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Article

Statistical Dynamical Closures and Subgrid Modeling for Inhomogeneous QG and 3D Turbulence

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Abstract: Statistical dynamical closures for inhomogeneous turbulence described by multi-field equations are derived based on renormalized perturbation theory. Generalizations of the computationally tractable quasi-diagonal direct interaction approximation for inhomogeneous barotropic turbulent flows over topography are developed. Statistical closures are also formulated for large eddy simulations including subgrid models that ensure the same large scale statistical behavior as higher resolution closures. The focus is on baroclinic quasigeostrophic and three-dimensional inhomogeneous turbulence although the framework is generally applicable to classical field theories with quadratic nonlinearity.

Keywords: inhomogeneous closures; turbulence; subgrid modeling; statistical dynamics

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1. Introduction

In recent years there have been significant advances in the formulation and applications of subgrid-scale parameterizations to quasigeostrophic (QG) and three-dimensional (3D) turbulence as reviewed by Frederiksen and O'Kane [1]. Statistical dynamical closure theory or stochastic modeling methods form the basis of much of this work. Results from the two methods may be very similar as discussed by Frederiksen and Kepert [2]. Perhaps this is to be expected since closures such as Kraichnan's [3] direct interaction approximation (DIA) for homogeneous turbulence and Frederiksen's [4] quasi-diagonal direct interaction approximation (QDIA) for inhomogeneous turbulence have underpinning exact generalized Langevin representations that guarantee realizability and positive definite energy spectra.

Kraichnan's DIA closure represented a major advance in the statistical theory of homogeneous turbulence. Herring [5] and McComb [6,7] independently developed the self consistent field theory (SCFT) and local energy transfer theory (LET) respectively. These three non-Markovian closures for homogeneous turbulence have subsequently been shown to form a class of renormalized closures that differ only in whether and how a fluctuation dissipation theorem (FDT) is applied [8,9]. Kraichnan [10] also developed a DIA closure for inhomogeneous turbulence but recognized that, because this general theory required the computation of the full covariance and response function matrices as well as the full three-point functions, it was computationally intractable at any reasonable resolution.

Frederiksen [4] formulated a computationally tractable non-Markovian closure theory, the quasi-diagonal direct interaction approximation (QDIA), for inhomogeneous turbulent flows and applied it to the subgrid modeling problem for barotropic flows over topography on an *f*-plane. Generalizations of the theory, to inhomogeneous turbulent flows on a β-plane with applications to Rossby wave dispersion and predictability, were made by Frederiksen and O'Kane [11]. The statistical closure has been implemented numerically and extensively tested and applied to problems in dynamics, predictability, data assimilation and subgrid modeling. O'Kane and Frederiksen [12] compared the performance of the *f*-plane QDIA closure with the statistics of large ensembles of direct numerical simulations in a suite of studies for inhomogeneous turbulent flows over topography. They found that their numerically implemented QDIA closure has similar performance to the DIA for homogeneous turbulence and is only a few times more computationally intensive. As in earlier homogeneous DIA, SCFT and LET closure calculations [8,13,14] they employed a cumulant update restart procedure [15] to enhance the performance of the QDIA. They also explored the efficacy of a regularization procedure, similar to that employed by Frederiksen and Davies [14] for homogeneous turbulence, which corresponds to an empirical vertex renormalization and ensures that the QDIA has the right power law behavior. The regularized QDIA was found to be in excellent agreement with the statistics of direct numerical simulation (DNS), both at the large scales and in the enstrophy cascading inertial range. Further, the homogeneous and inhomogeneous closure studies indicate that the regularization parameter that localizes interactions is essentially universal.

The β-plane QDIA closure [11] was found to be as successful as the *f*-plane QDIA in applications in a series of studies. Frederiksen and O'Kane [11] studied Rossby wave dispersion due to the interaction of eastward zonal flows with isolated topography in a turbulent environment; they found pattern correlations as high as 0.9999 between the closure and the statistics of 1800 DNSs for the mean Rossby wave trains in 10 day simulations. They also applied the β-plane QDIA closure to the issue of predictability and this was further pursued in a detailed examination of ensemble prediction during blocking regime transitions by O'Kane and Frederiksen [16]. O'Kane and Frederiksen [17] applied the β-plane QDIA closure to data assimilation while O'Kane and Frederiksen [18] and Frederiksen and O'Kane [1] examined the numerical evaluation of the QDIA subgrid model of Frederiksen [4] for equilibrium and non-equilibrium turbulent flows on *f*- and β-planes.

Given the success of the QDIA closure, for inhomogeneous turbulent barotropic flows, it may be fruitful to generalize the theory to more complex multi-field equations such as those for inhomogeneous QG and 3D turbulent flows. This is a primary aim of this paper. We consider generic prognostic equations that take the form:

$$\frac{\partial}{\partial t}\zeta_{\mathbf{k}}^{a}(t)+D_{0}^{a\beta}(\mathbf{k})\zeta_{\mathbf{k}}^{\beta}(t)=\sum_{\mathbf{p}}\sum_{\mathbf{q}}\left[\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)\zeta_{-\mathbf{q}}^{c}(t)+\mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}\right]+f_{0}^{a}(\mathbf{k},t). \tag{1}$$

Here $\zeta_{\mathbf{k}}^{a}(t)$ is a field variable depending on time t, the level or field type a and the vector $\mathbf{k}$; typically $\mathbf{k}$ is a vector of wavenumbers in spectral space. The equations are quadratic in the fields and also contain forcing $f_{0}^{a}(\mathbf{k},t)$, a linear term with coefficients $D_{0}^{a\beta}(\mathbf{k})$ and a term bilinear in the dynamical fields $\zeta_{\mathbf{k}}^{a}(t)$ and in constant fields $h_{\mathbf{k}}^{a}$. Here, $h_{\mathbf{k}}^{a}$ typically specifies the topography and $\mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ and $\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ are interaction coefficients. Throughout this paper we assume summation over repeated superscripts. We suppose that the interaction coefficients $\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ satisfy the relationship:

$$\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})=\mathcal{K}^{acb}(\mathbf{k},\mathbf{q},\mathbf{p}). \tag{2}$$

For particular systems considered in this paper Equation (2) is satisfied because:

$$\mathcal{K}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})=\frac{1}{2}[\mathcal{A}^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})+\mathcal{A}^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})]. \tag{3}$$

A further aim of this paper is to formulate the equations for subgrid-scale parameterizations of unresolved interactions needed for large eddy simulations (LES) of inhomogeneous turbulent flows when the resolution is reduced. We build on the extensive theoretical framework and insights of many researchers. In Kraichnan's [3] DIA the tendency of the second order cumulant, or two-point function, is driven by the three-point function which consists of a nonlinear damping term and a nonlinear noise term. Thus from the basic structure of the DIA closure one would expect the subgrid modeling of eddy-eddy interactions would consist of an eddy damping term and a stochastic backscatter term. Kraichnan [19] also developed the theory of eddy viscosity and noted a cusp in the eddy viscosity near the cut-off wavenumber. Rose [20] was one of the first to point out the importance of eddy noise in subgrid modeling while Leith [21] examined the effects of an empirical stochastic backscatter parameterization in conjunction with the Smagorinsky [22] empirical eddy viscosity model, in studies of a turbulent shear mixing layer. Subsequent subgrid modeling studies for three-dimensional homogeneous turbulence were carried out by Leslie and Quarini [23], Chollet and Lesieur [24], Chasnov [25], Domaradzki *et al.* [26], McComb *et al.* [27–29], Schilling and Zhou [30].

For the case of two-dimensional turbulent flows on the sphere, Frederiksen and Davies [31] developed self consistent representations of eddy viscosity and stochastic backscatter based on eddy damped quasi-normal Markovian (EDQNM) and DIA closures. Their subgrid model cured the resolution dependence of atmospheric energy spectra with LES; the spectra were in close agreement with higher resolution barotropic DNS at each resolution. Frederiksen [4] derived general expressions for the eddy-topographic force, eddy viscosity and stochastic backscatter based on the QDIA closure. O'Kane and Frederiksen [18] and Frederiksen and O'Kane [1] calculated and analysed inhomogeneous subgrid-scale parameterizations for observed atmospheric flows over global topography and compared the strengths of the subgrid-scale eddy-topographic, eddy-mean field, quadratic mean and mean field-topographic terms.

The QDIA closure also motivated the direct stochastic modeling approach to subgrid processes, based on the statistics of DNS, employed by Frederiksen and Kepert [2]. They found that their method gave very similar subgrid-scale parameterizations and barotropic model LES energy spectra to the closure based approach. Zidikheri and Frederiksen [32] also successfully applied this stochastic modeling methodology to two-level QG model studies with typical atmospheric flows. Baroclinic oceanic flows were considered by Zidikheri and Frederiksen [33,34]; they applied the stochastic modeling approach and showed that it can be successfully used to maintain the correct LES spectra for both simple flows and more complex flows characteristic of the Antarctic circumpolar current. In very recent work, Kitsios, Frederiksen and Zidikheri [35] derived universal scaling laws for subgrid models of eddy-eddy interactions applicable to baroclinic atmospheric flows.

In the last decade there has been increasing interest in exploring how parameterizations of stochastic backscatter may improve simulations and predictions of weather and climate [36,37]. Shutts [38], O'Kane and Frederiksen [16] and Berner *et al.* [39] have considered the role of stochastic backscatter in weather prediction studies where it increases ensemble spread. Berner *et al.* [40] also found that stochastic backscatter resulted in reductions in systematic errors and improvements in seasonal forecasts. Seiffert and von Storch [41] found that model climate sensitivity to increased carbon dioxide concentrations depends on whether a stochastic backscatter parameterization is employed.

More realistic subgrid-scale parameterizations are increasingly being applied and this partly forms the motivation for the current study where we develop statistical closures and subgrid models for multi-field equations. The general formulism presented covers a wide variety of equations but we give concrete examples in Section 2, where the QG equations are outlined, and in Section 3, where the equations for 3D turbulence are summarized. Further examples will be discussed in the text and the appendices. In Section 4, the QDIA closure equations are derived for general fields satisfying Equations (1) to (3). The generalized Langevin equation that underpins the closure, and guarantees realizability of the elements of the covariance matrices that are diagonal in spectral space, is presented in Section 5.

The statistical dynamical QDIA closure for large eddy simulations, including a derived subgrid model that ensures the same large scale statistical behavior as higher resolution QDIA closures, is formulated in Section 6. Here the stochastic and mean subgrid forcing function are derived as well as two-time dissipation elements. In Subsection 6.3 the generalized Langevin equation that underpins the QDIA closure with subgrid-scale parameterizations is presented. The effective dissipation and viscosity parameterizations for subgrid processes entering the mean field equation and the single time covariance equation are presented in Section 7. In Section 8 we discuss how the QDIA closure and subgrid models would change if a fluctuation dissipation theorem is employed to form quasi-diagonal SCFT and quasi-diagonal LET closures and subgrid models. There the issues of regularization and non-Gaussian initial conditions are also discussed and concluding remarks are presented.

Appendix A contains a derivation of expressions relating the general inhomogeneous elements of the covariance and response function matrices to those that are diagonal in spectral space. The derivation is based on renormalized perturbation theory. In appendix B the equations for a QG model with continuous vertical variations are presented and in Appendix C the equations for the two-level QG equations expressed in terms of barotropic and baroclinic components are summarized.

2. Flow over Topography in a Baroclinic Quasigeostrophic Model

Taking suitable length and time scales, the nondimensional equation for 2-level baroclinic quasigeostrophic flow over topography on an *f*-plane may be written in the form:

$$\frac{\partial q^a}{\partial t} = -J(\psi^a, q^a + h^a) - D_0^{ab} q^b + f_0^a, \tag{4}$$

Here, $a = 1$ or 2, ψ^a is the streamfunction and $q^a = \nabla^2 \psi^a + (-1)^a F_L(\psi^1 - \psi^2)$ is the reduced potential vorticity, $\omega^a = \nabla^2 \psi^a$ is the relative vorticity, $h^2 = h$, $h^1 = 0$ where h is the scaled topography, D_0^{ab} are dissipation operators to be specified below and f_0^a are forcing functions. Also, F_L is the layer coupling parameter [32], which is inversely proportional to the static stability. In planar geometry:

$$J(\psi, \zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}. \tag{5}$$

The analysis in this paper can be generalized to flows in a channel, in a bounded domain and in an infinite domain; it can also be generalized to flow on a β-plane. However, to be specific and to simplify the presentation we shall consider flow on the periodic *f*-plane $0 \le x \le 2\pi, 0 \le y \le 2\pi$. In Appendix C we discuss how the results may be generalized to flow on a periodic β-plane; they may also be applied to flows on the infinite domain as in the internal gravity wave closure theory of Carnevale and Frederiksen [42].

Spectral equations corresponding to the baroclinic system may be obtained by first expanding each of the functions in Equation (4) in a Fourier series:

$$\zeta^a(\mathbf{x}, t) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}}^a(t) \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{6a}$$

where:

$$\zeta_{\mathbf{k}}^a(t) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d^2\mathbf{x}\, \zeta^a(\mathbf{x}, t) \exp(-i\mathbf{k} \cdot \mathbf{x}) \tag{6b}$$

and $\mathbf{x} = (x, y), \mathbf{k} = (k_x, k_y)$. Then, multiplying Equation (4) by $\exp(-i\mathbf{k} \cdot \mathbf{x})$ and integrating over the (x, y) domain, we find that with the identification $q_{\mathbf{k}}^a(t) \Rightarrow \zeta_{\mathbf{k}}^a(t)$,

$$\begin{aligned} \frac{\partial}{\partial t}\zeta_{\mathbf{k}}^a(t) + D_0^{\alpha\beta}(\mathbf{k})\zeta_{\mathbf{k}}^\beta(t) = &\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})\left[A(\mathbf{k},\mathbf{p},\mathbf{q})\omega_{-\mathbf{p}}^a(t)\zeta_{-\mathbf{q}}^a(t) + A(\mathbf{k},\mathbf{p},\mathbf{q})\omega_{-\mathbf{p}}^a(t)h_{-\mathbf{q}}^a\right] \\ &+ f_0^a(\mathbf{k}, t). \end{aligned} \tag{7a}$$

Alternatively, we can write the spectral equations in the standard form:

$$\begin{aligned} \frac{\partial}{\partial t}\zeta_{\mathbf{k}}^a(t) + D_0^{\alpha\beta}(\mathbf{k})\zeta_{\mathbf{k}}^\beta(t) = &\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})\left[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^b(t)\zeta_{-\mathbf{q}}^c(t) + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^b(t)h_{-\mathbf{q}}^c\right] \\ &+ f_0^a(\mathbf{k}, t). \end{aligned} \tag{7b}$$

Here, we suppose that the dissipation may be related to the viscosity through:

$$D_0^{\alpha\beta}(\mathbf{k}) = \nu_0^{\alpha\beta}(\mathbf{k})k^2 \tag{8}$$

where we refer to $\nu_0^{\alpha\beta}(\mathbf{k})$ as the bare viscosity, although more general forms can be considered including wave frequencies; we shall also refer to $f_0^a(\mathbf{k},t)$ as the bare forcing.

In the above spectral equations:

$$k = |\mathbf{k}|, \tag{9a}$$

$$\zeta_{-\mathbf{k}}^a = \zeta_{\mathbf{k}}^{a*}, \tag{9b}$$

$$A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \begin{cases} A(\mathbf{k},\mathbf{p},\mathbf{q})V^{ab}(p) \text{ if } c = a \\ 0 \text{ otherwise,} \end{cases} \tag{9c}$$

$$V^{11}(p) = V^{22}(p) = \frac{p^2 + F_L}{p^2 + 2F_L}; \quad V^{12}(p) = V^{21}(p) = \frac{F_L}{p^2 + 2F_L}, \tag{9d}$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)/p^2, \tag{9e}$$

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\left[A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) + A^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})\right], \tag{9f}$$

$$\delta(\mathbf{k},\mathbf{p},\mathbf{q}) = \begin{cases} 1 \text{ if } \mathbf{k}+\mathbf{p}+\mathbf{q} = 0 \\ 0 \text{ otherwise.} \end{cases} \tag{9g}$$

Note that here we have represented the triangle sum rule of wave vectors by $\delta(\mathbf{k},\mathbf{p},\mathbf{q})$, for future convenience, rather than the $\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})$ used in Frederiksen [4] and related papers; the current notation allows for more general relationships between the wave vectors such as those in spherical geometry. We note that the interaction coefficients $K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})$ satisfy the relationship:

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = K^{acb}(\mathbf{k},\mathbf{q},\mathbf{p}). \tag{10}$$

Generalization to multi-level QG equations is straightforward.

3. Navier Stokes Equations for Three-Dimensional Inhomogeneous Turbulence

The Navies Stokes equations for three-dimensional inhomogeneous turbulence in a periodic box, $0 \le x \le 2\pi, 0 \le y \le 2\pi, 0 \le z \le 2\pi$, may be written in the form [7]:

$$\frac{\partial}{\partial t}u_{\mathbf{k}}^a(t) + D_0^{\alpha\beta}(\mathbf{k})u_{\mathbf{k}}^\beta(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})M^{abc}(\mathbf{k})u_{-\mathbf{p}}^b(t)u_{-\mathbf{q}}^c(t) + f_0^a(\mathbf{k},t). \tag{11}$$

Similar equations apply on the infinite domain with the sums replaced by integrals [42]. Here, $u_{\mathbf{k}}^a(t), a = 1,2,3$ are the spectral components of the velocity fields in the x, y, z directions respectively that depend on time t and the vector $\mathbf{k} = (k^1, k^2, k^3) = (k_x, k_y, k_z)$. The linear term involving $D_0^{\alpha\beta}(\mathbf{k})$ generally represents dissipation and $f_0^a(\mathbf{k},t)$ is a forcing term. Also, the interaction coefficient:

$$M^{abc}(\mathbf{k}) = i/2\{\Delta^{ab}(\mathbf{k})k^c + \Delta^{ac}(\mathbf{k})k^b\}, \tag{12a}$$

$$\Delta^{ab}(\mathbf{k}) = \delta^{ab} - k^a k^b / |\mathbf{k}|^2. \tag{12b}$$

We next change the notation to put Equations (11) and (12) into the standard form (1) and (2). We make the identifications:

$$u^a_{\mathbf{k}}(t) \Rightarrow \zeta^a_{\mathbf{k}}(t), \tag{13a}$$

$$A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = i\Delta^{ab}(\mathbf{k})k^c, \tag{13b}$$

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\left[A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) + A^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})\right] \tag{13c}$$

Thus Equation (11) is in the form (1) with $h^a_{\mathbf{k}} = 0$ and the following analysis applies equally to the Navier Stokes equations where $\mathbf{k}$ is a three-dimensional vector as it does to the baroclinic quasigeostrophic equations where $\mathbf{k}$ is a two-dimensional wave vector.

4. Quasi-Diagonal DIA Closure Equations

Our purpose here is to generalize the approach of Frederiksen [4] to formulate a tractable closure theory for prognostic equations of the form (1) including for baroclinic QG and 3D inhomogeneous turbulent flows. Further examples will be discussed in the conclusions.

We consider an ensemble of flows satisfying Equation (1) where the ensemble mean is denoted by $<\zeta^a_{\mathbf{k}}>$ and angle brackets denote expectation value. We express the field component for a given realization by:

$$\zeta^a_{\mathbf{k}} = <\zeta^a_{\mathbf{k}}> + \hat{\zeta}^a_{\mathbf{k}} \tag{14}$$

where $\hat{\zeta}^a_{\mathbf{k}}$ denotes the deviation from the ensemble mean. The spectral Equation (1) can then be expressed in terms of $<\zeta^a_{\mathbf{k}}>$ and $\hat{\zeta}^a_{\mathbf{k}}$ as follows:

$$\begin{aligned} &\frac{\partial}{\partial t} <\zeta^a_{\mathbf{k}}(t)> + D_0^{\alpha\beta}(\mathbf{k}) <\zeta^\beta_{\mathbf{k}}(t)> = \\ &\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\left\{<\zeta^b_{-\mathbf{p}}(t)><\zeta^c_{-\mathbf{q}}(t)> + C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t)\right\} \\ &+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) <\zeta^b_{-\mathbf{p}}(t)> h^c_{-\mathbf{q}}] + \bar{f}^a_0(\mathbf{k},t), \end{aligned} \tag{15a}$$

$$\begin{aligned} &\frac{\partial}{\partial t} \hat{\zeta}^a_{\mathbf{k}}(t) + D_0^{\alpha\beta}(\mathbf{k})\hat{\zeta}^\beta_{\mathbf{k}}(t) = \\ &\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{<\zeta^b_{-\mathbf{p}}(t)>\hat{\zeta}^c_{-\mathbf{q}}(t) + \hat{\zeta}^b_{-\mathbf{p}}(t)<\zeta^c_{-\mathbf{q}}(t)> \\ &+ \hat{\zeta}^b_{-\mathbf{p}}(t)\hat{\zeta}^c_{-\mathbf{q}}(t) - C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,t)\} + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\zeta}^b_{-\mathbf{p}}(t)h^c_{-\mathbf{q}}] + \hat{f}^a_0(\mathbf{k},t). \end{aligned} \tag{15b}$$

Here:

$$f^a_0(\mathbf{k}) = \bar{f}^a_0(\mathbf{k}) + \hat{f}^a_0(\mathbf{k}), \tag{16a}$$

$$\bar{f}^a_0(\mathbf{k}) = < f^a_0(\mathbf{k})>, \tag{16b}$$

$$C^{bc}_{-\mathbf{p},-\mathbf{q}}(t,s) = <\hat{\zeta}^b_{-\mathbf{p}}(t)\hat{\zeta}^c_{-\mathbf{q}}(s)>, \tag{16c}$$

are the mean and fluctuating forcing functions and two-time covariance matrix elements.

We assume that the initial $\hat{\zeta}_{\mathbf{k}}^{a}(t_o)$ have a Gaussian distribution for which the initial covariance matrix is diagonal in wavenumber ($\mathbf{k}$) space but fully inhomogeneous in the index (a) (Equation (A.6a)). We also suppose that (prior to renormalization) the general inhomogeneous elements of the two-time covariance and response function matrices are small compared with the elements that are diagonal in wavenumber space (Equations (A.6b), (A.7), (A.10) and (A.11)). The methods of deriving the expressions for the general inhomogeneous elements in terms of the elements that are diagonal in wavenumber space for the QDIA closure equations are described in Appendix A.

In order to close Equation (15a) we need an expression for the two-point cumulant $C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)$. We proceed by first expressing $C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)$ in terms of the corresponding elements that are diagonal in spectral space, through Equation (A.9), and then derive equations for the elements of the cumulants and response functions that are diagonal in spectral space. Thus, using Equation (A.9) in Equation (4.2a) yields:

$$\begin{aligned}
&\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\\
&=2\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\int_{t_o}^{t}ds R_{-\mathbf{p}}^{b\beta}(t,s)C_{-\mathbf{q}}^{c\gamma}(t,s).\\
&\cdot\left[A^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k})h_{\mathbf{k}}^{\alpha}+2K^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k})<\zeta_{\mathbf{k}}^{\alpha}(s)>\right]\\
&=-\int_{t_o}^{t}ds\eta_{\mathbf{k}}^{a\beta}(t,s)<\zeta_{\mathbf{k}}^{\beta}(s)>+f_H^{a}(\mathbf{k},t)
\end{aligned}\tag{17}$$

where the nonlinear eddy-eddy damping, eddy-topographic force and eddy-topographic interaction are given by:

$$\eta_{\mathbf{k}}^{a\alpha}(t,s)=-4\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k})R_{-\mathbf{p}}^{b\beta}(t,s)C_{-\mathbf{q}}^{c\gamma}(t,s),\tag{18a}$$

$$f_H^{a}(\mathbf{k},t)=h_{\mathbf{k}}^{\alpha}\int_{t_o}^{t}ds\chi_{\mathbf{k}}^{a\alpha}(t,s),\tag{18b}$$

$$\chi_{\mathbf{k}}^{a\alpha}(t,s)=2\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})A^{\beta\gamma\alpha}(-\mathbf{p},-\mathbf{q},-\mathbf{k})R_{-\mathbf{p}}^{b\beta}(t,s)C_{-\mathbf{q}}^{c\gamma}(t,s).\tag{18c}$$

In Equations (14) and (18) we have used the shortened notation:

$$R_{\mathbf{k}}^{ab}(t,t')\equiv R_{\mathbf{k},\mathbf{k}}^{ab}(t,t'),\tag{19a}$$

$$C_{\mathbf{k}}^{ab}(t,t')\equiv C_{\mathbf{k},-\mathbf{k}}^{ab}(t,t').\tag{19b}$$

Substituting Equation (17) into Equation (15a) yields:

$$\begin{aligned}
&\frac{\partial}{\partial t}<\zeta_{\mathbf{k}}^{a}(t)>+D_0^{a\beta}(\mathbf{k})<\zeta_{\mathbf{k}}^{\beta}(t)>=\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})<\zeta_{-\mathbf{p}}^{b}(t)><\zeta_{-\mathbf{q}}^{c}(t)>\\
&+\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})<\zeta_{-\mathbf{p}}^{b}(t)>h_{-\mathbf{q}}^{c}\\
&-\int_{t_o}^{t}ds\eta_{\mathbf{k}}^{a\beta}(t,s)<\zeta_{\mathbf{k}}^{\beta}(s)>+f_0^{a}(\mathbf{k},t)+f_H^{a}(\mathbf{k},t).
\end{aligned}\tag{20}$$

As noted above this equation is for initial conditions that are homogeneous in the horizontal. Inhomogeneous initial conditions can also be treated following the method of Frederiksen and Davies [14] and O'Kane and Frederiksen [12].

From Equation (15b), we can obtain an equation for the two-time cumulant, needed in Equations (18) to (20), by multiplying by $\hat{\zeta}^{\alpha}_{-\mathbf{k}}(t')$:

$$\begin{aligned}\frac{\partial}{\partial t}C^{a\alpha}_{\mathbf{k}}(t,t') + D^{a\beta}_{0}(\mathbf{k})C^{\beta\alpha}_{\mathbf{k}}(t,t') &= \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})[<\zeta^{b}_{-\mathbf{p}}(t)> C^{c\alpha}_{-\mathbf{q},-\mathbf{k}}(t,t') \\ &+ C^{b\alpha}_{-\mathbf{p},-\mathbf{k}}(t,t') <\zeta^{c}_{-\mathbf{q}}(t)> + <\hat{\zeta}^{b}_{-\mathbf{p}}(t)\hat{\zeta}^{c}_{-\mathbf{q}}(t)\hat{\zeta}^{\alpha}_{-\mathbf{k}}(t')>] \\ &+ \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})C^{b\alpha}_{-\mathbf{p},-\mathbf{k}}(t,t')h^{c}_{-\mathbf{q}} + \int_{t_o}^{t'} ds F^{a\beta}_{0}(\mathbf{k},t,s)R^{\alpha\beta}_{-\mathbf{k}}(t',s).\end{aligned} \tag{21}$$

Again, Equation (A.9) can be used to express the general inhomogeneous elements of the two-time cumulant in terms of the elements that are diagonal in wavenumber space. Also, Equation (A.13) gives the expression for the three-point cumulant. Using both of these expressions we find that:

$$\begin{aligned}\frac{\partial}{\partial t}C^{ab}_{\mathbf{k}}(t,t') + D^{a\beta}_{0}(\mathbf{k})C^{\beta b}_{\mathbf{k}}(t,t') &= \int_{t_o}^{t'} ds\{S^{a\beta}_{\mathbf{k}}(t,s) + P^{a\beta}_{\mathbf{k}}(t,s) + F^{a\beta}_{0}(\mathbf{k},t,s)\}R^{b\beta}_{-\mathbf{k}}(t',s) \\ &- \int_{t_o}^{t} ds\{\eta^{a\beta}_{\mathbf{k}}(t,s) + \pi^{a\beta}_{\mathbf{k}}(t,s)\}C^{b\beta}_{-\mathbf{k}}(t',s).\end{aligned} \tag{22}$$

This equation is valid for Gaussian initial conditions and can be generalized to non-Gaussian and inhomogeneous initial conditions following the approach of O'Kane and Frederiksen [12].

In Equation (4.9):

$$F^{a\alpha}_{0}(\mathbf{k},t,s) = <\hat{f}^{a}_{0}(\mathbf{k},t)\hat{f}^{\alpha}_{0}(-\mathbf{k},s)>, \tag{23a}$$

$$S^{a\alpha}_{\mathbf{k}}(t,s) = 2\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})K^{\alpha\beta\gamma}(-\mathbf{k},-\mathbf{p},-\mathbf{q})C^{b\beta}_{-\mathbf{p}}(t,s)C^{c\gamma}_{-\mathbf{q}}(t,s) \tag{23b}$$

$$\begin{aligned}P^{a\alpha}_{\mathbf{k}}(t,s) &= \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})C^{b\beta}_{-\mathbf{p}}(t,s)[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) <\zeta^{c}_{-\mathbf{q}}(t)> + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h^{c}_{-\mathbf{q}}] \\ &\cdot[2K^{\alpha\beta\gamma}(-\mathbf{k},-\mathbf{p},-\mathbf{q}) <\zeta^{\gamma}_{\mathbf{q}}(s)> + A^{\alpha\beta\gamma}(-\mathbf{k},-\mathbf{p},-\mathbf{q})h^{\gamma}_{\mathbf{q}}],\end{aligned} \tag{23c}$$

$$\begin{aligned}\pi^{a\alpha}_{\mathbf{k}}(t,s) &= -\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})R^{b\beta}_{-\mathbf{p}}(t,s)[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) <\zeta^{c}_{-\mathbf{q}}(t)> + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h^{c}_{-\mathbf{q}}] \\ &\cdot[2K^{\beta\alpha\gamma}(-\mathbf{p},-\mathbf{k},-\mathbf{q}) <\zeta^{\gamma}_{\mathbf{q}}(s)> + A^{\beta\alpha\gamma}(-\mathbf{p},-\mathbf{k},-\mathbf{q})h^{\gamma}_{\mathbf{q}}]\end{aligned} \tag{23d}$$

and $\eta^{a\alpha}_{\mathbf{k}}(t,s)$ is given in Equation (18a). Both $S^{a\alpha}_{\mathbf{k}}(t,s)$ and $P^{a\alpha}_{\mathbf{k}}(t,s)$ are positive semi-definite in the sense of Equation (19) of Bowman *et al.* [43]. The equation for the diagonal response function is derived in a similar way using Equation (A.12). We find:

$$\frac{\partial}{\partial t}R^{ab}_{\mathbf{k}}(t,t') + D^{a\beta}_{0}(\mathbf{k})R^{\beta b}_{\mathbf{k}}(t,t') = -\int_{t'}^{t} ds\{\eta^{a\beta}_{\mathbf{k}}(t,s) + \pi^{a\beta}_{\mathbf{k}}(t,s)\}R^{\beta b}_{\mathbf{k}}(s,t') \tag{24}$$

with $R^{ab}_{\mathbf{k}}(t,t) = \delta^{ab}$ and δ^{ab} is the Kronecker delta function.

The single-time cumulant equation may be obtained from the expression:

$$\begin{aligned}\frac{\partial}{\partial t}C_{\mathbf{k}}^{ab}(t,t) &= \lim_{t' \to t}\{\frac{\partial}{\partial t}C_{\mathbf{k}}^{ab}(t,t') + \frac{\partial}{\partial t'}C_{\mathbf{k}}^{ab}(t,t')\} \\ &= \lim_{t' \to t}\{\frac{\partial}{\partial t}C_{\mathbf{k}}^{ab}(t,t') + \frac{\partial}{\partial t'}C_{-\mathbf{k}}^{ba}(t',t)\}.\end{aligned} \tag{25a}$$

This leads to the equation:

$$\begin{aligned}&\frac{\partial}{\partial t}C_{\mathbf{k}}^{ab}(t,t) + D_0^{a\beta}(\mathbf{k})C_{\mathbf{k}}^{\beta b}(t,t) + D_0^{b\beta}(-\mathbf{k})C_{-\mathbf{k}}^{\beta a}(t,t) \\ &= \int_{t_o}^{t} ds\{S_{\mathbf{k}}^{a\beta}(t,s) + P_{\mathbf{k}}^{a\beta}(t,s) + F_0^{a\beta}(\mathbf{k},t,s)\}R_{-\mathbf{k}}^{b\beta}(t,s) \\ &+ \int_{t_o}^{t} ds\{S_{-\mathbf{k}}^{b\beta}(t,s) + P_{-\mathbf{k}}^{b\beta}(t,s) + F_0^{b\beta}(-\mathbf{k},t,s)\}R_{\mathbf{k}}^{a\beta}(t,s) \\ &- \int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{a\beta}(t,s) + \pi_{\mathbf{k}}^{a\beta}(t,s)\}C_{-\mathbf{k}}^{b\beta}(t,s) - \int_{t_o}^{t} ds\{\eta_{-\mathbf{k}}^{b\beta}(t,s) + \pi_{-\mathbf{k}}^{b\beta}(t,s)\}C_{\mathbf{k}}^{a\beta}(t,s).\end{aligned} \tag{25b}$$

The inhomogeneous, or off-diagonal, elements of the covariance and response function matrices are given by Equations (A.9) and (A.12) of Appendix A respectively. Further the three-point function is expressed by Equation (A.13). This completes the generalized QDIA theory under the conditions described in Appendix A.

5. Langevin Equation for QDIA Closure

The generalized Langevin equation which exactly reproduces the QDIA closure equations is:

$$\begin{aligned}&\frac{\partial}{\partial t}\tilde{\zeta}_{\mathbf{k}}^{a}(t) + D_{\mathbf{k}}^{a\beta}\tilde{\zeta}_{\mathbf{k}}^{\beta}(t) \\ &= -\int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{a\beta}(t,s) + \pi_{\mathbf{k}}^{a\beta}(t,s)\}\tilde{\zeta}_{\mathbf{k}}^{\beta}(s) + \hat{f}_0^{a}(\mathbf{k},t) + f_S^{a}(\mathbf{k},t) + f_P^{a}(\mathbf{k},t)\end{aligned} \tag{26}$$

where:

$$f_S^{a}(\mathbf{k},t) = \sqrt{2}\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\rho_{-\mathbf{p}}^{(1)b}(t)\rho_{-\mathbf{q}}^{(2)c}(t), \tag{27a}$$

$$f_P^{a}(\mathbf{k},t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{q}}^{c}(t) > + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c}]\rho_{-\mathbf{p}}^{(3)b}(t)\sigma_{-\mathbf{q}}^{c}(t). \tag{27b}$$

Here $\rho_{\mathbf{k}}^{(j)a}(t)$, where j = 1, 2 or 3, and $\sigma_{\mathbf{k}}^{a}(t)$ are statistically independent random variables such that:

$$< \rho_{\mathbf{k}}^{(j)a}(t)\rho_{-\mathbf{l}}^{(j')b}(t') >= \delta^{jj'}\delta_{\mathbf{kl}}C_{\mathbf{k}}^{ab}(t,t'), \tag{28a}$$

$$< \sigma_{\mathbf{k}}^{a}(t)\sigma_{-\mathbf{l}}^{b}(t') >= \delta_{\mathbf{kl}} \tag{28b}$$

and:

$$< \tilde{\zeta}_{\mathbf{k}}^{a}(t)\tilde{\zeta}_{-\mathbf{k}}^{b}(t') >= C_{\mathbf{k}}^{ab}(t,t'). \tag{29}$$

In Equation (28), δ is the Kronecker delta function.

The Langevin Equation (26) guarantees realizability for the elements of the covariance matrices that are diagonal in spectral space in the quasi-diagonal closure equations.

6. Subgrid-Scale Parameterizations

We are now in a position to derive expressions for subgrid scale terms when the resolution is reduced from C_T to $C_R < C_T$, where C_R is the resolution of the resolved scales. In the previous sections, the summations over **p** and **q** are such that $p \le C_T, q \le C_T$ or $(\mathbf{p},\mathbf{q}) \in \mathcal{T}$ where the set:

$$\mathcal{T} = \{\mathbf{p},\mathbf{q} | p \le C_T, q \le C_T\}. \tag{30}$$

We also define the set $\mathcal{R}$ of resolves scales by:

$$\mathcal{R} = \{\mathbf{p},\mathbf{q} | p \le C_R, q \le C_R\}. \tag{31}$$

and the set $\mathcal{S}$ of subgrid scales by:

$$\mathcal{S} = \mathcal{T} - \mathcal{R}. \tag{32a}$$

Here $\mathcal{S}$ can be written in the form:

$$\mathcal{S} = \mathcal{C} \cup \mathcal{G} \tag{32b}$$

where $\mathcal{C}$ is the set of cross terms defined by:

$$\mathcal{C} = \{\mathbf{p},\mathbf{q} | (p \le C_R, C_R < q \le C_T) \text{ or } (C_R < p \le C_T, q \le C_R)\} \tag{32c}$$

and $\mathcal{G}$ is the set for which both **p** and **q** are subgrid:

$$\mathcal{G} = \{\mathbf{p},\mathbf{q} | C_R < p \le C_T, C_R < q \le C_T\}. \tag{32d}$$

Thus, for $(\mathbf{p},\mathbf{q}) \in \mathcal{S}$, *one or both* of the inequalities $C_R < p \le C_T, C_R < q \le C_T$ holds. Each of the functions defined in the previous sections, which involve summations over **p** and **q**, can then be split into resolved scale terms for which $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$ and subgrid scale terms for which $(\mathbf{p},\mathbf{q}) \in \mathcal{S}$. For $(\mathbf{p},\mathbf{q}) \in \mathcal{S}$ we define $\eta_{\mathbf{k}}^{\mathcal{S}ab}(t,s), f_H^{\mathcal{S}ab}(\mathbf{k},t), \chi_{\mathbf{k}}^{\mathcal{S}ab}(t,s)$ by right hand side of Equations (18a), (18b), (18c), $S_{\mathbf{k}}^{\mathcal{S}ab}(t,s), \pi_{\mathbf{k}}^{\mathcal{S}ab}(t,s), P_{\mathbf{k}}^{\mathcal{S}ab}(t,s)$ by (23b), (23c), (23d), and $f_S^{\mathcal{S}ab}(\mathbf{k},t)$ and $f_P^{\mathcal{S}ab}(\mathbf{k},t)$ by Equations (27a) and (27b) respectively. Similar expressions with superscript $\mathcal{R}$ may be defined for $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$.

6.1. Mean Field

The dynamical equation for the mean resolved scale vorticity, including subgrid scale terms, may then be derived as follows. For $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$, we use the original Equation (15a) for $< \zeta_{\mathbf{k}}^a >$ and the subgrid scale contributions are taken from the closure based Equation (20). Thus, for $k \le C_R$ we have:

$$\begin{aligned}
&\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^a(t) > + D_0^{\alpha\beta}(\mathbf{k}) < \zeta_{\mathbf{k}}^\beta(t) > + \int_{t_o}^{t} ds \eta_{\mathbf{k}}^{\mathcal{S}\alpha\beta}(t,s) < \zeta_{\mathbf{k}}^\beta(s) > \\
&= \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{R}}\sum \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{< \zeta_{-\mathbf{p}}^b(t) >< \zeta_{-\mathbf{q}}^c(t) > + C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\} \\
&+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^b(t) > h_{-\mathbf{q}}^c] + \bar{f}_0^a(\mathbf{k},t) + f_H^{\mathcal{S}a}(\mathbf{k},t) \\
&+ \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{S}}\sum \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^b(t) >< \zeta_{-\mathbf{q}}^c(t) > + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^b(t) > h_{-\mathbf{q}}^c].
\end{aligned} \tag{33a}$$

This can also be written in the form:

$$\begin{aligned}
&\frac{\partial}{\partial t} < \zeta_{\mathbf{k}}^{a}(t) > + \int_{t_o}^{t} ds \bar{d}_r^{a\beta}(\mathbf{k},t,s) < \zeta_{\mathbf{k}}^{\beta}(s) > \\
&= \sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\left\{ < \zeta_{-\mathbf{p}}^{b}(t) >< \zeta_{-\mathbf{q}}^{c}(t) > + C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t) \right\} \\
&+ A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) > h_{-\mathbf{q}}^{c}] + \bar{f}_r^{a}(\mathbf{k},t)
\end{aligned} \tag{33b}$$

where the two-time renormalized dissipation elements:

$$\bar{d}_r^{a\beta}(\mathbf{k},t,s) = D_0^{\alpha\beta}(\mathbf{k})\delta(t-s) + \bar{d}_d^{a\beta}(\mathbf{k},t,s) \tag{34a}$$

and $\delta(t-s)$ is the Dirac delta function. Here we have defined the two-time drain dissipation elements by:

$$\bar{d}_d^{a\beta}(\mathbf{k},t,s) = \eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s). \tag{34b}$$

As well, the renormalized mean force is defined by:

$$\bar{f}_r^{a}(\mathbf{k},t) = \bar{f}_0^{a}(\mathbf{k},t) + \bar{f}_h^{a}(\mathbf{k},t) + j^{a}(\mathbf{k},t), \tag{34c}$$

where the 'residual Jacobian' term is given by:

$$j^{a}(\mathbf{k},t) = \sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{S}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{b}(t) >< \zeta_{-\mathbf{q}}^{c}(t) > + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \zeta_{-\mathbf{p}}^{a}(t) > h_{-\mathbf{q}}^{b}]. \tag{34d}$$

Here, we have also denoted the subgrid eddy-topographic force by:

$$\bar{f}_h^{a}(\mathbf{k},t) \equiv f_H^{\mathcal{S}a}(\mathbf{k},t). \tag{34e}$$

6.2. Fluctuating Field

Again, for $(\mathbf{p},\mathbf{q})\in\mathcal{R}$ the equation for the resolved scale vorticity fluctuations is taken from Equation (15b) and for the subgrid scale terms the Langevin Equation (26) is used with $\tilde{\zeta}_{\mathbf{k}}^{a} \to \hat{\zeta}_{\mathbf{k}}^{a}$ to give:

$$\begin{aligned}
&\frac{\partial}{\partial t}\hat{\zeta}_{\mathbf{k}}^{a}(t) + D_0^{\alpha\beta}(\mathbf{k})\hat{\zeta}_{\mathbf{k}}^{\beta}(t) + \int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)\}\hat{\zeta}_{\mathbf{k}}^{\beta}(s) \\
&= \sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{ < \zeta_{-\mathbf{p}}^{b}(t) > \hat{\zeta}_{-\mathbf{q}}^{c}(t) + \hat{\zeta}_{-\mathbf{p}}^{b}(t) < \zeta_{-\mathbf{q}}^{c}(t) > \\
&+ \hat{\zeta}_{-\mathbf{p}}^{b}(t)\hat{\zeta}_{-\mathbf{q}}^{c}(t) - C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\} + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\zeta}_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}] \\
&+ \hat{f}_0^{a}(\mathbf{k},t) + f_S^{\mathcal{S}a}(\mathbf{k},t) + f_P^{\mathcal{S}a}(\mathbf{k},t).
\end{aligned} \tag{35a}$$

This can also be written as:

$$\begin{aligned}
&\frac{\partial}{\partial t}\hat{\zeta}_{\mathbf{k}}^{a}(t) + \int_{t_o}^{t} ds \hat{d}_r^{a\beta}(\mathbf{k},t,s)\hat{\zeta}_{\mathbf{k}}^{\beta}(s) \\
&= \sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\{ < \zeta_{-\mathbf{p}}^{b}(t) > \hat{\zeta}_{-\mathbf{q}}^{c}(t) + \hat{\zeta}_{-\mathbf{p}}^{b}(t) < \zeta_{-\mathbf{q}}^{c}(t) > \\
&+ \hat{\zeta}_{-\mathbf{p}}^{b}(t)\hat{\zeta}_{-\mathbf{q}}^{c}(t) - C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\} + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\zeta}_{-\mathbf{p}}^{b}(t)h_{-\mathbf{q}}^{c}] + \hat{f}_r^{a}(\mathbf{k},t)
\end{aligned} \tag{35b}$$

where the two-time renormalized dissipation elements:

$$\hat{d}_r^{\alpha\beta}(\mathbf{k},t,s) = D_0^{\alpha\beta}(\mathbf{k})\delta(t-s) + \hat{d}_d^{\alpha\beta}(\mathbf{k},t,s). \tag{36a}$$

Here, the two-time drain dissipation elements are given by:

$$\hat{d}_d^{\alpha\beta}(\mathbf{k},t,s) = \eta_{\mathbf{k}}^{\mathcal{S}\alpha\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{S}\alpha\beta}(t,s). \tag{36b}$$

As well, the renormalized random force is defined by:

$$\hat{f}_r^{a}(\mathbf{k},t) = \hat{f}_0^{a}(\mathbf{k},t) + \hat{f}_b^{a}(\mathbf{k},t) \tag{36c}$$

where:

$$\hat{f}_b^{a}(\mathbf{k},t) = f_S^{\mathcal{S}a}(\mathbf{k},t) + f_P^{\mathcal{S}a}(\mathbf{k},t). \tag{36d}$$

These mean field and fluctuation equations are generalizations of the original Equations (15a) and (15b) with additional forcing contributions and linear terms modifying the bare viscous dissipation; these additional terms are due to the subgrid scale eddies. Note also that the linear terms now have an integral representation. That is, our parameterization of the subgrid-scale eddies changes the original coupled ordinary differential Equations (15a) and (15b) to coupled integro-differential equations for the resolved scales. This is due to the subgrid scales having memory effects.

6.3. Response Function and Covariance

The equation for the response function, including subgrid terms, follows from Equation (4.11):

$$\begin{aligned}
&\frac{\partial}{\partial t}R_{\mathbf{k}}^{ab}(t,t') + \int_{t'}^{t} ds\{D_0^{a\beta}(\mathbf{k})\delta(t-s) + \eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)\}R_{\mathbf{k}}^{\beta b}(s,t') \\
&= -\int_{t'}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)\}R_{\mathbf{k}}^{\beta b}(s,t')
\end{aligned} \tag{37a}$$

or, equivalently:

$$\begin{aligned}
&\frac{\partial}{\partial t}R_{\mathbf{k}}^{ab}(t,t') + \int_{t'}^{t} ds\,\hat{d}_r^{a\beta}(\mathbf{k},t,s)R_{\mathbf{k}}^{\beta b}(s,t') \\
&= -\int_{t'}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)\}R_{\mathbf{k}}^{\beta b}(s,t').
\end{aligned} \tag{37b}$$

Similarly, from Equation (22), the two-time cumulant equation including subgrid terms is:

$$\begin{aligned}
&\frac{\partial}{\partial t}C_{\mathbf{k}}^{ab}(t,t') + \int_{t_o}^{t} ds\{D_0^{a\beta}(\mathbf{k})\delta(t-s) + \eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)\}C_{-\mathbf{k}}^{b\beta}(t',s) \\
&- \int_{t_o}^{t'} ds\{S_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + P_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + F_0^{a\beta}(\mathbf{k},t,s)\}R_{-\mathbf{k}}^{b\beta}(t',s) \\
&= \int_{t_o}^{t'} ds\{S_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + P_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)\}R_{-\mathbf{k}}^{b\beta}(t',s) - \int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)\}C_{-\mathbf{k}}^{b\beta}(t',s)
\end{aligned} \tag{38a}$$

or, more compactly:

$$\begin{aligned}
&\frac{\partial}{\partial t} C^{ab}_{\mathbf{k}}(t,t') + \int_{t_o}^{t} ds\, \hat{d}_r^{a\beta}(\mathbf{k},t,s) C^{b\beta}_{-\mathbf{k}}(t',s) - \int_{t_o}^{t'} ds\, F_r^{a\beta}(\mathbf{k},t,s) R^{b\beta}_{-\mathbf{k}}(t',s) \\
&= \int_{t_o}^{t'} ds\{S^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+P^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\} R^{b\beta}_{-\mathbf{k}}(t',s) - \int_{t_o}^{t} ds\{\eta^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+\pi^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\}C^{b\beta}_{-\mathbf{k}}(t',s).
\end{aligned} \tag{38b}$$

Finally, from Equation (25b), the single-time cumulant equations including subgrid terms is:

$$\begin{aligned}
&\frac{\partial}{\partial t} C^{ab}_{\mathbf{k}}(t,t) + D_0^{a\beta}(\mathbf{k}) C^{\beta b}_{\mathbf{k}}(t,t) + D_0^{b\beta}(-\mathbf{k}) C^{\beta a}_{-\mathbf{k}}(t,t) \\
&+ \int_{t_o}^{t} ds\{\eta^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s)+\pi^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s)\}C^{b\beta}_{-\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{S^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s)+P^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s)+F_0^{a\beta}(\mathbf{k},t,s)\}R^{b\beta}_{-\mathbf{k}}(t,s) \\
&+ \int_{t_o}^{t} ds\{\eta^{\mathcal{S}b\beta}_{-\mathbf{k}}(t,s)+\pi^{\mathcal{S}b\beta}_{-\mathbf{k}}(t,s)\}C^{a\beta}_{\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{S^{\mathcal{S}b\beta}_{-\mathbf{k}}(t,s)+P^{\mathcal{S}b\beta}_{-\mathbf{k}}(t,s)+F_0^{b\beta}(-\mathbf{k},t,s)\}R^{a\beta}_{\mathbf{k}}(t,s) \\
&= \int_{t_o}^{t} ds\{S^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+P^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\}R^{b\beta}_{-\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{\eta^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+\pi^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\}C^{b\beta}_{-\mathbf{k}}(t,s) \\
&+ \int_{t_o}^{t} ds\{S^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)+P^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)\}R^{a\beta}_{\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{\eta^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)+\pi^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)\}C^{a\beta}_{\mathbf{k}}(t,s).
\end{aligned} \tag{39a}$$

This may also be written as:

$$\begin{aligned}
&\frac{\partial}{\partial t} C^{ab}_{\mathbf{k}}(t,t) + \int_{t_o}^{t} ds\, \hat{d}_r^{a\beta}(\mathbf{k},t,s) C^{b\beta}_{-\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\, F_r^{a\beta}(\mathbf{k},t,s) R^{b\beta}_{-\mathbf{k}}(t,s) \\
&+ \int_{t_o}^{t} ds\, \hat{d}_r^{b\beta}(-\mathbf{k},t,s) C^{a\beta}_{\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\, F_r^{b\beta}(-\mathbf{k},t,s) R^{a\beta}_{\mathbf{k}}(t,s) \\
&= \int_{t_o}^{t} ds\{S^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+P^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\}R^{b\beta}_{-\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{\eta^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)+\pi^{\mathcal{R}a\beta}_{\mathbf{k}}(t,s)\}C^{b\beta}_{-\mathbf{k}}(t,s) \\
&+ \int_{t_o}^{t} ds\{S^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)+P^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)\}R^{a\beta}_{\mathbf{k}}(t,s) - \int_{t_o}^{t} ds\{\eta^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)+\pi^{\mathcal{R}b\beta}_{-\mathbf{k}}(t,s)\}C^{a\beta}_{\mathbf{k}}(t,s).
\end{aligned} \tag{39b}$$

Here, we have defined renormalized noise covariance matrix elements by:

$$\begin{aligned}
F_r^{a\beta}(\mathbf{k},t,s) &= < \hat{f}_r^{a}(\mathbf{k},t) \hat{f}_r^{\beta}(-\mathbf{k},s) > \\
&= F_0^{a\beta}(\mathbf{k},t,s) + F_b^{a\beta}(\mathbf{k},t,s).
\end{aligned} \tag{40a}$$

The renormalized random force $\hat{f}_r^{a}(\mathbf{k},t)$ is given by Equation (36c), $F_0^{a\beta}(\mathbf{k},t,s)$ is given in Equation (23a) and the backscatter covariance matrix elements by:

$$F_b^{a\beta}(\mathbf{k},t,s) = S^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s) + P^{\mathcal{S}a\beta}_{\mathbf{k}}(t,s) \equiv < \hat{f}_b^{a}(\mathbf{k},t) \hat{f}_b^{\beta}(-\mathbf{k},s) > . \tag{40b}$$

As well, the stochastic backscatter noise $\hat{f}_b^{a}(\mathbf{k},t)$ is given by Equation (36d).

6.4. Langevin Equation for QDIA with Subgrid-Scale Parameterizations

The generalized Langevin equation which exactly reproduces the QDIA closure equations with subgrid-scale parameterizations is as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{\zeta}_{\mathbf{k}}^{a}(t)+\int_{t_o}^{t} ds\hat{d}_r^{a\beta}(\mathbf{k},t,s)\tilde{\zeta}_{\mathbf{k}}^{\beta}(s) \\ &=-\int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)+\pi_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s)\}\tilde{\zeta}_{\mathbf{k}}^{\beta}(s)+\hat{f}_r^{a}(\mathbf{k},t)+f_S^{\mathcal{R}a}(\mathbf{k},t)+f_P^{\mathcal{R}a}(\mathbf{k},t) \end{aligned} \tag{41}$$

where:

$$f_S^{\mathcal{R}a}(\mathbf{k},t)=\sqrt{2}\sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\rho_{-\mathbf{p}}^{(1)b}(t)\rho_{-\mathbf{q}}^{(2)c}(t), \tag{42a}$$

$$f_P^{\mathcal{R}a}(\mathbf{k},t)=\sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})[2K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})<\zeta_{-\mathbf{q}}^{c}(t)>+A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})h_{-\mathbf{q}}^{c}]\rho_{-\mathbf{p}}^{(3)b}(t)\sigma_{-\mathbf{q}}^{c}(t). \tag{42b}$$

Again, $\rho_{\mathbf{k}}^{(j)a}(t)$, where j = 1, 2 or 3, and $\sigma_{\mathbf{k}}^{a}(t)$ are statistically independent random variables that satisfy Equation (28) and Equation (29) also holds. The Langevin Equation (41) guarantees realizability for the elements of the covariance matrices that are diagonal in spectral space in the quasi-diagonal closure equations.

7. Effective Dissipation and Viscosity Parameterizations

We can also write Equation (33) in the form:

$$\begin{aligned} &\frac{\partial}{\partial t}<\zeta_{\mathbf{k}}^{a}(t)>+\overline{D}_r^{a\beta}(\mathbf{k})<\zeta_{\mathbf{k}}^{\beta}(t)> \\ &=\sum\sum_{(\mathbf{p},\mathbf{q})\in\mathcal{R}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\left\{<\zeta_{-\mathbf{p}}^{b}(t)><\zeta_{-\mathbf{q}}^{c}(t)>+C_{-\mathbf{p},-\mathbf{q}}^{bc}(t,t)\right\} \\ &+A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})<\zeta_{-\mathbf{p}}^{a}(t)>h_{-\mathbf{q}}^{b}]+\bar{f}_r^{a}(\mathbf{k},t), \end{aligned} \tag{43}$$

where renormalized generalized drain dissipation matrices appearing in Equation (43) are given by:

$$\overline{D}_r^{ab}(\mathbf{k})=D_0^{ab}(\mathbf{k})+\overline{D}_d^{ab}(\mathbf{k}). \tag{44}$$

Here the drain dissipation matrix elements for the mean field are given by:

$$\overline{D}_d^{a\beta}(\mathbf{k})<\zeta_{\mathbf{k}}^{\beta}(t)>=\int_{t_o}^{t} ds\overline{d}_d^{a\beta}(\mathbf{k},t,s)<\zeta_{\mathbf{k}}^{\beta}(s)>=\int_{t_o}^{t} ds\eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)<\zeta_{\mathbf{k}}^{\beta}(s)>. \tag{45}$$

In general $\overline{D}_r^{ab}(\mathbf{k})$ and $\overline{D}_d^{ab}(\mathbf{k})$ are time-dependent but in our previous studies we have primarily been interested in their properties at statistical steady state.

We also define the drain dissipation matrix elements for fluctuations by:

$$\hat{D}_d^{a\beta}(\mathbf{k})C_{\mathbf{k}}^{\beta b}(t,t)=\int_{t_o}^{t} ds\hat{d}_d^{a\beta}(\mathbf{k},t,s)C_{-\mathbf{k}}^{b\beta}(t,s)=\int_{t_o}^{t} ds\{\eta_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)+\pi_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s)\}C_{-\mathbf{k}}^{b\beta}(t,s). \tag{46}$$

Thus:

$$
\begin{aligned}
&\frac{\partial}{\partial t} C_{\mathbf{k}}^{ab}(t,t) + \hat{D}_r^{a\beta}(\mathbf{k}) C_{\mathbf{k}}^{\beta b}(t,t) + \hat{D}_r^{b\beta}(-\mathbf{k}) C_{-\mathbf{k}}^{\beta a}(t,t) \\
&= \int_{t_o}^{t} ds \{ S_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + P_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + F_r^{a\beta}(\mathbf{k},t,s) \} R_{-\mathbf{k}}^{b\beta}(t,s) - \int_{t_o}^{t} ds \{ \eta_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) + \pi_{\mathbf{k}}^{\mathcal{R}a\beta}(t,s) \} C_{-\mathbf{k}}^{b\beta}(t,s) \\
&+ \int_{t_o}^{t} ds \{ S_{-\mathbf{k}}^{\mathcal{R}b\beta}(t,s) + P_{-\mathbf{k}}^{\mathcal{R}b\beta}(t,s) + F_r^{b\beta}(-\mathbf{k},t,s) \} R_{\mathbf{k}}^{a\beta}(t,s) - \int_{t_o}^{t} ds \{ \eta_{-\mathbf{k}}^{\mathcal{R}b\beta}(t,s) + \pi_{-\mathbf{k}}^{\mathcal{R}b\beta}(t,s) \} C_{\mathbf{k}}^{a\beta}(t,s).
\end{aligned}
\tag{47}
$$

where:

$$
\hat{D}_r^{ab}(\mathbf{k}) = D_0^{ab}(\mathbf{k}) + \hat{D}_d^{ab}(\mathbf{k}). \tag{48}
$$

We can define the backscatter dissipation matrix elements by:

$$
\hat{D}_b^{a\beta}(\mathbf{k}) C_{\mathbf{k}}^{\beta b}(t,t) = -\int_{t_o}^{t} ds \{ S_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) + P_{\mathbf{k}}^{\mathcal{S}a\beta}(t,s) \} R_{-\mathbf{k}}^{b\beta}(t,s) = -\int_{t_o}^{t} ds F_b^{a\beta}(\mathbf{k},t,s) R_{-\mathbf{k}}^{b\beta}(t,s). \tag{49a}
$$

Further, we define the net dissipation matrix elements by:

$$
\hat{D}_n^{ab}(\mathbf{k}) = \hat{D}_d^{ab}(\mathbf{k}) + \hat{D}_b^{ab}(\mathbf{k}), \tag{49b}
$$

and the renormalized net dissipation matrix elements by:

$$
\hat{D}_{rn}^{ab}(\mathbf{k}) = D_0^{ab}(\mathbf{k}) + \hat{D}_n^{ab}(\mathbf{k}). \tag{49c}
$$

Again, generalized viscosity matrix elements may be defined as follows:

$$
\bar{\nu}_{\bullet}^{ab}(\mathbf{k}) = \bar{D}_{\bullet}^{ab}(\mathbf{k}) k^{-2}, \tag{50a}
$$

and:

$$
\hat{\nu}_{\bullet}^{ab}(\mathbf{k}) = \hat{D}_{\bullet}^{ab}(\mathbf{k}) k^{-2}, \tag{50b}
$$

where $\bullet$ denotes any of $0, b, d, n, r, rn$.

In general, and specifically if we include the beta-effect in our analysis, the dissipation matrix elements $\bar{D}_{\bullet}^{ab}(\mathbf{k})$, $\hat{D}_{\bullet}^{ab}(\mathbf{k})$, and the viscosities $\bar{\nu}_{\bullet}^{ab}(\mathbf{k})$, $\hat{\nu}_{\bullet}^{ab}(\mathbf{k})$ are complex; both the viscosity and wave frequency are renormalized by the subgrid scale eddies. If our system is quasi-isotropic, as well as quasi-diagonal, then these terms are real viscosities and only depend on k, the magnitude of $\mathbf{k}$. This is the case, in particular, near canonical equilibrium.

We note that Equation (43) is identical to Equation (15a) except that the bare dissipation $D_0^{a\beta}(\mathbf{k})$ is replaced by the renormalized dissipation $\bar{D}_r^{a\beta}(\mathbf{k})$, the bare force $\bar{f}_0^{a}(\mathbf{k},t)$ is replaced by the renormalized force $\bar{f}_r^{a}(\mathbf{k},t)$ and the summation is now for $(\mathbf{p},\mathbf{q}) \in \mathcal{R}$ rather than over the whole space $(\mathbf{p},\mathbf{q}) \in \mathcal{T}$. Similarly Equation (47) is the same as Equation (25b) but with $D_0^{a\beta}(\mathbf{k}) \to \bar{D}_r^{a\beta}(\mathbf{k})$, $F_0^{a\beta}(\mathbf{k},t,s) \to F_r^{a\beta}(\mathbf{k},t,s)$ (or $\bar{f}_0^{a}(\mathbf{k},t) \to \bar{f}_r^{a}(\mathbf{k},t)$) and $\mathcal{T} \to \mathcal{R}$. Our analysis also shows that the subgrid-scale eddies affect the mean and fluctuating-parts of the field variables differently. In general, the mean and fluctuating parts of the fields relax at different rates as seen from Equations (44) and (48).

8. Discussion and Conclusions

We have generalized the computationally tractable QDIA closures of Frederiksen [4] and Frederiksen and O'Kane [11] for inhomogeneous barotropic turbulent flows over topography to multi-field classical field theories with quadratic nonlinearity. Next, we note that it is also possible to obtain from the QDIA closure generalizations of the SCFT [5] and LET [6,7] statistical closures.

8.1. Quasi-Diagonal LET and SCFT Inhomogeneous Closures

As discussed in detail by Frederiksen *et al.* [8] both the SCFT and LET closures for homogeneous turbulence may be formally obtained from the DIA by invoking the fluctuation-dissipation theorem (FDT). In a similar way we can obtain quasi-diagonal SCFT (QSCFT) and LET (QLET) closures for inhomogeneous turbulent flows from the QDIA. For our multi-field equations we use the FDT:

$$C^{ab}_{\mathbf{k}}(t,t')\Theta(t-t') = R^{a\beta}_{\mathbf{k}}(t,t')C^{\beta b}_{\mathbf{k}}(t,t) \tag{51}$$

for the diagonal elements in spectral space. Here $\Theta(x)$ is the Heaviside theta function which is unity for x positive and otherwise vanishes. The QSCFT is then obtained from the QDIA by replacing the prognostic equation for the two-time cumulant, Equation (22), by Equation (51). Similarly, the QLET closure is obtained from the QDIA by replacing the equation for the response function, Equation (24), by the expression in Equation (51) in terms of the single- and two-time cumulants.

8.2. Regularization and Non-Gaussian initial Conditions

We can also follow the approach of Frederiksen and Davies [14] and O'Kane and Frederiksen [12] and introduce a regularization of the QDIA closure. This corresponds to an empirical vertex renormalization in which the interactions are localized in wavenumber space in a manner specified by an interaction cut-off parameter α_c. For the current multi-field QDIA this is achieved by using $\Theta(x)$, the Heaviside theta function, and making the replacements:

$$A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \rightarrow \Theta(p-k/\alpha_c)\Theta(q-k/\alpha_c)A^{abc}(\mathbf{k},\mathbf{q},\mathbf{p}), \tag{52a}$$

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \rightarrow \Theta(p-k/\alpha_c)\Theta(q-k/\alpha_c)K^{abc}(\mathbf{k},\mathbf{q},\mathbf{p}) \tag{52b}$$

in the two-time cumulant and response function equations but not in the single-time cumulant equations. For suitable α_c we expect that the multi-field QDIA will again have the correct power laws as well as accurately capturing the evolution of the energy containing scales.

We can also include the effects of non-Gaussian initial conditions in the multi-field QDIA by following the approaches of Frederiksen *et al.* [8] and O'Kane and Frederiksen [12]. This again allows the employment of more computationally efficient cumulant update variants of the closure equations. However, for the sake of brevity these straightforward generalizations will be left for the future numerical implementation of the closures.

8.3. Concluding Comments

We have derived statistical dynamical closures for inhomogeneous turbulent flows described by multi-field equations with quadratic nonlinearity. We have generalized the computationally tractable

QDIA closure for inhomogeneous barotropic flows over topography and focused our analysis on QG and 3D turbulence. The analysis applies equally to other equations such as the primitive equations for atmospheric and oceanic circulations [44], to internal gravity wave turbulence [42], to continuous [42] rather than discrete representations of the fields, and more generally to classical field theories [50]. Statistical dynamical closures have also been formulated for large eddy simulations and subgrid models have been presented that ensure that the LESs have the same large scale statistical behavior as the higher resolution closures. These subgrid models include general expressions for all the subgrid terms required in the equations for the mean and fluctuating fields.

The multi-field QDIA and its underpinning Langevin equation also provide further support for the direct stochastic modeling approach to subgrid-scale parameterizations developed by Frederiksen and Kepert [2] and applied also to the baroclinic QG equations for atmospheric and oceanic flows by Zidikheri and Frederiksen [32–34] and Kitsios, Frederiksen and Zidikheri [35]. In some of these studies, particularly for higher resolution LESs of atmospheric flows [32,33,35], where the baroclinic instability is resolved, the renormalized mean forcing is essentially just the bare forcing: $\bar{f}_r^a(\mathbf{k},t) \approx \bar{f}_0^a(\mathbf{k},t)$. However, as in the formulations and studies of Frederiksen [4], Frederiksen and Kepert [2], Franzke, Majda and Branstator [45], O'Kane and Frederiksen [18], Frederiksen and O'Kane [1], Zidikheri and Frederiksen [34], the subgrid contribution to the mean field forcing is in general significant and the renormalized forcing is given by $\bar{f}_r^a(\mathbf{k},t) = \bar{f}_0^a(\mathbf{k},t) + \bar{f}_h^a(\mathbf{k},t) + j^a(\mathbf{k},t)$ as in Equation (34c).

The multi-field QDIA, like the barotropic QDIA and the DIA closures, has an underpinning generalized Langevin equation with memory effects that guarantees relizability of the covariance elements that are diagonal in spectral space and positive definite energy spectra. The direct stochastic modeling approach to subgrid-scale parameterizations by Frederiksen and Kepert [2] also accounts for memory effects. It differs in this respect from many commonly used stochastic models [46,47] based on linear regression that may not always be realizable as discussed by Majda, Gershgorin and Yuan [48] (and references therein). Frederiksen and Kepert [2] have found that their approach gave very similar subgrid-scale parameterizations, and successful LES, to the closure based method of Frederiksen and Davies [31]. Unlike the case for the QDIA, we know of no general proof of the realizability of the direct stochastic approach with memory effects and stochastic backscatter [2] but all subsequent applications have yielded realizable positive definite energy spectra [2,32–35].

Acknowledgements

The aim of this theoretical study has been to generalize the QDIA closure and subgrid-scale parameterizations to multi-field equations and to underpin recent work by collaborators and colleagues on stochastic modeling and backscatter. I thank them for providing the motivation.

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Appendix A: Perturbation Theory

In this appendix, we derive some expressions relating general inhomogeneous elements of the covariance and response function matrices to the corresponding elements that are diagonal in spectral space. These are needed in the formulation of the QDIA closure equations presented in Section 4. The closure equations are formulated by doing a formal perturbation theory. We suppose that the terms on the right hand side of Equation (15b) (apart from $\hat{f}_0^a(\mathbf{k},t)$) are multiplied by small parameter λ. The closure equations are then formally renormalized and λ restored back to unity.

We begin by expanding $\hat{\zeta}_{\mathbf{k}}^a$ in Equation (15b) in a perturbation series:

$$\hat{\zeta}_{\mathbf{k}}^a = \hat{\zeta}_{\mathbf{k}}^{(0)a} + \lambda \hat{\zeta}_{\mathbf{k}}^{(1)a} + \ldots \tag{A.1}$$

Then, to zero order, we have from Equation (15b) (with the right hand side, apart from $\hat{f}_0^a(\mathbf{k},t)$, multiplied by λ) that:

$$\frac{\partial \hat{\zeta}_{\mathbf{k}}^{(0)a}(t)}{\partial t} + D_{\mathbf{k}}^{a\beta} \hat{\zeta}_{\mathbf{k}}^{(0)\beta}(t) = \hat{f}_0^a(\mathbf{k},t). \tag{A.2}$$

To first order we have:

$$\begin{aligned}
&\frac{\partial \hat{\zeta}_{\mathbf{k}}^{(1)a}(t)}{\partial t} + D_{\mathbf{k}}^{a\beta} \hat{\zeta}_{\mathbf{k}}^{(1)\beta}(t) = \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \hat{\zeta}_{-\mathbf{p}}^{(0)b}(t) h_{-\mathbf{q}}^c \\
&+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) [< \zeta_{-\mathbf{p}}^b(t) > \hat{\zeta}_{-\mathbf{q}}^{(0)c}(t) + \hat{\zeta}_{-\mathbf{p}}^{(0)b}(t) < \zeta_{-\mathbf{q}}^c(t) > \\
&+ \hat{\zeta}_{-\mathbf{p}}^{(0)b}(t) \hat{\zeta}_{-\mathbf{q}}^{(0)c}(t) - < \hat{\zeta}_{-\mathbf{p}}^{(0)b}(t) \hat{\zeta}_{-\mathbf{q}}^{(0)c}(t) >].
\end{aligned} \tag{A.3}$$

Then the formal solution to (A.3) can be written, using the Greens function $R_{\mathbf{k},\mathbf{k}}^{(0)a\beta}(t,s)$ corresponding to Equation (A.2), as follows:

$$\hat{\zeta}^{(1)a}_{\mathbf{k}}(t) = \int_{t_o}^{t} ds R^{(0)a\alpha}_{\mathbf{k},\mathbf{k}}(t,s)\Big\{\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})A^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)h^{\gamma}_{-\mathbf{q}}$$
$$+\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q})[<\zeta^{\beta}_{-\mathbf{p}}(s)>\hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s)+\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)<\zeta^{\gamma}_{-\mathbf{q}}(s)> \tag{A.4}$$
$$+\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s) - <\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s)>]\Big\}$$

We can also express the two-time cumulant as:

$$C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t') = <\hat{\zeta}^{a}_{\mathbf{k}}(t)\hat{\zeta}^{b}_{-\mathbf{l}}(t')>$$
$$= <\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')> + \lambda <\hat{\zeta}^{(1)a}_{\mathbf{k}}(t)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')> \tag{A.5}$$
$$+\lambda <\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)\hat{\zeta}^{(1)b}_{-\mathbf{l}}(t')> + \ldots$$

We assume the initial $\hat{\zeta}^{a}_{\mathbf{k}}(t_o)$ have a Gaussian distribution that is diagonal in spectral space. In particular, this implies:

$$<\hat{\zeta}^{a}_{k}(t_o)\hat{\zeta}^{b}_{-\mathbf{l}}(t_o)> = \delta_{\mathbf{kl}} <\hat{\zeta}^{a}_{\mathbf{k}}(t_o)\hat{\zeta}^{b}_{-\mathbf{k}}(t_o)>. \tag{A.6a}$$

Moreover, we suppose diagonal dominance in spectral space so that the zero order fields satisfy:

$$<\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')> = \delta_{\mathbf{kl}} <\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)\hat{\zeta}^{(0)b}_{-\mathbf{k}}(t')>. \tag{A.6b}$$

That is, to zero order $C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t')$ is homogeneous in the horizontal but inhomogeneous in the vertical. To first order in λ we have the general inhomogeneous contribution:

$$C^{(1)ab}_{\mathbf{k},-\mathbf{l}}(t,t') = <\hat{\zeta}^{(1)a}_{\mathbf{k}}(t)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')> + <\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)\hat{\zeta}^{(1)b}_{-\mathbf{l}}(t')>. \tag{A.7}$$

Using Equation (A.4) then gives:

$$C^{(1)ab}_{\mathbf{k},-\mathbf{l}}(t,t') = \int_{t_o}^{t} ds R^{(0)a\alpha}_{\mathbf{k},\mathbf{k}}(t,s)\Big\{\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})A^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q}) <\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')> h^{\gamma}_{-\mathbf{q}}$$
$$+\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q})[<\zeta^{\beta}_{-\mathbf{p}}(s)><\hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')>$$
$$+<\hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)b}_{-\mathbf{l}}(t')><\zeta^{\gamma}_{-\mathbf{q}}(s)>]\Big\} \tag{A.8}$$
$$+\int_{t_o}^{t'} ds R^{(0)b\beta}_{-\mathbf{l},-\mathbf{l}}(t',s)\Big\{\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(-\mathbf{l},\mathbf{p},\mathbf{q})A^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{p},\mathbf{q}) <\hat{\zeta}^{(0)\alpha}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)> h^{\gamma}_{-\mathbf{q}}$$
$$+\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(-\mathbf{l},\mathbf{p},\mathbf{q})K^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{p},\mathbf{q})[<\zeta^{\alpha}_{-\mathbf{p}}(s)><\hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s)\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)>$$
$$+<\hat{\zeta}^{(0)\alpha}_{-\mathbf{p}}(s)\hat{\zeta}^{(0)a}_{\mathbf{k}}(t)><\zeta^{\gamma}_{-\mathbf{q}}(s)>]\Big\}.$$

Next, we implement Equation (A.6) and perform the formal renormalizations $\lambda \to 1, R^{(0)ab}_{\mathbf{k},\mathbf{k}} \to R^{ab}_{\mathbf{k},\mathbf{k}}, C^{(0)ab}_{\mathbf{k},-\mathbf{k}} \to C^{ab}_{\mathbf{k},-\mathbf{k}}, C^{(1)ab}_{\mathbf{k},-\mathbf{l}} \to C^{ab}_{\mathbf{k},-\mathbf{l}}$. Finally, the general inhomogeneous two-point cumulant can be expressed in terms of the cumulants and response functions that are diagonal in spectral space as:

$$
\begin{aligned}
C^{ab}_{\mathbf{k},-\mathbf{l}}(t,t') = &\int_{t_o}^{t} ds R^{a\alpha}_{\mathbf{k},\mathbf{k}}(t,s) C^{b\beta}_{-\mathbf{l},\mathbf{l}}(t',s)[A^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) h^{\gamma}_{(\mathbf{k}-\mathbf{l})} \\
&+ 2K^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) < \zeta^{\gamma}_{(\mathbf{k}-\mathbf{l})}(s) >] \\
&+ \int_{t_o}^{t'} ds R^{b\beta}_{-\mathbf{l},-\mathbf{l}}(t',s) C^{a\alpha}_{\mathbf{k},\mathbf{k}}(t,s)[A^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k}) h^{\gamma}_{(\mathbf{k}-\mathbf{l})} \\
&+ 2K^{\beta\alpha\gamma}(-\mathbf{l},\mathbf{k},\mathbf{l}-\mathbf{k}) < \zeta^{\gamma}_{(\mathbf{k}-\mathbf{l})}(s) >].
\end{aligned} \tag{A.9}
$$

In a similar way, we can express the general inhomogeneous response function in terms of the response functions that are diagonal in spectral space. We define:

$$
R^{ab}_{\mathbf{k},\mathbf{l}}(t,t') = \left\langle \frac{\delta \hat{\zeta}^{a}_{\mathbf{k}}(t)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle \tag{A.10a}
$$

and performing a perturbation expansion we have:

$$
R^{ab}_{\mathbf{k},\mathbf{l}}(t,t') = \left\langle \frac{\delta \hat{\zeta}^{(0)a}_{\mathbf{k}}}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle + \lambda \left\langle \frac{\delta \hat{\zeta}^{(1)a}_{\mathbf{k}}}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle + \ldots \tag{A.10b}
$$

where, again we suppose homogeneity in the horizontal to order zero, so that:

$$
\left\langle \frac{\delta \hat{\zeta}^{(0)a}_{\mathbf{k}}(t)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle = \delta_{\mathbf{k},\mathbf{l}} \left\langle \frac{\delta \hat{\zeta}^{(0)a}_{\mathbf{k}}(t)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle = \delta_{\mathbf{k},\mathbf{l}} R^{(0)ab}_{\mathbf{k},\mathbf{l}}(t,t'). \tag{A.10c}
$$

Thus, to zero order $R^{ab}_{\mathbf{k},\mathbf{l}}(t,t')$ is homogeneous in the horizontal but inhomogeneous in the vertical. To first order in λ, the general inhomogeneous contribution is:

$$
\begin{aligned}
R^{(1)ab}_{\mathbf{k},\mathbf{l}}(t,t') = &\int_{t'}^{t} ds R^{(0)a\alpha}_{\mathbf{k},\mathbf{k}}(t,s). \\
&\left\{ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) A^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q}) \left\langle \frac{\delta \hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle h^{\gamma}_{-\mathbf{q}} \right. \\
&+ \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q}) K^{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q}) [< \zeta^{\beta}_{-\mathbf{p}}(s) > \left\langle \frac{\delta \hat{\zeta}^{(0)\gamma}_{-\mathbf{q}}(s)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle \\
&\left. + \left\langle \frac{\delta \hat{\zeta}^{(0)\beta}_{-\mathbf{p}}(s)}{\delta \hat{f}^{b}_{\mathbf{l}}(t')} \right\rangle < \zeta^{\gamma}_{-\mathbf{q}}(s) >] \right\}.
\end{aligned} \tag{A.11}
$$

We now implement Equation (A.10c) and perform the formal renormalizations $\lambda \to 1$, $R^{(0)ab}_{\mathbf{k},\mathbf{k}} \to R^{ab}_{\mathbf{k},\mathbf{k}}$, $R^{(1)ab}_{\mathbf{k},\mathbf{l}} \to R^{ab}_{\mathbf{k},\mathbf{l}}$. This yields:

$$
\begin{aligned}
R^{ab}_{\mathbf{k},\mathbf{l}}(t,t') = &\int_{t'}^{t} ds R^{a\alpha}_{\mathbf{k},\mathbf{k}}(t,s) R^{\beta b}_{\mathbf{l},\mathbf{l}}(s,t'). \\
&\left\{ A^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) h^{\gamma}_{(\mathbf{k}-\mathbf{l})} + 2K^{\alpha\beta\gamma}(\mathbf{k},-\mathbf{l},\mathbf{l}-\mathbf{k}) < \zeta^{\gamma}_{(\mathbf{k}-\mathbf{l})}(s) > \right\}.
\end{aligned} \tag{A.12}
$$

In the quasi-diagonal approximation (in spectral space), the treatment of the three-point cumulant in Equation (21) follows closely the approach of Kraichnan [3] for homogeneous turbulence (see also

Frederiksen [49] for a pedagogical approach) and Kraichnan [10] and Martin *et al.* [50] for general inhomogeneous turbulence. In fact, the approach is closest to that of Carnevale and Frederiksen [42] who considered the DIA closure for internal gravity wave turbulence. We find that for Gaussian initial conditions:

$$\sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) < \hat{\zeta}^{b}_{-\mathbf{p}}(t)\hat{\zeta}^{c}_{-\mathbf{q}}(t)\hat{\zeta}^{\alpha}_{-\mathbf{k}}(t') > \\ = \int_{t_o}^{t'} ds S^{a\beta}_{\mathbf{k}}(t,s)R^{\alpha\beta}_{-\mathbf{k}}(t',s) - \int_{t_o}^{t} ds\eta^{a\beta}_{\mathbf{k}}(t,s)C^{\alpha\beta}_{-\mathbf{k}}(t',s) \tag{A.13}$$

where the nonlinear eddy-eddy damping $\eta^{a\alpha}_{\mathbf{k}}(t,s)$ is given in Equation (18a) and the nonlinear eddy-eddy noise $S^{a\alpha}_{\mathbf{k}}(t,s)$ is given in Equation (23b).

Appendix B: Quasigeostrophic Model with Continuous Vertical Variations

Taking suitable length and time scales, the nondimensional equation for flow over topography on an *f*-plane may be written in the form:

$$\frac{\partial\zeta}{\partial t} = -J(\psi,\zeta+h) - D_0\zeta + f_0 \ , \tag{B.1a}$$

Here ψ is the streamfunction, and $\zeta = \nabla^2\psi \ - F_L\dfrac{\partial^2}{\partial z^2}\psi$ is the reduced potential vorticity, $h(z)$ is the scaled topography, D_0 is a dissipation operator to be specified below and f_0 is a forcing function. Again, F_L is the vertical coupling parameter and, in planar geometry:

$$J(\psi,\zeta) = \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}. \tag{B.1b}$$

Spectral equations corresponding to this system may be obtained by first expanding each of the functions in a Fourier series, e.g.,

$$\zeta(\mathbf{x},t) = \sum_{\mathbf{k}}\zeta_{\mathbf{k}}(t)\exp(i\mathbf{k}.\mathbf{x}) \tag{B.2a}$$

where:

$$\zeta_{\mathbf{k}}(t) = \frac{1}{(2\pi)^3}\int_o^{2\pi} d^3\mathbf{x}\,\zeta(\mathbf{x},t)\exp(-i\mathbf{k}.\mathbf{x}) \tag{B.2b}$$

and $\mathbf{x} = (x,y,z), \mathbf{k} = (k_x,k_y,k_z)$. Then, multiplying Equation (B.1a) by $\exp(-i\mathbf{k}.\mathbf{x})$ and integrating over the (x,y,z) domain, we find that:

$$\frac{\partial}{\partial t}\zeta_{\mathbf{k}}(t) + \sum_{\mathbf{k}'}D_0(\mathbf{k},\mathbf{k}')\zeta_{\mathbf{k}'}(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}}\delta(\mathbf{k},\mathbf{p},\mathbf{q})\left[K(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}(t)\zeta_{-\mathbf{q}}(t) + A(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}(t)h_{-\mathbf{q}}\right] \\ + f_0(\mathbf{k},t). \tag{B.3}$$

Here:

$$D_0(\mathbf{k},\mathbf{k}') = \nu_0(\mathbf{k},\mathbf{k}')k^2 \tag{B.4}$$

where we refer to $\nu_0(\mathbf{k},\mathbf{k}')$ as the bare viscosity and we shall also refer to $f_0(\mathbf{k},t)$ as the bare forcing. In the above spectral equations:

$$\zeta_{-\mathbf{k}} = \zeta_{\mathbf{k}}^*, \tag{B.5a}$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)/(p_x^2 + p_y^2 + F_L p_z^2), \tag{B.5b}$$

$$K(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\left[A(\mathbf{k},\mathbf{p},\mathbf{q}) + A(\mathbf{k},\mathbf{q},\mathbf{p})\right], \tag{B.5c}$$

$$\delta(\mathbf{k},\mathbf{p},\mathbf{q}) = \begin{cases} 1 \text{ if } \mathbf{k}+\mathbf{p}+\mathbf{q}=0 \\ 0 \text{ otherwise.} \end{cases} \tag{B.5d}$$

Appendix C: Quasigeostrophic Model in Terms of Barotropic and Baroclinic Components

The nondimensional equation for 2-level baroclinic quasigeostrophic flow over topography on an *f*-plane may be written in the form:

$$\begin{aligned} \frac{\partial \zeta^0}{\partial t} &= -J(\psi^0, \zeta^0 + h^0) - J(\psi^1, \zeta^1 + h^1) - D_0^{0b}\zeta^b + f_0^0, \\ \frac{\partial \zeta^1}{\partial t} &= -J(\psi^0, \zeta^1 + h^1) - J(\psi^1, \zeta^0 + h^0) - D_0^{1b}\zeta^b + f_0^1 \end{aligned} \tag{C.1}$$

where we have taken suitable length and space scales. Here, ψ^0 and ψ^1 are the barotropic and baroclinic streamfunctions, $\zeta^0 = \nabla^2\psi^0$ and $\zeta^1 = \nabla^2\psi^1 - \Gamma\psi^1$ are the barotropic and baroclinic vorticity, $h^1 = -h^0$ and h^0 is *one half* the scaled topography, D_0^{0b} and D_0^{1b} are linear operators to be specified below and f_0^0 and f_0^1 are forcing functions. Also $\Gamma = 2F_L$ is twice the layer coupling parameter and is the inverse of the Rossby radius of deformation squared [32]. Again we obtain the spectral equations in the standard form:

$$\begin{aligned} \frac{\partial}{\partial t}\zeta_{\mathbf{k}}^a(t) + D_0^{\alpha\beta}(\mathbf{k})\zeta_{\mathbf{k}}^\beta(t) = &\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k},\mathbf{p},\mathbf{q})\left[K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^b(t)\zeta_{-\mathbf{q}}^c(t) + A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}^b(t)h_{-\mathbf{q}}^c\right] \\ &+ f_0^a(\mathbf{k},t) \end{aligned} \tag{C.2}$$

where now the superscripts are 0 or 1. In the above spectral equations:

$$k = |\mathbf{k}|, \tag{C.3a}$$

$$\zeta_{-\mathbf{k}}^a = \zeta_{\mathbf{k}}^{a*}, \tag{C.3b}$$

$$A^{000}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{101}(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)/p^2, \tag{C.3c}$$

$$A^{011}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{110}(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)/(p^2 + \Gamma), \tag{C.3d}$$

$$A^{010}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{001}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{100}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{111}(\mathbf{k},\mathbf{p},\mathbf{q}) = 0, \tag{C.3e}$$

$$K^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\left[A^{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) + A^{acb}(\mathbf{k},\mathbf{q},\mathbf{p})\right], \tag{C.3f}$$

$$\delta(\mathbf{k},\mathbf{p},\mathbf{q}) = \begin{cases} 1 \text{ if } \mathbf{k}+\mathbf{p}+\mathbf{q}=0 \\ 0 \text{ otherwise.} \end{cases} \tag{C.3g}$$

were * denotes complex conjugate.

As for the case of barotropic flow we can also generalize the equations to turbulent flow on a β-plane. The small scales are again coupled to a large scale streamfunction $-U^a(t)y$ at each level, where $U^a(t)$ are the large scale zonal velocities. The large scale flow can again be regarded as the (0,0) vorticity component and the equations written in the form (1) by generalizing the interaction coefficients as in Frederiksen and O'Kane [11]. In this case $A^{000}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{101}(\mathbf{k},\mathbf{p},\mathbf{q})$ are as given in Equations (18) to (23) of Frederiksen and O'Kane [11]. Also $A^{000}(\mathbf{k},\mathbf{p},\mathbf{q}) = A^{101}(\mathbf{k},\mathbf{p},\mathbf{q})$ have the same expressions but with $p^2 \to (p^2 + \Gamma)$ in the denominator. On the β-plane the linear operator $D_0^{a\beta}(\mathbf{k})$ includes the Rossby wave frequency as well as dissipation (see the corresponding barotropic results in Equations (15) and (16) of Frederiksen and O'Kane [11]).