_{\epsilon \in \mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \rho_{-\mathbf{p}}^{(1)b}(t) \rho_{-\mathbf{q}}^{(2)c}(t)$$
(57a)

$$f_{P}^{\mathcal{R}a}(\mathbf{k},t) = \sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}} \sum_{\sigma\in\mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c}]\rho_{-\mathbf{p}}^{(3)b}(t)$$
(57b)

Again, $\rho_{k}^{(j)a}(t)$, where j = 1, 2 or 3, are statistically independent random variables that satisfy Equation (42) and Equation (43) also holds. The Langevin Equation (56) guarantees realizability for the elements of the covariance matrices in the SE closure equations.

6. Effective Dissipation and Viscosity Parameterizations

We can also write Equation (34) in the form:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > +\sum_{\mathbf{k}'} \overline{D}_{r}^{a\beta}(\mathbf{k},\mathbf{k}') < \zeta_{\mathbf{k}'}^{\beta}(t) >
= \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{R}} \sum_{\boldsymbol{k}'} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) [K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \left\{ < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) > +C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t) \right\}
+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{a}(t) > h^{b}_{-\mathbf{q}}] + \overline{f}_{r}^{a}(\mathbf{k},t)$$
(58)

where renormalized generalized drain dissipation matrices appearing in Equation (58) are given by:

$$\overline{D}_{r}^{ab}(\mathbf{k},\mathbf{k}') = D_{0}^{ab}(\mathbf{k},\mathbf{k}') + \overline{D}_{d}^{ab}(\mathbf{k},\mathbf{k}')$$
(59)

Here the drain dissipation matrix elements for the mean-field are given by:

$$\overline{D}_{d}^{a\beta}(\mathbf{k},\mathbf{k}') < \zeta_{\mathbf{k}'}^{\beta}(t) >= \int_{t_{o}}^{t} ds \, \overline{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s) < \zeta_{\mathbf{k}'}^{\beta}(s) > = \int_{t_{o}}^{t} ds \, \eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{s}a\beta}(t,s) < \zeta_{\mathbf{k}'}^{\beta}(s) >$$
(60)

In general $\overline{D}_r^{ab}(\mathbf{k},\mathbf{k}')$ and $\overline{D}_d^{ab}(\mathbf{k},\mathbf{k}')$ are time-dependent but in our previous studies we have primarily been interested in their properties at statistical steady state.

We also define the drain dissipation matrix elements for fluctuations by:

$$\hat{D}_{d}^{a\beta}(\mathbf{k},\mathbf{k}')C_{\mathbf{k}',-\mathbf{l}}^{\beta b}(t,t) = \int_{t_{o}}^{t} ds \,\hat{d}_{d}^{a\beta}(\mathbf{k},\mathbf{k}',t,s)C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t,s) = \int_{t_{o}}^{t} ds \,\{\eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{s}a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{\mathbf{s}a\beta}(t,s)\}C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t,s)$$
(61)

Thus:

$$\frac{\partial}{\partial t} C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t) + \sum_{\mathbf{k}'} \hat{D}_{r}^{a\beta}(\mathbf{k},\mathbf{k}') C_{\mathbf{k}',-\mathbf{l}}^{\beta b}(t,t) + \sum_{\mathbf{k}'} \hat{D}_{r}^{b\beta}(-\mathbf{l},-\mathbf{k}') C_{-\mathbf{k}',\mathbf{k}}^{\beta a}(t,t)
= \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S_{\mathbf{k},-\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + P_{\mathbf{k},-\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + F_{r}^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)\} R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t,s)
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s) + \pi_{\mathbf{k},\mathbf{k}'}^{\mathbf{z}a\beta}(t,s)\} C_{-\mathbf{l},\mathbf{k}'}^{b\beta}(t,s)
+ \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + P_{-\mathbf{l},\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + F_{r}^{b\beta}(-\mathbf{l},\mathbf{k}',t,s)\} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,s)
- \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{\eta_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s) + \pi_{-\mathbf{l},-\mathbf{k}'}^{\mathbf{z}b\beta}(t,s)\} C_{\mathbf{k},-\mathbf{k}'}^{a\beta}(t,s)$$
(62)

where:

$$\hat{D}_r^{ab}(\mathbf{k},\mathbf{k}') = D_0^{ab}(\mathbf{k},\mathbf{k}') + \hat{D}_d^{ab}(\mathbf{k},\mathbf{k}')$$
(63)

We can define the backscatter dissipation matrix elements by:

$$\hat{D}_{b}^{a\beta}(\mathbf{k},\mathbf{k}')C_{\mathbf{k},-\mathbf{l}}^{\betab}(t,t) = -\int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} \{S_{\mathbf{k},-\mathbf{k}'}^{sa\beta}(t,s) + P_{\mathbf{k},-\mathbf{k}'}^{sa\beta}(t,s)\}R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t,s)$$

$$= -\int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} F_{b}^{a\beta}(\mathbf{k},-\mathbf{k}',t,s)R_{-\mathbf{l},-\mathbf{k}'}^{b\beta}(t,s)$$
(64a)

Further, we define the net dissipation matrix elements by:

$$\hat{D}_{n}^{ab}(\mathbf{k},\mathbf{k}') = \hat{D}_{d}^{ab}(\mathbf{k},\mathbf{k}') + \hat{D}_{b}^{ab}(\mathbf{k},\mathbf{k}')$$
(64b)

and the renormalized net dissipation matrix elements by:

$$\hat{D}_{rn}^{ab}(\mathbf{k},\mathbf{k}') = D_0^{ab}(\mathbf{k},\mathbf{k}') + \hat{D}_n^{ab}(\mathbf{k},\mathbf{k}')$$
(64c)

Again, generalized viscosity matrix elements may be defined as follows:

$$\overline{\nu}_{\bullet}^{ab}(\mathbf{k},\mathbf{k}') = \overline{D}_{\bullet}^{ab}(\mathbf{k},\mathbf{k}')k^{-2}$$
(65a)

and:

$$\hat{v}_{\bullet}^{ab}(\mathbf{k},\mathbf{k}') = \hat{D}_{\bullet}^{ab}(\mathbf{k},\mathbf{k}')k^{-2}$$
(65b)

where • denotes any of 0, b, d, n, r, rn.

In general, and specifically if we include the beta-effect in our analysis, the dissipation matrix elements $\overline{D}_{\bullet}^{ab}(\mathbf{k})$, $\hat{D}_{\bullet}^{ab}(\mathbf{k})$, and the viscosities $\overline{v}_{\bullet}^{ab}(\mathbf{k})$, $\hat{v}_{\bullet}^{ab}(\mathbf{k})$ are complex; both the viscosity and wave frequency are renormalized by the subgrid scale eddies. If our system is quasi-isotropic, as well as quasi-diagonal, then these terms are real viscosities and only depend on *k*, the magnitude of **k**. This is the case, in particular, near canonical equilibrium.

We note that Equation (58) is identical to Equation (11) except that the bare dissipation $D_0^{a\beta}(\mathbf{k})$ is replaced by the renormalized dissipation $\overline{D}_r^{a\beta}(\mathbf{k})$, the bare force $\overline{f}_0^a(\mathbf{k},t)$ is replaced by the

renormalized force $\bar{f}_r^a(\mathbf{k},t)$ and the summation is now for $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$ rather than over the whole space $(\mathbf{p},\mathbf{q}) \in \mathbf{7}$. Similarly Equation (62) is the same as Equation (39) but with $D_0^{a\beta}(\mathbf{k}) \rightarrow \hat{D}_r^{a\beta}(\mathbf{k})$, $F_0^{a\beta}(\mathbf{k},t,s) \rightarrow F_r^{a\beta}(\mathbf{k},t,s)$ (or $\hat{f}_0^a(\mathbf{k},t) \rightarrow \hat{f}_r^a(\mathbf{k},t)$) and $\mathbf{7} \rightarrow \mathcal{R}$. Our analysis also shows that the subgrid-scale eddies affect the mean and fluctuating-parts of the field variables differently. In general, the mean and fluctuating parts of the fields relax at different rates as seen from Equations (60) and (61).

7. Derivation of QDIA Closure from Self-Energy Closure

The Self-Energy closure provides a general formalism for subgrid modeling for the mean-field equation as well as the transients. It is also convenient for deriving simpler, more computationally tractable closures, for different geometries and boundary conditions. In this section we apply it to derive the QDIA closure theory for planar geometry. The QDIA may be obtained from the SE closure equations as follows. In the SE closure equations of Section 5 we suppose that the two-point cumulants and response functions are diagonal in spectral space to lowest order. Thus, to lowest order:

$$R_{k,k'}^{ab}(t,t') = R_k^{ab}(t,t')\delta_{k,k'}$$
(66a)

$$R_{\mathbf{k}}^{ab}(t,t') \equiv R_{\mathbf{k},\mathbf{k}}^{ab}(t,t') \tag{66b}$$

$$C_{\mathbf{k},\mathbf{k}'}^{ab}(t,t') = C_{\mathbf{k}}^{ab}(t,t')\delta_{\mathbf{k},\mathbf{k}'}$$
(66c)

$$C_{\mathbf{k}}^{ab}(t,t') \equiv C_{\mathbf{k},-\mathbf{k}}^{ab}(t,t') \tag{66d}$$

Then the mean-field Equation (34) becomes:

$$\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > + D_{0}^{a\beta}(\mathbf{k}) < \zeta_{\mathbf{k}}^{\beta}(t) > = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > < \zeta_{-\mathbf{q}}^{c}(t) >
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}, \mathbf{p}, \mathbf{q}) A^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}
- \int_{t_{o}}^{t} ds \, \eta_{\mathbf{k}}^{a\beta}(t, s) < \zeta_{\mathbf{k}}^{\beta}(s) > + \overline{f}_{0}^{a}(\mathbf{k}, t) + f_{H}^{a}(\mathbf{k}, t)$$
(67)

where the nonlinear eddy-eddy damping, eddy-topographic force and eddy-topographic interaction are given by:

$$\eta_{\mathbf{k}}^{a\alpha}(t,s) = -4\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k})R^{b\beta}_{-\mathbf{p}}(t,s)C^{c\gamma}_{-\mathbf{q}}(t,s)$$
(68a)

$$f_{H}^{a}(\mathbf{k},t) = h_{\mathbf{k}}^{\alpha} \int_{t_{o}}^{t} ds \, \chi_{\mathbf{k}}^{a\alpha}(t,s)$$
(68b)

$$\chi_{\mathbf{k}}^{a\alpha}(t,s) = 2\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) A^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k}) R^{b\beta}_{-\mathbf{p}}(t,s) C^{c\gamma}_{-\mathbf{q}}(t,s)$$
(68c)

This equation is for initial conditions that are homogeneous in the horizontal. Inhomogeneous initial conditions can also be treated following the method of O'Kane and Frederiksen [13].

The two-point SE Equation (35) reduces to:

$$\frac{\partial}{\partial t} C_{\mathbf{k}}^{ab}(t,t') + D_{0}^{a\beta}(\mathbf{k}) C_{\mathbf{k}}^{\beta b}(t,t') = \int_{t_{o}}^{t} ds \{ S_{\mathbf{k}}^{a\beta}(t,s) + P_{\mathbf{k}}^{a\beta}(t,s) + F_{0}^{a\beta}(\mathbf{k},t,s) \} R_{-\mathbf{k}}^{b\beta}(t',s) - \int_{t_{o}}^{t} ds \{ \eta_{\mathbf{k}}^{a\beta}(t,s) + \pi_{\mathbf{k}}^{a\beta}(t,s) \} C_{-\mathbf{k}}^{b\beta}(t',s)$$
(69)

This equation is again valid for Gaussian initial conditions and can be generalized to non-Gaussian and inhomogeneous initial conditions following the approach of O'Kane and Frederiksen [13]. In Equation (69):

$$F_0^{a\alpha}(\mathbf{k},t,s) = \langle \hat{f}_0^a(\mathbf{k},t) \hat{f}_0^\alpha(-\mathbf{k},s) \rangle$$
(70a)

$$S_{\mathbf{k}}^{a\alpha}(t,s) = 2\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\alpha\beta\gamma}(-\mathbf{k},-\mathbf{p},-\mathbf{q})C_{-\mathbf{p}}^{b\beta}(t,s)C_{-\mathbf{q}}^{c\gamma}(t,s)$$
(70b)

$$P_{\mathbf{k}}^{a\alpha}(t,s) = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) C_{-\mathbf{p}}^{b\beta}(t,s) [2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c}]$$

$$(70c)$$

$$(70c)$$

$$\pi_{\mathbf{k}}^{a\alpha}(t,s) = -\sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) R_{-\mathbf{p}}^{b\beta}(t,s) [2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > +A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c}]$$

$$\cdot [2K^{\beta\alpha\gamma}(-\mathbf{p},-\mathbf{k},-\mathbf{q}) < \zeta_{\mathbf{q}}^{\gamma}(s) > +A^{\beta\alpha\gamma}(-\mathbf{p},-\mathbf{k},-\mathbf{q})h_{\mathbf{q}}^{\gamma}]$$
(70d)

and $\eta_{\mathbf{k}}^{a\alpha}(t,s)$ is given in Equation (68a).

Again, the SE response function Equation (38) reduces to:

$$\frac{\partial}{\partial t}R_{\mathbf{k}}^{ab}(t,t') + D_{0}^{a\beta}(\mathbf{k})R_{\mathbf{k}}^{\beta b}(t,t') = -\int_{t'}^{t} ds \{\eta_{\mathbf{k}}^{a\beta}(t,s) + \pi_{\mathbf{k}}^{a\beta}(t,s)\}R_{\mathbf{k}}^{\beta b}(s,t')$$
(71)

with $R_{k}^{ab}(t,t) = \delta^{ab}$ and δ^{ab} is the Kronecker delta function. Further, the single-time two-point cumulant Equation (39) becomes:

$$\frac{\partial}{\partial t} C_{\mathbf{k}}^{ab}(t,t) + D_{0}^{a\beta}(\mathbf{k}) C_{\mathbf{k}}^{\beta b}(t,t) + D_{0}^{b\beta}(-\mathbf{k}) C_{-\mathbf{k}}^{\beta a}(t,t)
= \int_{t_{o}}^{t} ds \{S_{\mathbf{k}}^{a\beta}(t,s) + P_{\mathbf{k}}^{a\beta}(t,s) + F_{0}^{a\beta}(\mathbf{k},t,s)\} R_{-\mathbf{k}}^{b\beta}(t,s)
+ \int_{t_{o}}^{t} ds \{S_{-\mathbf{k}}^{b\beta}(t,s) + P_{-\mathbf{k}}^{b\beta}(t,s) + F_{0}^{b\beta}(-\mathbf{k},t,s)\} R_{\mathbf{k}}^{a\beta}(t,s)
- \int_{t_{o}}^{t} ds \{\eta_{\mathbf{k}}^{a\beta}(t,s) + \pi_{\mathbf{k}}^{a\beta}(t,s)\} C_{-\mathbf{k}}^{b\beta}(t,s) - \int_{t_{o}}^{t} ds \{\eta_{-\mathbf{k}}^{b\beta}(t,s) + \pi_{-\mathbf{k}}^{b\beta}(t,s)\} C_{\mathbf{k}}^{a\beta}(t,s)$$
(72)

reduces to:

$$C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') = \int_{t_o}^{t} ds \, R_{\mathbf{k},\mathbf{k}}^{a\alpha}(t,s) C_{-\mathbf{l},\mathbf{l}}^{b\beta}(t',s) [A^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k})h_{(\mathbf{k}-\mathbf{l})}^{\gamma} +2K^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) < \zeta_{(\mathbf{k}-\mathbf{l})}^{\gamma}(s) >] +\int_{t_o}^{t'} ds \, R_{-\mathbf{l},-\mathbf{l}}^{b\beta}(t',s) C_{\mathbf{k},\mathbf{k}}^{a\alpha}(t,s) [A^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k})h_{(\mathbf{k}-\mathbf{l})}^{\gamma} +2K^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k}) < \zeta_{(\mathbf{k}-\mathbf{l})}^{\gamma}(s) >]$$
(73)

This is just the QDIA expression in Equation (A.9) of Frederiksen [12]. As well, the first order expression for the response function in Equation (A.12) becomes:

$$R_{\mathbf{k},\mathbf{l}}^{ab}(t,t') = \int_{t'}^{t} ds R_{\mathbf{k},\mathbf{k}}^{a\alpha}(t,s) R_{\mathbf{l},\mathbf{l}}^{\beta b}(s,t').$$

$$\left\{ A^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) h_{(\mathbf{k}-\mathbf{l})}^{\gamma} + 2K^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) < \zeta_{(\mathbf{k}-\mathbf{l})}^{\gamma}(s) > \right\}$$
(74)

This is the QDIA expression in Equation (A.12) of [12].

8. Closure Equations for Quasigeostrophic Turbulent Flow on a Sphere

Next, we consider dynamical and closure equations in spherical geometry as a specific example of the generality of the closure equations.

8.1. Quasigeostrophic Equations for Flow on the Sphere

The two-level quasigeostrophic equations for geophysical flows on the sphere take the same form as in Equation (4). However, the x and y coordinates for planar geometry are replaced by $\lambda =$ longitude and $\mu =$ sin(latitude). The derivation of the corresponding spectral equations is as outlined in Appendix with the result:

$$\frac{\partial}{\partial t} \zeta_{\mathbf{k}}^{a}(t) + D_{0}^{a\beta}(\mathbf{k})\zeta_{\mathbf{k}}^{\beta}(t)
= \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta_{Sphere}(\mathbf{k},\mathbf{p},\mathbf{q}) \Big[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)\zeta_{-\mathbf{q}}^{c}(t) + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c} \Big] + f_{0}^{a}(\mathbf{k},t)$$
(75)

Here, $D_0^{a\beta}(\mathbf{k}) = v_0^{a\beta}(\mathbf{k})k(k+1)$ and we use the notation:

$$\mathbf{k} = (m_k, n_k) = (m, n),$$

- $\mathbf{k} = (-m_k, n_k),$ (76)
 $k = n$

where m is zonal wavenumber and n is total wavenumber. Also, in the above spectral equations:

$$\zeta_{-k}^{a} \equiv \zeta_{(-m,n)}^{a} = \zeta_{(m,n)}^{a^{*}} \equiv \zeta_{k}^{a^{*}}$$
(77a)

$$\delta_{Sphere}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } m_k + m_p + m_q = 0, \\ \text{and } n_k + n_p + n_q & \text{is odd}, \\ \text{and } |n_p - n_q| & < n_k < n_p + n_q, \\ 0 & \text{otherwise} \end{cases}$$
(77b)

Also $K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ satisfies Equation (7c) and the interaction coefficients are given in Appendix A. The selection rules for non-zero interaction coefficients are as given in Equation (2.3) of Frederiksen and Sawford [52]. These equations also apply when $D_0^{a\beta}(\mathbf{k})$ includes the Rossby wave frequency due to differential rotation.

8.2. Self-Energy Closure for Inhomogeneous Turbulent Flow on the Sphere

For the case of quasigeostrophic turbulent flows on a sphere the SE closure equations are again as given in Section 4, as is the corresponding Langevin equation, with $\delta_{Sphere}(\mathbf{k}, \mathbf{p}, \mathbf{q})$ in Equation (77b) replacing $\delta(\mathbf{k}, \mathbf{p}, \mathbf{q})$.

8.3. Quasi-Diagonal Self-Energy Closure for Inhomogeneous Turbulent Flow on the Sphere

We can again consider simplified closure equations for flow on the sphere by assuming that to lowest order the covariance and response functions are diagonal as in Equation (66). This yields a reduced system of realizable closure equations that are related to the QDIA closure but differ in that there is still more complex coupling in the total wavenumber because of the different selection rules (77b).

8.4. QDIA Closure for Inhomogeneous Turbulent Flow on the Sphere

Although applying the quasi-diagonal approximation at lowest order in spherical geometry does not yield equations with the simplicity of the QDIA closure of Section 7, the QDIA closure is still a realizable system for flow on the sphere with $\delta_{Sphere}(\mathbf{k}, \mathbf{p}, \mathbf{q})$ in Equation (77b) replacing $\delta(\mathbf{k}, \mathbf{p}, \mathbf{q})$. Its performance would of course need to be tested against direct numerical simulations.

9. Discussion and Conclusions

We have formulated the Self-Energy closure theory for inhomogeneous turbulent flow and more generally for classical fields with quadratic nonlinearities. The SE closure is a generalization of the computationally tractable QDIA closure [10–12]. It has a similar structure to the QDIA in that the nonlinear interactions are expressed as integrals of self-energy terms multiplied by the mean-fields, topography, covariance and response functions. The renormalization of the bare dissipation and forcing by eddy-eddy interactions is thus transparent in both the mean-field equation and in the equations for the propagators, the two-point cumulant and the response function. The SE closure equations are also second order in perturbation theory and in the interaction coefficients like the QDIA. However, the SE closure does not employ the quasi-diagonal approximation but allows more complex interactions like Kraichnan's IDIA [22]. The IDIA differs from the SE closure in that the covariance of eddy-eddy transient interactions is not expressed as integrals over self-energy terms

multiplied by the mean-field and topography that makes renormalization transparent. In the IDIA the propagator equations are second order in perturbation theory and in interaction coefficients. Thus when the second order expression for the covariance is inserted in the mean-field equation it becomes third order in the interaction coefficient.

We have shown how the SE closure can be used to formulate subgrid-scale parameterizations for both the mean-field and transient equations needed for large eddy simulations. It provides a general formalism for tackling the particularly difficult problem of determining the subgrid contribution to the mean-field dissipation and forcing [10,12,16,17,35,38,54].

9.1. Self-Energy LET and SCFT Inhomogeneous Closures

Both the SCFT [3] and LET [4,1] closures for homogeneous turbulence may be formally obtained from the DIA by invoking the fluctuation-dissipation theorem [5–7], although this was not the way they were originally derived. In a similar way we can obtain Self-Energy SCFT (SE-SCFT) and LET (SE-LET) closures for inhomogeneous turbulent flows from the SE. For our multi-field equations we use the fluctuation-dissipation theorem:

$$C_{\mathbf{k},\mathbf{l}}^{ab}(t,t')\Theta(t-t') = \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{a\beta}(t,t')C_{\mathbf{k}',\mathbf{l}}^{\beta b}(t,t)$$
(78)

which relates the two-time cumulant to the response function and the single-time cumulant. Here $\Theta(x)$ is the Heaviside theta function which is unity for x positive and otherwise vanishes. The SE-SCFT is then obtained from the SE by replacing the prognostic equation for the two-time cumulant, Equation (35), by Equation (78). Similarly, the SE-LET closure is obtained from the SE by replacing the equation for the response function, Equation (38), by the expression in Equation (78) in terms of the single- and two-time cumulants.

9.2. Regularization, Vertex Renormalization and Non-Gaussian Initial Conditions

The SE closure can also be adapted to incorporate a regularization procedure following the approach of Frederiksen and Davies [14] and O'Kane and Frederiksen [13]. This corresponds to an empirical vertex renormalization in which the interactions are localized in wavenumber space. For the SE closure this is achieved by making the replacements:

$$A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \to \Theta(p-k/\alpha_c)\Theta(q-k/\alpha_c)A^{abc}(\mathbf{k},\mathbf{q},\mathbf{p})$$
(79a)

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \to \Theta(p-k/\alpha_c)\Theta(q-k/\alpha_c)K^{abc}(\mathbf{k},\mathbf{q},\mathbf{p})$$
(79b)

in the two-time cumulant and response function equations, but not in the single-time cumulant equations, where $\Theta(x)$ is again the Heaviside theta function. For suitable interaction cut-off parameter α_c we expect that the SE closure will again have the correct power laws as well as accurately capturing the evolution of the energy containing scales. Again, non-Gaussian initial conditions can be included in the SE closure following the methods of Rose [18], Frederiksen *et al.* [5] and O'Kane and Frederiksen [13].

9.3. Maximum Entropy States of Canonical Equilibrium

Statistical mechanics equilibrium states [52,55–58] corresponding to maximum entropy are exact solutions of the inviscid QDIA closure for the case of barotropic turbulent flows [10]. In the case of inviscid quasigeostrophic baroclinic turbulent flows canonical equilibrium states [59,60] are again exact solutions of the QDIA. These maximum entropy states are also exact solutions of the SE closure for the corresponding inviscid equations. So it is perhaps not surprising that near canonical equilibrium states the QDIA closure would perform well and could replace the SE or IDIA closures. What is more surprising is that the QDIA performs so well in all the studies mentioned in the Introduction including for far from equilibrium phenomena like Rossby wave dispersion in a turbulent environment. Thus employing the diagonal or homogeneous conjecture (66) to lowest order appears to be a good approximation. It is as if the only inhomogeneities that survive in the covariance and response functions are those generated by the mean-field and topography through Equations (73) and (74).

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Appendices

Appendix A: Spectral Equations for Inhomogeneous Fluid Turbulence

A.1. Spectral Equations for Quasigeostrophic Turbulent Flow in Planar Geometry

Here, we summarize the conversion of the quasigeostrophic Equations (4) and (5) in physical space to spectral space. We focus on flow on the doubly periodic *f*-plane $0 \le x \le 2\pi, 0 \le y \le 2\pi$. However, the analysis can equally be generalized to flows in a channel, in a bounded domain, in an infinite domain [53] and to flow on a β -plane.

The spectral equations are obtained by expanding each of the functions in Equation (4) in a Fourier series:

$$\zeta^{a}(\mathbf{x},t) = \sum_{\mathbf{k}} \zeta^{a}_{\mathbf{k}}(t) \exp(i\mathbf{k} \cdot \mathbf{x})$$
(A.1)

where:

$$\zeta_{\mathbf{k}}^{a}(t) = \frac{1}{(2\pi)^{2}} \int_{b}^{2\pi} d^{2}\mathbf{x} \zeta^{a}(\mathbf{x}, t) \exp(-i\mathbf{k} \cdot \mathbf{x})$$
(A.1b)

and $\mathbf{x} = (x, y)$, $\mathbf{k} = (k_x, k_y)$. Then, multiplying Equation (4) by $\exp(-i\mathbf{k} \cdot \mathbf{x})$ and integrating over the (x, y) domain, we find that with the identification $q_k^a(t) \Rightarrow \zeta_k^a(t)$, that the spectral equation takes the form given in Equation (6). Here, the interaction coefficients are defined by:

$$A^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \begin{cases} A(\mathbf{k}, \mathbf{p}, \mathbf{q}) V^{ab}(p) & \text{if } c = a \\ 0 & \text{otherwise} \end{cases}$$
(A.2a)

$$V^{11}(p) = V^{22}(p) = \frac{p^2 + F_L}{p^2 + 2F_L}; \quad V^{12}(p) = V^{21}(p) = \frac{F_L}{p^2 + 2F_L}$$
(A2.b)

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x) / p^2$$
(A2.c)

Also $K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ is then determined by Equation (7c).

A.2. Spectral Equations for Three-Dimensional Turbulent Flow

We can again convert the physical space Navier Stokes Equation (9) into spectral space by expanding the fields in Fourier series. The Navies Stokes equations for three-dimensional inhomogeneous turbulence in a periodic box, $0 \le x = x^1 \le 2\pi$, $0 \le y = x^2 \le 2\pi$, $0 \le z = x^3 \le 2\pi$, may then be written in the form [1]:

$$\frac{\partial}{\partial t}u_{\mathbf{k}}^{a}(t) + D_{0}^{a\beta}(\mathbf{k})u_{\mathbf{k}}^{\beta}(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})M^{abc}(\mathbf{k})u_{-\mathbf{p}}^{b}(t)u_{-\mathbf{q}}^{c}(t) + f_{0}^{a}(\mathbf{k},t)$$
(A.3)

Similar equations apply on the infinite domain with the sums replaced by integrals [49]. Here, $u_{\mathbf{k}}^{a}(t), a = 1,2,3$ are the spectral components of the velocity fields in the *x*, *y*, *z* directions respectively that depend on time *t* and the vector $\mathbf{k} = (k^{1}, k^{2}, k^{3}) = (k_{x}, k_{y}, k_{z})$. The linear term involving $D_{0}^{a\beta}(\mathbf{k})$ generally represents dissipation and $f_{0}^{a}(\mathbf{k}, t)$ is a forcing term. Also, the interaction coefficient:

$$M^{abc}(\mathbf{k}) = i/2\{\Delta^{ab}(\mathbf{k})k^{c} + \Delta^{ac}(\mathbf{k})k^{b}\}$$
(A.4a)

$$\Delta^{ab}(\mathbf{k}) = \delta^{ab} - k^a k^b / \left| \mathbf{k} \right|^2 \tag{A.4b}$$

We next change the notation to put Equations (A.3) and (A.4) into the standard form (1) and (2). We make the identifications:

$$u_{\mathbf{k}}^{a}(t) \Rightarrow \zeta_{\mathbf{k}}^{a}(t)$$
 (A.5a)

$$A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = i\Delta^{ab}(\mathbf{k})k^{c}$$
(A.5b)

Also $K^{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q})$ is then determined by Equation (7c). Thus Equation (A.3) is in the form (1) or (6) with $h_{\mathbf{k}}^{a} = 0$.

A.3. Spectral Equations for Quasigeostrophic Turbulent Flow on the Sphere

In spherical geometry the quasigeostrophic equations are again given by Equation (4) and the Jacobian is given by:

$$J(\psi,\zeta) = \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}$$
(A.6)

Spectral equations corresponding to this system may be obtained by first expanding each of the fields in spherical harmonics in the horizontal, e.g.,

$$\zeta^{a}(\lambda,\mu,t) = \sum_{m} \sum_{n} \zeta^{a}_{(m,n)}(t) P_{n}^{m}(\mu) \exp(im\lambda)$$

=
$$\sum_{k} \zeta^{a}_{k}(t) P_{k}(\mu) \exp(im_{k}\lambda)$$
 (A.7)

where:

$$P_{\mathbf{k}}(\mu) = P_{n_{k}}^{m_{k}}(\mu) ,$$

$$\mathbf{k} = (m_{k}, n_{k}) = (m, n) ,$$

$$-\mathbf{k} = (-m_{k}, n_{k}) ,$$

$$k = n$$

(A.8)

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Then, multiplying Equation (4) by $P_n^m(\mu)\exp(-im\lambda)$ and integrating over the (λ,μ) domain, we find that the spectral equations are as given in Equation (75). Here the interaction coefficients take the form:

$$A^{000}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{101}(\mathbf{k},\mathbf{p},\mathbf{q}) = -i[n_p(n_p+1)]^{-1} \cdot \cdot \int_{-1}^{1} d\mu P_{\mathbf{k}} \left(m_p P_{\mathbf{p}} \frac{dP_{\mathbf{q}}}{d\mu} - m_q P_{\mathbf{q}} \frac{dP_{\mathbf{p}}}{d\mu} \right)$$
(A.9a)

$$A^{011}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{110}(\mathbf{k},\mathbf{p},\mathbf{q}) = -i[n_p(n_p+1)+\Gamma]^{-1} \cdot \int_{-1}^{1} d\mu P_{\mathbf{k}} \left(m_p P_{\mathbf{p}} \frac{dP_{\mathbf{q}}}{d\mu} - m_q P_{\mathbf{q}} \frac{dP_{\mathbf{p}}}{d\mu} \right)$$
(A.9b)

$$A^{010}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = A^{001}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = A^{100}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = A^{111}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = 0$$
(A.9c)

Also $K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ is then determined by Equation (7c).

Appendix B: Perturbation Theory

Here, we derive expressions that relate the off-diagonal elements of the covariance and response function matrices to the mean-field and topography. These are needed in the formulation of the SE closure equations presented in Section 4. The closure equations are formulated by doing a formal perturbation theory. We suppose that the interaction coefficients on the right hand side of Equation (12) are multiplied by small parameter λ . The closure equations are then formally renormalized and λ restored back to unity.

We begin by expanding $\hat{\zeta}_k^a$ in Equation (12) in a perturbation series:

$$\hat{\zeta}_{k}^{a} = \hat{\zeta}_{k}^{(0)a} + \lambda \hat{\zeta}_{k}^{(1)a} + \dots$$
(B.1)

where $\hat{\zeta}_{k}^{(0)a}(t)$ has a multivariate Gaussian distribution. Then, to zero order, we have from Equation (12) that:

$$\frac{\partial \hat{\zeta}_{\mathbf{k}}^{(0)a}(t)}{\partial t} + \sum_{\mathbf{k}'} D_0^{a\beta}(\mathbf{k},\mathbf{k}') \hat{\zeta}_{\mathbf{k}'}^{(0)\beta}(t) = \hat{f}_0^a(\mathbf{k},t)$$
(B.2)

Here, we note that $D_0^{\alpha\beta}(\mathbf{k},\mathbf{k}') = D_0^{\alpha\beta}(\mathbf{k})$ when $\mathbf{k}' = \mathbf{k}$ and is zero otherwise. However, we keep the more general form since it makes the renormalization of the first order covariance and response functions more transparent.

To first order we have:

$$\frac{\partial \hat{\zeta}_{\mathbf{k}}^{(1)a}(t)}{\partial t} + \sum_{\mathbf{k}'} D_{0}^{a\beta}(\mathbf{k},\mathbf{k}')\hat{\zeta}_{\mathbf{k}'}^{(1)\beta}(t) = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \hat{\zeta}_{-\mathbf{p}}^{(0)b} h_{-\mathbf{q}}^{c} \\
+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) [\langle \zeta_{-\mathbf{p}}^{b} \rangle \hat{\zeta}_{-\mathbf{q}}^{(0)c} + \hat{\zeta}_{-\mathbf{p}}^{(0)b} \langle \zeta_{-\mathbf{q}}^{c} \rangle \\
+ \hat{\zeta}_{-\mathbf{p}}^{(0)b} \hat{\zeta}_{-\mathbf{q}}^{(0)c} - \langle \hat{\zeta}_{-\mathbf{p}}^{(0)b} \hat{\zeta}_{-\mathbf{q}}^{(0)c} \rangle]$$
(B.3)

Then the formal solution to (B.3) can be written, using the Greens function $R_{\mathbf{k},\mathbf{k}'}^{(0)a\beta}(t,s)$ corresponding to equation (B.2), as follows:

$$\hat{\zeta}_{\mathbf{k}}^{(1)a}(t) = \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{(0)a\alpha}(t,s) \left\{ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) h_{-\mathbf{q}}^{\gamma} \right. \\
\left. + \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) [\langle \zeta_{-\mathbf{p}}^{\beta}(s) \rangle \hat{\zeta}_{-\mathbf{q}}^{(0)\gamma}(s) + \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \langle \zeta_{-\mathbf{q}}^{\gamma}(s) \rangle \\
\left. + \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{-\mathbf{q}}^{(0)\gamma}(s) - \langle \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{-\mathbf{q}}^{(0)\gamma}(s) \rangle] \right\}$$
(B.4)

We can also express the two-time cumulant as:

$$C_{\mathbf{k},-\mathbf{l}}^{ab}(t,t') = \langle \hat{\zeta}_{\mathbf{k}}^{a}(t) \hat{\zeta}_{-\mathbf{l}}^{b}(t') \rangle$$

$$= \langle \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(0)b}(t') \rangle + \lambda \langle \hat{\zeta}_{\mathbf{k}}^{(1)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(0)b}(t') \rangle + \lambda \langle \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(1)b}(t') \rangle + \dots$$
(B.5)

To lowest order in perturbation theory:

$$C_{\mathbf{k},-\mathbf{l}}^{(0)ab}(t,s) = \langle \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(0)b}(t') \rangle$$
(B.6)

To first order in λ we have the general inhomogeneous contribution:

$$C_{\mathbf{k},-\mathbf{l}}^{(1)ab}(t,t') = \langle \hat{\zeta}_{\mathbf{k}}^{(1)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(0)b}(t') \rangle + \langle \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) \hat{\zeta}_{-\mathbf{l}}^{(1)b}(t') \rangle$$
(B.7)

Using equation (B.4) then gives:

$$C_{\mathbf{k},-\mathbf{l}}^{(1)ab}(t,t') = \int_{t_{o}}^{t} ds \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{(0)a\alpha}(t,s) \{ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) < \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{-1}^{(0)b}(t') > h_{-\mathbf{q}}^{\gamma} + \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^{\beta}(s) > \langle \hat{\zeta}_{-\mathbf{q}}^{(0)\gamma}(s) \hat{\zeta}_{-1}^{(0)b}(t') > + \langle \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{-1}^{(0)b\alpha}(t') > \langle \zeta_{-\mathbf{q}}^{\gamma}(s) >] \}$$

$$+ \int_{t_{o}}^{t'} ds \sum_{\mathbf{k}'} R_{-\mathbf{l},\mathbf{k}'}^{(0)b\alpha}(t',s) \{ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) < \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) > h_{-\mathbf{q}}^{\gamma} + \sum_{t_{o}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) [\langle \zeta_{-\mathbf{p}}^{\beta}(s) \rangle \langle \hat{\zeta}_{-\mathbf{q}}^{(0)\beta}(s) \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) > h_{-\mathbf{q}}^{\gamma} + \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) [\langle \zeta_{-\mathbf{p}}^{\beta}(s) \rangle \langle \hat{\zeta}_{-\mathbf{q}}^{(0)\beta}(s) \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) > + \langle \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s) \hat{\zeta}_{\mathbf{k}}^{(0)a}(t) \rangle \langle \zeta_{-\mathbf{q}}^{\gamma}(s) \rangle] \}$$
(B.8)

Next, we implement Equation (B.6) and perform the formal renormalizations $\lambda \to 1, R_{k,l}^{(0)ab} \to R_{k,l}^{ab}, C_{k,-l}^{(0)ab} \to C_{k,-l}^{ab}$. Finally, we have:

$$C_{\mathbf{k},-\mathbf{l}}^{(1)ab}(t,t') = \int_{t_o}^{t'} ds \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) \{ \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') [A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}')h_{-\mathbf{q}'}^{\gamma} + 2K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}') < \zeta_{-\mathbf{q}'}^{\gamma}(s) >]C_{-\mathbf{p}',-\mathbf{l}}^{\betab}(s,t') \}$$

$$+ \int_{t_o}^{t'} ds \sum_{\mathbf{k}'} R_{-\mathbf{l},\mathbf{k}'}^{b\alpha}(t',s) \{ \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') [A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}')h_{-\mathbf{q}'}^{\gamma} + 2K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}') < \zeta_{-\mathbf{q}'}^{\gamma}(s) >]C_{\mathbf{k},-\mathbf{p}'}^{a\beta}(t,s) \}$$

$$(B.9)$$

This completes the derivation of the expression for $C_{k,-l}^{(1)ab}$. We recall that $C_{k,-l}^{(0)ab} = \delta_{k,l}C_{k,-k}^{(0)ab}$ is in fact diagonal in spectral space and does not contribute to the Jacobian terms.

In a similar way, we treat the response function. We define:

$$R_{\mathbf{k},\mathbf{l}}^{ab}(t,t') = \left\langle \frac{\delta \hat{\zeta}_{\mathbf{k}}^{a}(t)}{\delta \hat{f}_{\mathbf{l}}^{b}(t')} \right\rangle$$
(B.10a)

and performing a perturbation expansion we have:

$$R_{\mathbf{k},\mathbf{l}}^{ab}(t,t') = \left\langle \frac{\delta \hat{\zeta}_{\mathbf{k}}^{(0)a}(t)}{\delta \hat{f}_{\mathbf{l}}^{b}(t')} \right\rangle + \lambda \left\langle \frac{\delta \hat{\zeta}_{\mathbf{k}}^{(1)a}(t)}{\delta \hat{f}_{\mathbf{l}}^{b}(t')} \right\rangle + \dots$$
(B.10b)

To lowest order in perturbation theory:

$$R_{\mathbf{k},\mathbf{l}}^{(0)ab}(t,t') = \left\langle \frac{\delta \hat{\zeta}_{\mathbf{k}}^{(0)a}(t)}{\delta \hat{f}_{\mathbf{l}}^{b}(t')} \right\rangle \tag{B.10c}$$

To first order in λ , the general inhomogeneous contribution is:

$$R_{\mathbf{k},\mathbf{l}}^{(1)ab}(t,t') = \int_{t'}^{t} ds \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{(0)a\alpha}(t,s) \cdot \left\{ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) \left\langle \frac{\delta \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s)}{\delta \hat{f}_{1}^{b}(t')} \right\rangle h_{-\mathbf{q}}^{\gamma} + \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k}',\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p},\mathbf{q}) [\langle \zeta_{-\mathbf{p}}^{\beta}(s) \rangle \left\langle \frac{\delta \hat{\zeta}_{-\mathbf{q}}^{(0)\gamma}(s)}{\delta \hat{f}_{1}^{b}(t')} \right\rangle + \left\langle \frac{\delta \hat{\zeta}_{-\mathbf{p}}^{(0)\beta}(s)}{\delta \hat{f}_{1}^{b}(t')} \right\rangle \langle \zeta_{-\mathbf{q}}^{\gamma}(s) \rangle \right] \right\}$$
(B.11)

We then perform the formal renormalizations $\lambda \to 1$, $R_{k,l}^{(0)ab} \to R_{k,l}^{ab}$. This procedure yields:

$$R_{\mathbf{k},\mathbf{l}}^{(1)ab}(t,t') = \int_{t'}^{t'} ds \sum_{\mathbf{k}'} R_{\mathbf{k},\mathbf{k}'}^{a\alpha}(t,s) \{ \sum_{\mathbf{p}'} \sum_{\mathbf{q}'} \delta(\mathbf{k}',\mathbf{p}',\mathbf{q}') [A^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}')h_{-\mathbf{q}'}^{\gamma} + 2K^{\alpha\beta\gamma}(\mathbf{k}',\mathbf{p}',\mathbf{q}') < \zeta_{-\mathbf{q}'}^{\gamma}(s) >] \} R_{-\mathbf{p}',\mathbf{l}}^{\betab}(s,t')$$
(B.12)

Again at zero order $R_{k,l}^{(0)ab} = \delta_{k,l} R_{k,k}^{(0)ab}$ is diagonal in spectral space.

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