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*Letter*

# Radiation Entropy Bound from the Second Law of Thermodynamics

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**Abstract:** It has been suggested heuristically by Unruh and Wald, and independently by Page, that at given energy and volume, thermal radiation has the largest entropy. The suggestion leads to the corresponding universal bound on entropy of physical systems. Using a gedanken experiment we show that the bound follows from the Second Law of Thermodynamics if the CPT symmetry is assumed and a general condition on matter holds. The experiment suggests that a wide class of Lorentz invariant local quantum field theories obeys a bound on the density of states.

**Keywords:** entropy; Second Law of Thermodynamics; CPT

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## 1. Introduction

The demand that total entropy of a closed system never decreases imposes important constraints on the structure of microscopic theories. A particularly interesting consequence of the demand is the existence of universal upper bounds on entropy of physical systems. Such bounds can be inferred by considering gedanken experiments that involve a reference thermodynamical system interacting universally with systems from the considered class. The first example of such a reference system was provided by black holes. Using that any physical system interacts with a black hole, Bekenstein proposed a universal entropy bound for the entropy of weakly gravitating systems:

$$S \leq \frac{2\pi E R}{\hbar c} \tag{1}$$

where S is the system entropy, E is the energy (that includes the rest mass for non-relativistic systems) and R is the characteristic linear size of the system (for macroscopic systems where energy and entropy are shape-independent one can assume with no loss a spherical shape so R is the radius) (see [1]). The bound is based upon the demand that the total entropy never decreases in gedanken experiments that involve a macroscopic system absorption by a black hole. The derivation employs an extended version of the second law (the so-called generalized second law or the GSL) that includes the contribution of the black holes in the total entropy. Since the entropy of black holes is determined uniquely by the hole's energy in the spherically symmetric case, then the entropy of the black hole after the absorption of the system is determined uniquely by the system's energy and the energy of the original black hole. The application of the GSL produces a non-trivial constraint on the possible entropy of the system expressed by Equation (1).

Despite some objections on the realizability of the considered gedanken experiment (see [2]), no physical counterexamples to the bound were found since its proposition. Moreover, the bound was rederived by an alternative approach (see [3]). Here we wish to address another bound on entropy, related to a universal agent different from a black hole, namely the radiation. In their discussion of the Bekenstein bound, Unruh and Wald [2,4,5] and Page [6], proposed heuristically that at given energy and system size, the entropy of the radiation is the largest so that:

$$S \leq \mathrm{Srad}(\mathrm{E},\mathrm{R}) \propto [2\,\pi\,\mathrm{E}\,\mathrm{R}/(\hbar\,\mathrm{c})]^{3/4} \tag{2}$$

with coefficient of proportionality of order one. Following [7] we will refer to the above bound as the Page-Unruh-Wald (PUW) bound. The PUW bound expresses the expectation that at given E and R the state with the largest entropy is a gas (*cf.* [8]), and it takes into account that thermal radiation has the maximal concentration of about one particle per volume with size of the Compton wavelength. The bound is stronger than the Bekenstein bound and often the difference is large: the system size can be much larger than its Compton wavelength, $R \gg R_{Comp} \equiv \hbar c/E$ (see [9] for the discussion of the relation between the two bounds). The bound in Equation (2) may fail in situations with strong gravity, such as black holes that saturate Equation (1). Still, the bound can be expected to hold quite generally and for its use it is important to understand what is needed for the bound to hold. Here we show that the second law of thermodynamics allows to shed light on the underlying assumptions.

## 2. Results and Discussion

We consider a weakly self-gravitating thermodynamic system with given energy E, volume V and entropy S. We assume that the CPT symmetry holds and the antimatter system obtained upon the CPT transformation of the original system is also physically admissible. Then, the antimatter partner is in the same thermodynamic state as the original system and it has the same energy E, volume V and entropy S. We now let the two systems, matter and antimatter one, interact. It can be assumed that before the moment of the interaction the two systems are enclosed in a somewhat larger box, so that there is a spacing both between both the systems and the box.

The volume of the box can be chosen close to 2V. Before the interaction the entropy inside the box is given by (the entropy of the box is not essential in the following considerations):

$$S_{in} = 2S \tag{3}$$

After the interaction the system and the "antisystem" annihilate giving rise to a certain product of the reaction. Designating the entropy of the product by $S_{an}(2E, 2V)$, we find that the second law's demand that the entropy of the whole system does not decrease as a result of the reaction produces the inequality:

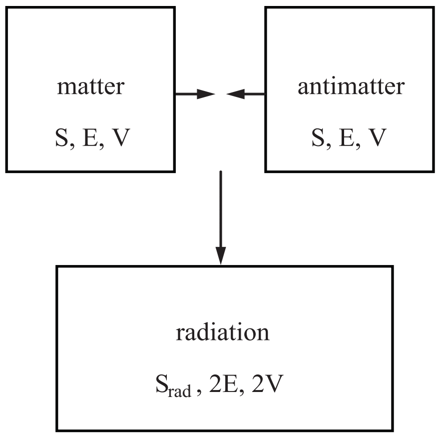
$$2S \leq S_{an}(2E, 2V) \quad (4)$$

In a wide class of situations the product of the annihilation reaction is the thermal radiation and we will have:

$$S_{an}(2E, 2V) = S_{rad}(2E, 2V) = 2S_{rad}(E, V) \quad (5)$$

where we used that $S_{rad}(E, V)$ is proportional to $E^{3/4}V^{1/4}$, *cf.* Equation (2). The scheme of the experiment in this case is shown in Figure 1.

**Figure 1.** The demand that the entropy does not decrease in the annihilation process gives the radiation bound: initial entropy 2S of the matter-antimatter couple must be not greater than the final entropy of radiation at energy 2E and volume 2V.




Then the Second Law condition described by Equation (4) produces the sought for Page–Unruh–Wald bound:

$$S \leq S_{rad}(E, V) = g[E/(\hbar c)]^{3/4} V^{1/4} \quad (6)$$

where the effective number of degrees of freedom (see [6,8]) and numerical constants are absorbed in the g-factor which is of the order of a few. It should be noted that we use the term "thermal radiation" in the spirit of the original works (2) and (4), where "radiation" includes matter with the rest energy smaller or comparable with the temperature (see also [8]). In particular, the g-factor in Equation (6) is energy density dependent near temperatures corresponding to a pair creation threshold.

We see that once the idea of the gedanken experiment is conceived, the demonstration of the bound is rather immediate. A possible objection to the consideration could concern the existence of an enclosure able to retain the products of the annihilation reaction. However, the assumption of the existence of such an enclosure can be avoided by noticing that we deal here with the volume (where the reaction occurs) *versus* the area (where the boundary is) problem. By making, if necessary, a large number of copies of the system-antisystem pairs and letting them interact in their common volume one can see that for the Second Law of Thermodynamics to hold, the entropy should not decrease in the volume (the volume entropy grows faster with the number of copies than the contribution to the entropy from the boundary). It is the latter demand that is expressed by Equations (5) and (6).

Let us note that the bound in the form (6) may fail for containers where one dimension of size d is much smaller than the other two of size D. If d is much smaller than the thermal wavelength λ of the photons, while $D \gg \lambda$, then we deal with effectively two-dimensional gas of photons where $S \propto (DE)^{2/3}$.The latter value exceeds $(D^2 d)^{1/4} E^{3/4}$ following from Equation (6) if d is sufficiently small. This and similar counterexamples to the bound can be avoided be rewriting the latter in its more fundamental form:

$$S \leq S'_{rad}(E, V) \tag{7}$$

where $S'_{rad}(E, V)$ is defined as the entropy of radiation that takes into account the given shape of the container. In the form (7) the bound holds also for special shapes of container such as described above.

In the cases where the effects of gravity become important, the bound's modifications are needed. In particular, the PUW bound has a different form for small system volumes, $V \lesssim R_g^4 R^{-1}_{Comp}$, where $R_g \equiv G E/c^4$ is the gravitational radius of the system [6]. Here the equilibrium state of thermal radiation includes a black hole that exchanges radiation with the system via the Hawking radiation process [9]. As a result, at these V one should use for $S_{rad}(E, V)$ in Equation (6) an expression that involves a sum of the entropies of the black hole and the remaining radiation [6].

As with any entropy bound there is a question of how precisely the bound works and how the situations where it could fail are avoided. An example of a challenge to entropy bounds is the zeromode argument proposed in [10]. It is argued that a zero mode possibly present in a field can accommodate arbitrarily large entropy at zero energy cost that would lead to a violation of an entropy bound. This argument against a bound was refuted in [11] where it was shown that in fact a zero mode contributes vanishing entropy to the system. A general discussion of how entropy bounds work can be found in [12].

## 3. Conclusions

The proposed gedanken experiment is informative about any system whatever the details of the product of annihilation reaction are. However, it seems most interesting where one can say that the product is radiation thus obtaining the PUW bound for the considered system. Then the physical reason for the bound is the expected instability of the matter-antimatter couple and the irreversibility of the resulting annihilation reaction. While radiation is expected to result from annihilation rather generally, there are situations where the reaction results are different such as the case of hot systems where pair creation by radiation occurs. In this case and possible other cases the bound in the form described by Equation (7) still works and one has to calculate the entropy of the annihilation product.

Thus the main drawback of the proposed consideration is the absence of a condition to decide the product of the annihilation reaction. It is clear however that the product is radiation for most known real systems. Thus the present work demonstrates the bound for a wide class of systems occurring in nature. Moreover, it shows for the first time what may go wrong if the bound does not hold. By establishing that the assumption that the annihilation produces radiation implies the bound, we get a new point of view on the bound that might eventually lead to the significant progress in its understanding. Below we indicate two directions for future studies implied by the present work.

Our consideration shows that the bound can be expected to hold rather generally in theories that incorporate antimatter. Then the CPT theorem (see e.g., [13]) implies that a local Lorentz invariant quantum field theory (below LLIQFT) obeys the bound (7). Furthermore, in a general class of situations where the product of the annihilation reaction is given by the radiation, a LLIQFT must also obey the PUW bound. Using the Boltzmann formula for the entropy S(E, V) = \ln g(E, V) Δ E, where g(E, V) is the density of states and Δ E is the energy window, we conclude that density of states of a wide class of LLIQFT should grow with E not faster than $\exp[S_{rad}(E, V)]$ (the factor in front of the exponent gives a subleading dependence on E). This sheds new light on the calculations of [14,15], aimed to demonstrate the validity of the holographic entropy bound for LLIQFT (note that the calculations of [14] were criticized in [16] for containing a possible mathematical flaw). The holographic bound $S \leq \pi c^3 R^2/\hbar G$ was introduced by 't Hooft and Susskind [8,17], and it can be obtained from Equation (6) by using $R_g < R$. The above leads to the conjecture that under the assumptions of LLIQFT the PUW bound must hold. The study of this conjecture is a subject for further work.

Another direction that the present work suggests is the generalization of the PUW bound to include gravity. As mentioned in the introduction, the PUW bound does not hold in the situations with strong gravity where one should rather use the Bekenstein bound in Equation (1). Is it possible to derive a more general bound that would reproduce the two bounds in the limits of weak and strong gravity? Since the suggested gedanken experiment can be considered in situations where the self-gravity is not negligible, it is not unlikely that such a generalized bound exists. The study of this possibility is an important subject for further studies.

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