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Black Hole Horizons and Thermodynamics: A Quantum Approach

Valter Moretti ^{1,*} and Nicola Pinamonti ²

¹ Dipartimento di Matematica, Università di Trento and Istituto Nazionale di Fisica Nucleare–Gruppo Collegato di Trento, via Sommarive 14 I-38050 Povo (TN), Italy

² Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica 1, I-00133 Roma, Italy; E-Mail: pinamont@mat.uniroma2.it

* Author to whom correspondence should be addressed; E-Mail: moretti@science.unitn.it.

Received: 24 June 2010; in revised form: 20 July 2010 / Accepted: 21 July 2010 /

Published: 23 July 2010

Abstract: We focus on quantization of the metric of a black hole restricted to the Killing horizon with universal radius r_0 . After imposing spherical symmetry and after restriction to the Killing horizon, the metric is quantized employing the chiral currents formalism. Two “components of the metric” are indeed quantized: The former behaves as an affine scalar field under changes of coordinates, the latter is instead a proper scalar field. The action of the symplectic group on both fields is realized in terms of certain horizon diffeomorphisms. Depending on the choice of the vacuum state, such a representation is unitary. If the reference state of the scalar field is a coherent state rather than a vacuum, spontaneous breaking of conformal symmetry arises and the state contains a Bose-Einstein condensate. In this case the order parameter fixes the actual size of the black hole with respect to r_0 . Both the constructed state together with the one associated with the affine scalar are thermal states (KMS) with respect to Schwarzschild Killing time when restricted to half horizon. The value of the order parameter fixes the temperature at the Hawking value as well. As a result, it is found that the quantum energy and entropy densities coincide with the black hole mass and entropy, provided the universal parameter r_0 is suitably chosen, not depending on the size of the actual black hole in particular.

Keywords: black hole; conformal field theory; quantum gravity

1. Introduction and Summary

After the work of Bekenstein and Hawking [1,2] defining the thermodynamical properties of black holes [3,4], microscopic explanations of thermodynamical features of black holes became a standard issue of modern theoretical physics. A microscopic explanation should throw some light on a possible quantum description of gravity. The holographic principle, proposed by 't Hooft and Susskind [5–8], suggests to search exactly on event horizons the quantum nature of the gravity. That is a reason why many scientists in the last decades have tackled the problem under different points of view obtaining some relevant but partial results [4,9,10].

Restricting to the case of spherical symmetry, we propose here an alternate approach, regarding canonical quantization (by means of a straightforward extension of the notion of chiral current) of some “components” of the metric when restricted to the horizon of a black hole. This approach is based on results previously obtained by the authors [12,13], using some of the ideas of [11], where the language and the mathematical tools of conformal nets of local observables were exploited profitably. It is worth remarking that the approach employed in this paper has recently been developed toward other very promising directions in quantum field theories in curved spacetime [14–18]. Herein, instead, we focus to quantum gravity. Our main idea is that the metric of a black hole, restricted to the Killing horizon with radius r_0 and represented in a suitable manner after imposing spherical symmetry, can be quantized with the procedures of chiral currents. We quantize two “components of the metric”: one behaves as an affine scalar fields under changes of coordinates and the other is a proper scalar field. However, the reference state of the latter, which, in fact fixes the black hole, is not a vacuum state but a coherent state arising from spontaneous breaking of conformal symmetry (which is a natural symmetry on null surfaces). That state (and the state associated with the affine field), when examined on a half horizon, turns out to be thermal (KMS) with respect to Schwarzschild Killing time (restricted to the horizon) and it contains a Bose-Einstein condensate. The value of order parameter individuates the actual size of the black hole with respect to r_0 , as well as the Hawking temperature. In spite of initial limitation to only treat the spherical case, we finally find that the densities, energy and entropy of this state coincide to the mass and the entropy of the black hole provided the universal parameter r_0 is fixed suitably (not depending on the size of the black hole).

Let us review some basic results of the theory of n -dimensional ($n \geq 3$) vacuum Einstein solution (with cosmological constant Λ) enjoying spherical symmetry. Such metrics have the following general form in a coordinate patch where $\eta > 0$

$$ds^2 = \frac{g_{ab}(x)}{\eta(x)^{\frac{n-3}{n-2}}} dx^a dx^b + \eta(x)^{\frac{2}{n-2}} d\Sigma^2(y). \quad (1)$$

Above $a, b = 0, 1$, moreover the “non-angular part of the metric” is given by the Lorentzian metric g (with signature $-, +$) which, together with the so-called *dilaton field* η , depends only on coordinates x^a . The “angular part of the metric” $d\Sigma^2 = h_{AB} dy^A dy^B$ with $A, B = 1, 2, \dots, n - 2$, depends on coordinates y^A only. h is the metric on a Riemannian $(n - 2)$ -dimensional space Σ and it is supposed to satisfy, if $R_{AB}[h]$ is Ricci tensor associated with h ,

$$R_{AB}[h] = \Gamma h_{AB}, \quad \text{with } \Gamma \text{ constant.} \quad (2)$$

Under these assumptions vacuum Einstein equations give rise to equations for the metric g and the dilaton η which can also be obtained by means of a variational principle. Starting from Hilbert-Einstein action for the complete metric (1) and integrating out the angular part discarding it, one obtains:

$$I[g, \eta] = \frac{2}{G} \int dx^0 dx^1 \sqrt{|detg|} \left\{ \eta \frac{R[g]}{2} + \mathcal{V}(\eta) \right\}. \tag{3}$$

That is the action of a 2-D dilatonic theory with dilatonic potential

$$\mathcal{V}(\eta) = \frac{\Gamma}{2\eta^{\frac{1}{n-2}}} - \Lambda \eta^{\frac{1}{n-2}}. \tag{4}$$

\mathcal{V} encodes all information about the n -dimensional original spacetime with cosmological constant Λ . The approach to Einstein equation based on the action (3) is called *dimensional reduction* [19–23]. We stress that, in spite of the reduction, the finally obtained 2-dimensional models share many properties with n -dimensional ones.

We can reduce the number of used fields by the following remark. It is well known that every Lorentzian 2-dimensional metric g is, at least locally, conformally equivalent to the flat metric $\gamma = diag(-1, 1)$ referred to Minkowski coordinates x^0, x^1 :

$$g_{ab}(x) = e^{\rho(x)} \gamma_{ab}(x). \tag{5}$$

(In several papers the exponent is defined as -2ρ instead of ρ). Working with null coordinates $x^\pm = x^0 \pm x^1$, so that $\gamma_{\pm\mp} = \gamma_{\mp\pm} = -1/2$ and $\gamma_{++} = \gamma_{--} = 0$, one has

$$g_{++} = g_{--} = 0, \quad g_{+-} = g_{-+} = -e^{\rho(x)}/2, \tag{6}$$

and vacuum Einstein equations become very simple:

$$\partial_+ \partial_- \eta + e^\rho \frac{\mathcal{V}(\eta)}{2} = 0, \quad \partial_\pm^2 \eta - \partial_{\pm\rho} \partial_\pm \eta = 0, \quad \partial_+ \partial_- \rho + \frac{e^\rho}{2} \frac{d\mathcal{V}(\eta)}{d\eta} = 0. \tag{7}$$

Starting from the action above as a functional of the fields η, ρ the variational procedure produces only two equations, the first and the last in (7), of the original four Einstein equations, the remaining ones can be imposed as constraints on the solutions. Equations (7) are completely integrable and the general solution depends on an arbitrary real function $\phi(x) = \phi^+(x^+) + \phi^-(x^-)$ and an arbitrary real constant C :

$$e^\rho = -\frac{F_C(\eta)}{2} \partial_+ \phi \partial_- \phi, \quad 2G_C(\eta) = \phi, \tag{8}$$

where η is assumed to satisfy $\eta > 0$ and

$$F_C(\eta) = \int_0^\eta \mathcal{V}(\alpha) d\alpha - C, \quad G_C(\eta) = \int \frac{1}{F_C(\eta)} d\eta. \tag{9}$$

The integration constant of the latter integral is included in the field ϕ in (8). The explicit expression for F_C reads

$$F_C(\eta) = \left(\frac{\Gamma}{2} \frac{n-2}{n-3} - \Lambda \frac{n-2}{n-1} \eta^{\frac{2}{n-2}} \right) \eta^{\frac{n-3}{n-2}} - C \quad \text{or} \quad F_C(\eta) = -\frac{\Lambda \eta^2}{2} - C \quad \text{if } n = 3. \tag{10}$$

With these definitions the metric (1) takes the following explicit form

$$ds^2 = \frac{F_C(\eta(\phi))}{2 \eta(\phi)^{\frac{n-3}{n-2}}} \partial_+ \phi \partial_- \phi \, dx^+ dx^- + \eta(\phi)^{\frac{2}{n-2}} d\Sigma^2. \tag{11}$$

In $I_+ \times I_- \times \Sigma$ (I_\pm being any pair of (open) segments where $\partial_\pm \phi \neq 0$, $F_C(\eta(\phi)) \neq 0$) the fields ϕ^\pm together with coordinates on Σ define a coordinate patch of the spacetime where

$$ds^2 = \frac{F_C(\eta(\phi))}{2 \eta(\phi)^{\frac{n-3}{n-2}}} d\phi^+ d\phi^- + \eta(\phi)^{\frac{2}{n-2}} d\Sigma^2. \tag{12}$$

As the metric depends only on $\phi^+ + \phi^-$, $\partial_{\phi^+} - \partial_{\phi^-}$ is a *Killing field*. This is a straightforward generalization of Birkhoff theorem [23]. The arbitrariness, due to an additive constant, in defining ϕ^+ and ϕ^- from ϕ does not affect the Killing field and it reduces to the usual arbitrariness of the origin of its integral curves.

As a comment, notice that in the three dimensional case ($n = 3$), the last equation in (7) becomes the field equation of a 2D *Liouville theory* for $\varrho = \rho/2$ with $k = -\Lambda/4 > 0$

$$-\partial_+ \partial_- \varrho + k e^{2\varrho} = 0. \tag{13}$$

Notice however that the action in (3) does not reduce to the usual Liouville action in this case. Restricting to the case $n = 4$ with $\Lambda = 0$ and $\Gamma = 2$, two relevant cases arise. If $C > 0$, the metric (12) is *Schwarzschild's* one with black-hole mass $M = C/4$, $r = \sqrt{\eta}$, $\phi/2$ is the ‘‘Regge-Wheeler tortoise coordinate’’ r_* and ϕ^\pm are the usual null coordinates. As F_C has a unique non-integrable zero, there are two inequivalent functions G_C corresponding to the internal singular metric and the external static metric respectively. $\partial_{\phi^+} - \partial_{\phi^-}$ defines Schwarzschild time in the external region. The case $C = 0$ is nothing but Minkowski spacetime. $\partial_{\phi^+} - \partial_{\phi^-}$ is the Killing field associated with Minkowski time, there is a unique function G_C , and coordinates ϕ^\pm are the usual global (radial) null coordinates with range $\phi^+ + \phi^- > 0$. Global structures are constructed gluing together solutions of Einstein equations. In particular, manifolds with *bifurcate Killing horizon* arise. Consider again the case $n = 4$, $C = 4M > 0$, $\Lambda = 0$, $\Gamma = 2$ (Schwarzschild black hole). In this cases one fixes global coordinates $X^+ \in \mathbb{R}$, $X^- \in \mathbb{R}$ such that the metric reduces to (12) in each of the four sectors $X^+ \leq 0, X^- \leq 0$. ϕ , ρ and η are functions of X^\pm defined as follows:

$$\phi(X^+, X^-) = 4M \left(1 + \ln \left| \frac{X^+ X^-}{32M^3} \right| \right), \quad \rho(X^+, X^-) = 1 - \frac{\sqrt{\eta(X^+, X^-)}}{2M}, \tag{14}$$

η is obtained by solving, for $0 < \eta < (2M)^2$ and $\eta > (2M)^2$ respectively, the equation

$$\sqrt{\eta(X^+, X^-)} + 2M \ln \left| \frac{\sqrt{\eta(X^+, X^-)}}{2M} - 1 \right| = \frac{\phi(X^+, X^-)}{2}. \tag{15}$$

The global metric (1) obtained in this way (with $x = X$) is smooth for $\eta(X^+, X^-) > 0$, $\eta = 0$ being the black/white-hole singularity. The spacetime obtained is maximally extended and $K = \partial_{\phi^+} - \partial_{\phi^-}$ turns out to be a Killing field smoothly defined globally which is light-like on a pair of 3-dimensional null hypersurfaces \mathbb{F} and \mathbb{P} . These hypersurfaces intersect at the compact 2-dimensional spacelike

submanifolds Σ , localized at $X_{\Sigma}^{\pm} = 0$, and are normal to it. Σ is called *bifurcation surface*. K vanishes exactly at the bifurcation surface. It turns out that $\eta|_{\mathbb{F}} = \eta|_{\mathbb{P}} = \eta_C > 0$ is *constant* and it is the *unique positive solution* of $F_C(\eta) = 0$. $\sqrt{\eta_C} = 2M$ is the Schwarzschild radius. Either \mathbb{F} and \mathbb{P} are diffeomorphic to $\mathbb{R} \times \Sigma$ with \mathbb{R} covered by the coordinate $X^+, X^- \in \mathbb{R}$ respectively. In coordinates X^{\pm} it holds:

$$\rho|_{\mathbb{F}} = \rho|_{\mathbb{P}} = 0. \tag{16}$$

Notice that ϕ, η, ρ depend on the product X^+X^- only. Therefore, passing to new global coordinates $X'^{\pm} = C_{\pm}X^{\pm}$ with constants C_{\pm} satisfying $C_+C_- = 1$, Equations (14)–(16) and $X_{\Sigma}^{\pm} = 0$ still hold for the considered metric replacing X^{\pm} with X'^{\pm} .

The point of view we wish to put forward is to promote (some of) the objects ρ, η and ϕ to quantum objects, *i.e.*, averaged values of associated quantum fields $\hat{\rho}, \hat{\eta}$ and $\hat{\phi}$ with respect to reference quantum states. Those quantum states do individuate the actual metric and in particular the mass of the black hole on a hand. On the other hand they should account for thermodynamical properties of black holes.

In the rest of the paper we adopt of Planck units so that $\hbar = c = G = k_B = 1$, in this way every physical quantity is a pure number.

2. Quantum Structures on the Horizon of a Black Holes

2.1. Geometrical Background and its Quantum Interpretation

To go on with our proposal we have to consider as separated objects part of the background manifold (not quantized) and part of the metric structure (at least partially quantized). More precisely we consider a 4-dimensional differentiable manifold \mathbb{M} diffeomorphic to $(\mathbb{R} \times \Sigma) \times (\mathbb{R} \times \Sigma) = \mathbb{R}^2 \times \Sigma$ such that a reference flat Lorentzian 2D metric γ is assigned in a global coordinate frame $(x^+, x^-) \in \mathbb{R} \times \mathbb{R}$ where the two factors \mathbb{R} are those in the decomposition of $\mathbb{M} = (\mathbb{R} \times \Sigma) \times (\mathbb{R} \times \Sigma)$. These coordinates, together with coordinates on Σ , describe respectively manifolds \mathbb{F} and \mathbb{P} . Every admissible metric on \mathbb{M} must be such that, **(C1)** it has the general structure (1) and in particular it enjoys S^2 -spherical symmetry, Σ being tangent to the associated Killing fields, **(C2)** it solves equations (7) with $C > 0$ (that is the mass M) fixed *a priori* in some way depending, at quantum level, on a quantum reference state as we shall discuss shortly, and **(C3)** $\mathbb{F} \cup \mathbb{P}$ is a bifurcate Killing horizon with bifurcation surface Σ .

A time orientation is also assumed for convenience by selecting one of the two disjoint parts of $\mathbb{F} \setminus \Sigma$ and calling it $\mathbb{F}_{>}$. The other will be denoted by $\mathbb{F}_{<}$. (In [12,13] we used notations \mathbb{F}_{\pm} instead of \mathbb{F}_{\gtrless} .)

There is quite a large freedom in choosing global coordinates $x^{\pm} \in \mathbb{R}$ on \mathbb{F}, \mathbb{P} respectively, such that the form of the metric (6) hold. In the following we call **admissible null global frames** those coordinate frames. It is simply proved that the following is the most general transformation between pairs of admissible null global frames provided η transforms as a scalar field (as we assume henceforth):

$$x'^+ = f_+(x^+), \quad x'^- = f_-(x^-) \quad \text{and} \quad \frac{df_+}{dx^+} \frac{df_-}{dx^-} > 0, \tag{17}$$

where the ranges of the functions f_{\pm} cover the whole real axis. We remark that preservation of the form of the metric (6) entails preservation of the form of equations (7). There are infinitely many possibilities to assign the metric fulfilling the constraints (C1), (C2), (C3). Considering Kruskal

spacetime, if $X_{\Sigma}^{\pm} = 0$ and if $f_{\pm} : \mathbb{R} \rightarrow \mathbb{R}$ are functions as in (17) with $f_{\pm}(0) = 0$, the fields $\phi'(X) = 4M \left(1 + \ln \left| \frac{f_+(X^+)f_-(X^-)}{16M^2} \right| \right)$ give rise to everywhere well-defined fields η and ρ using (8) and (9). The produced spacetime has a bifurcate Killing horizon with respect to the Killing vector $\partial_{\phi^+} - \partial_{\phi^-}$ (which has the temporal orientation of $\partial_{\phi^+} - \partial_{\phi^-}$) just determined by the initially assigned manifolds $\mathbb{F}, \mathbb{P}, \Sigma$. Different global metrics obtained from different choices of the functions f_{\pm} are however diffeomorphic, since they have the form of Kruskal-like metric (with the same mass) in admissible null global frame, $x^{\pm} = f_{\pm}(X^{\pm})$ in the considered case.

2.2. The Field ρ and the Interplay with ϕ on Killing Horizons

As a consequence of the decomposition $\phi(x) = \phi^+(x^+) + \phi^-(x^-)$, ϕ is a solution of d'Alembert equation $\square\phi = 0$, where \square is referred to the reference flat metric γ in any admissible null global frame. This field is a good candidate to start with a quantization procedure. In particular each component $\phi^{\pm}(x^{\pm})$ of ϕ could be viewed as a scalar quantum field on \mathbb{F} and \mathbb{P} respectively. Here we focus also attention of the field ρ and on the interplay between ρ and ϕ when restricted to the horizon.

Let us start by showing classical nontrivial properties of the field ρ and its restrictions $\rho|_{\mathbb{F}}$ and $\rho|_{\mathbb{P}}$. First of all consider transformations of coordinates (17) where, in general, we relax the requirement that coordinates x'^+, x'^- are global and we admit that the ranges of functions f_{\pm} may be finite intervals in \mathbb{R} . The field ρ transforms as

$$\rho(x'^+, x'^-) = \rho(x^+(x'^+), x^-(x'^-)) + \ln \frac{\partial(x^+, x^-)}{\partial(x'^+, x'^-)}, \tag{18}$$

where the argument of \ln is the Jacobian determinant of a transformation $x = x(x')$. (18) says that the field ρ transform as an affine scalar under changes of coordinates. (We notice *en passant* that, from (8) and (9) and the fact that η is a scalar field, (18) entails that ϕ is a scalar field as assumed previously.) A reason for the affine transformation rule (18) is that, for the metric g , only Christoffel symbols Γ_{++}^+ and Γ_{--}^- are non vanishing and

$$\partial_+\rho = \Gamma_{++}^+, \quad \partial_-\rho = \Gamma_{--}^-. \tag{19}$$

Remark. The reader should pay attention to the used notation. ρ should be viewed as a function of both the points of $\mathbb{F} \times \mathbb{P}$ and the used chart. If the chart \mathcal{C} is associated with the coordinate frame x^+, x^- , an appropriate notation to indicate the function representing ρ in \mathcal{C} could be $\rho(\mathcal{C}|x^+, x^-)$ or $\rho_{\mathcal{C}}(x^+, x^-)$. However we shall use the simpler, but a bit miss-understandable, notation $\rho(x^+, x^-)$. As a consequence, the reader should bear in his/her mind that, in general

$$\rho(x'^+, x'^-) \neq \rho(x^+(x'^+), x^-(x'^-)).$$

Form a classical point of view, on a hand $\rho|_{\mathbb{F}}$ and $\rho|_{\mathbb{P}}$ embody all information about the metric, since they determine completely it via Einstein equations, on the other hand these restrictions can be assigned freely. More precisely the following theorem holds.

Theorem 1. Working in a fixed admissible null global frame x^\pm on \mathbb{M} , if $\rho_+ = \rho_+(x^+)$, $\rho_- = \rho_-(x^-)$ are smooth bounded-below functions, there is a unique metric which satisfies (C1), (C2), (C3) (with assigned mass $M > 0$) and such that $\rho|_{\mathbb{F}} = \rho_+$ and $\rho|_{\mathbb{P}} = \rho_-$.

Proof. First of all we prove that if ds^2, \widetilde{ds}^2 are solutions of Einstein equation on \mathbb{M} satisfying (C1), (C2), (C3) (with a fixed value of the mass), they coincide if the restrictions of the respectively associated functions $\rho, \widetilde{\rho}$ to $\mathbb{F} \cup \mathbb{P}$ coincide working in some admissible null global frame x^\pm .

Indeed, using (18) one sees that if $\rho|_{\mathbb{F}} = \widetilde{\rho}|_{\mathbb{F}}$ and $\rho|_{\mathbb{P}} = \widetilde{\rho}|_{\mathbb{P}}$ in an admissible null global frame, these relations must hold in any other admissible null global frame. Hence we consider, for the metric ds^2 , the special admissible null global frame X^\pm introduced in the end of the introduction. In these coordinates it must hold $\rho|_{\mathbb{F}} = \widetilde{\rho}|_{\mathbb{F}} = \rho|_{\mathbb{P}} = \widetilde{\rho}|_{\mathbb{P}} = 0$. On the other hand, $\widetilde{\rho}|_{\mathbb{F}} = \widetilde{\rho}|_{\mathbb{P}} = 0$ is valid also in coordinates \widetilde{X}^\pm analog of X^\pm for the metric \widetilde{ds}^2 . Applying (18) for $\widetilde{\rho}$ with respect to the coordinate systems X^\pm and \widetilde{X}^\pm one easily finds (assuming that $\widetilde{X}_\Sigma^\pm = X_\Sigma^\pm = 0$ for convenience) $X^\pm = C_\pm \widetilde{X}^\pm$ such that the constants C_\pm satisfy $C_+ C_- = 1$. Therefore, in coordinates X^\pm, \widetilde{ds}^2 has the form individuated by (14) and (15) and thus it coincides with ds^2 . This facts is invariant under transformations (18) and so, in particular, the affine scalars ρ and ρ' coincide also in the initial reference frame. The $n = 3$ case is analog.

To conclude, let us prove the existence of a metric satisfying (C1), (C2), (C3) when restrictions of ρ to $\mathbb{F} \cup \mathbb{P}$ are assigned in a global null admissible coordinate frame. In the given hypotheses, by direct inspection one may build up a global transformation of coordinates $x^\pm \rightarrow X^\pm$ as in (17), such that $\rho_+(X^+) = \rho_-(X^-) = 0$ constantly. Now a well-defined metric compatible with the bifurcate Killing horizon structure can be defined as in (14). This metric is such that ρ reduces to $\rho_+(X^+)$ on \mathbb{F} and $\rho_-(X^-)$ on \mathbb{P} . Transforming back everything in the initial reference frame x^\pm , the condition $\rho|_{\mathbb{F}} = \rho_+, \rho|_{\mathbb{P}} = \rho_-$ turns out to be preserved trivially by transformations (17). The $n = 3$ case is analog. \square

Working in a fixed admissible global null coordinate frame, consider the restriction of ρ to \mathbb{F} : $\rho|_{\mathbb{F}}(x^+) = \rho(x^+, x_\Sigma^-)$. The transformation rule of $\rho|_{\mathbb{F}}(x^+)$ under changes of coordinates makes sense only if we consider a change of coordinates involving both x^+ and x^- . It is not possible to say how $\rho(x^+)$ transforms if only the transformation rule $x'^+ = f_+(x^+)$ is known whereas $x'^- = f_-(x^-)$ is not. This is because (18) entails

$$\rho(x'^+, x'_\Sigma^-) = \rho(x^+, x_\Sigma^-) + \ln \frac{dx^+}{dx'^+} + \ln \frac{dx^-}{dx'^-} \Big|_{x_\Sigma^-}. \tag{20}$$

However, since the last term in the right-hand side is constant on \mathbb{F} , the transformation rule for field $\partial_{x^+} \rho(x^+, x_\Sigma^-)$ is well-defined for changes of coordinates in \mathbb{F} only $x'^+ = f_+(x^+)$. This situation resembles that of ϕ . The restriction of ϕ to \mathbb{F} is ill defined due to the divergence of $\phi^-(x^-)$ at x_Σ^- (see (14)), whereas the restriction of $\partial_+ \phi$ is well-defined and it coincides $\partial_+ \phi^+$, the arbitrary additive constant in the definition of ϕ^+ being not relevant due to the presence of the derivative. The analogs hold replacing \mathbb{F} with \mathbb{P} .

2.3. Quantization

Quantization of ρ and ϕ in the whole spacetime would require a full quantum interpretation of Einstein equations, we shall not try to study that very difficult issue. Instead, we quantize $\rho|_{\mathbb{F}}$ (actually the derivatives of that field) which, classically, contains the full information of the metric but they are not constrained by any field equations. Similarly we quantize $\phi|_{\mathbb{F}} = \phi^+$ (actually its derivative) and we require that classical constraints hold for its mean value (with respect to that of $\hat{\rho}$), which is required to coincide with the classical field ϕ^+ (actually its derivative). We show that, in fact there are quantum states which fulfill this constraint and enjoy very interesting physical properties.

From now on we consider the quantization procedure for fields $\hat{\rho}_{\mathbb{F}}$ and $\hat{\phi}^+$ defined on the metrically degenerate hypersurface \mathbb{F} . Since we consider only quantization on \mathbb{F} and not on \mathbb{P} , for notational simplicity we omit the the indices \mathbb{F} and $+$ of $\hat{\rho}_{\mathbb{F}}$ and $\hat{\phi}^+$ respectively and we write $\hat{\rho}$ and $\hat{\phi}$ simply. Omitting complicated mathematical details, we adopt canonical quantization procedure on null manifolds introduced in [24,25] and developed in [12,13] for a real scalar field as ϕ based on Weyl algebra. This procedure gives rise to a nice interplay with conformal invariance studied in various contexts [12,13,24–26].

It is convenient to assume that $\hat{\rho}$ and $\hat{\phi}$ are function of x^+ but also of angular coordinates s on Σ : *The Independence from angular coordinates will be imposed at quantum level picking out a 2-dimensional spherically symmetric reference state.* Σ is supposed to be equipped with the metric of the 2-sphere with radius r_0 , it being a universal number to be fixed later. Notice that, as a consequence r_0 does not depend on the mass of any possible black hole. A black hole is selected by fixing a quantum state.

We assume that only transformations of coordinates which do not mix angular coordinates s and coordinate x^\pm are admissible. Under transformations of angular coordinates, $s' = s'(s)$, $\partial_{x^+}\hat{\rho}$ transform as a usual scalar field, whereas it transforms as a connection symbol under transformations of coordinates $x'^+ = x'^+(x^+)$ with positive derivative:

$$\partial_{x'^+}\hat{\rho}(x'^+, s') = \frac{dx^+}{dx'^+} \partial_{x^+}\hat{\rho}(x^+, s) + \frac{dx'^+}{dx^+} \frac{d^2x^+}{dx'^{+2}} I . \tag{21}$$

Conversely $\hat{\phi}$ transforms as a proper scalar field in both cases with the consequent transformation rule for $\partial_+\hat{\phi}$:

$$\partial_{x'^+}\hat{\phi}(x'^+, s') = \frac{dx^+}{dx'^+} \partial_{x^+}\hat{\phi}(x^+, s) . \tag{22}$$

Transformation rules for the field $\hat{\rho}$ are not completely determined from (21). However, as explained below it does not matter since the relevant object is $\partial_+\hat{\rho}$ either from a physical and mathematical point of view.

Fix an admissible (future oriented for convenience) null global frame (V, s) on $\mathbb{F} = \mathbb{R} \times \Sigma$. For sake of simplicity we assume that $V_\Sigma = 0$ so that the bifurcation surface is localized at the origin of the coordinate V on \mathbb{R} . In coordinate (V, s) Fock representations of $\hat{\phi}$ and $\hat{\rho}$ are obtained as follows [12,13]

in terms of a straightforward generalization of chiral currents (from now on, $n = 0, \pm 1, \pm 2, \dots$ and $j = 1, 2, \dots$ and where $\theta(V) = 2 \tan^{-1} V$):

$$\hat{\phi}(V, s) = \frac{1}{i\sqrt{4\pi}} \sum_{n,j} \frac{u_j(s)e^{-in\theta(V)}}{n} J_n^{(j)}, \tag{23}$$

$$\hat{\rho}(V, s) = \frac{1}{i\sqrt{4\pi}} \sum_{n,j} \frac{w_j(s)e^{-in\theta(V)}}{n} P_n^{(j)}. \tag{24}$$

As V ranges in \mathbb{R} , $\theta(V)$ ranges in $[-\pi, \pi]$. (The identification $-\pi \equiv \pi$ would make compact the horizon, which would become $\mathbb{S}^1 \times \Sigma$, by adding a point at infinity to every null geodesic on \mathbb{F} . This possibility will be exploited shortly in considering the natural action of the conformal group $PSL(2, \mathbb{R})$.) u_j and w_j are *real* and, separately, define Hilbert bases in $L^2(\Sigma, \omega_\Sigma)$ with measure $\omega_\Sigma = r_0^2 \sin \vartheta d\vartheta \wedge d\varphi$. There is no cogent reason to assume $u_j = w_j$ since the results are largely independent from the choice of that Hilbert basis. Operators $J_n^{(j)}, P_n^{(j)}$ are such that $J_0^{(j)} = P_0^{(j)} = 0$ and $J_n^{(j)\dagger} = J_{-n}^{(j)}, P_n^{(j)\dagger} = P_{-n}^{(j)}$ and oscillator commutation relations for two independent systems are valid

$$[J_n^{(j)}, J_{n'}^{(j')}] = n\delta^{jj'} \delta_{n,-n'} I, \tag{25}$$

$$[P_n^{(j)}, P_{n'}^{(j')}] = n\delta^{jj'} \delta_{n,-n'} I, \tag{26}$$

$$[J_n^{(j)}, P_{n'}^{(j')}] = 0. \tag{27}$$

The space of the representation is the tensor products of a pair of bosonic Fock spaces $\mathfrak{F}_\Psi \otimes \mathfrak{F}_\Upsilon$ built upon the vacuum states Ψ, Υ such that $J_n^{(j)}\Psi = 0, P_m^{(k)}\Upsilon = 0$ if $n, m \geq 0$, while the states with finite number of particles are obtained, in the respective Fock space, by the action of operators $J_n^{(j)}$ and $P_m^{(k)}$ on Ψ and Υ respectively for $n, m < 0$.

From a mathematical point of view it is important to say that the fields $\hat{\phi}(x^+, s)$ and $\hat{\rho}(x^+, s)$ have to be smeared by integrating the product of $\hat{\phi}(x^+, s)$, respectively $\hat{\rho}(x^+, s)$, and a differential form ω of shape

$$\omega = \frac{\partial f(x^+, s)}{\partial x^+} dx^+ \wedge \omega_\Sigma(y),$$

where f is a smooth real scalar field on \mathbb{F} compactly supported and ω_Σ is the volume-form on Σ defined above. There are several reasons [12,13,24,25] for justify this procedure, in particular the absence of a measure on the factor \mathbb{R} of $\mathbb{F} = \mathbb{R} \times \Sigma$: Notice that forms include a measure to be used to smear fields, for instance, the smearing procedure for $\hat{\phi}$ reads $\int_{\mathbb{F}} \hat{\phi}(V, s)\omega(V, s)$ simply. Moreover, this way gives rise to well-defined quantization procedure based on a suitable Weyl C^* -algebra [12,13]. Actually, concerning the field $\hat{\rho}$ another reason arises from the discussion about Equation (20) above. Using x^+ -derivatives of compactly supported functions to smear $\hat{\rho}$ it is practically equivalent, via integration by parts, to using actually the field $\partial_+ \hat{\rho}(x^+, s)$ which is well-defined concerning its transformation properties under changes of coordinates. Another consequence of the smearing procedure is the following. Relations (26) are equivalent to bosonic commutation relations for two independent systems

$$[\hat{\phi}(V_1, s_1), \hat{\phi}(V_2, s_2)] = \frac{-i}{4} \delta(s_1, s_2) \text{sign}(V_1 - V_2) I, \tag{28}$$

$$[\hat{\rho}(V_1, s_1), \hat{\rho}(V_2, s_2)] = \frac{-i}{4} \delta(s_1, s_2) \text{sign}(V_1 - V_2) I, \tag{29}$$

$$[\hat{\phi}(V_1, s_1), \hat{\rho}(V_2, s_2)] = 0. \tag{30}$$

Indeed, these relations arise from bosonic quantization procedure based on bosonic Weyl algebra constructed by a suitable symplectic form, see [12,13] for full details. Actually those relations have to be understood for fields smeared with forms as said above. Changing coordinates and using (21), these relations are preserved for the field $\hat{\rho}(x'^+, s')$ smeared with forms since the added term arising from (21) is a \mathbb{C} -number and thus it commutes with operators.

The mean values $\langle \Upsilon | \hat{\rho} | \Upsilon \rangle$, $\langle \Psi | \hat{\phi} | \Psi \rangle$ with respect quantum states Υ and Ψ respectively should correspond (modulo mathematical technicalities) to the classical function $\rho|_{\mathbb{F}}$ and $\phi|_{\mathbb{F}}$. Let us examine this case.

By construction $\langle \Upsilon | \hat{\rho}(V, s) \Upsilon \rangle = 0$. This suggest that the interpretation of the coordinate V must be the global coordinate along the future horizon X^+ introduced at the end of the introduction when the mean value of $\langle \Upsilon | \hat{\rho}(V, s) \Upsilon \rangle$ is interpreted as the restriction of the classical field ρ to \mathbb{F} . Indeed, in the coordinate X^+ , ρ vanishes on \mathbb{F} . Actually this interpretation should be weakened because the field must be smeared with forms to be physically interpreted. In this way $\langle \Upsilon | \hat{\rho}(V, s) \Upsilon \rangle = 0$ has to be interpreted more properly as $\langle \Upsilon | \partial_V \hat{\rho}(V, s) \Upsilon \rangle = 0$. Thus one cannot say that $V = X^+$ but only that $V = kX^+$ for some non vanishing constant k . However the coordinate X^+ is defined up to such a transformation provided the inverse transformation is performed on its companion X^- on \mathbb{P} (see the end of the introduction). Notice also that by construction $\rho|_{\mathbb{F}}(V) = \langle \Upsilon | \hat{\rho}(V, s) \Upsilon \rangle$ is spherically symmetric since it vanishes. From a semi-classical point of view at least, one may argue that the state Υ and the analog for quantization on \mathbb{P} referred to a global coordinate U , picks out a classical metric: It is the metric having the form determined by Equations (14),(15) in coordinates $X^+ = V, X^- = U$.

The interpretation of the mean value of $\hat{\phi}$ is much more intriguing. Working in coordinates V , from the interpretation of $\langle \Upsilon | \hat{\rho}(V, s) \Upsilon \rangle$ given above and using (14) one expects that the mean value of $\partial_V \hat{\phi}(V, s)$ coincides with ζ/V where $\zeta = 4M$. This is not possible if the reference state is Ψ . However, indicating the field $\hat{\phi}$ with $\hat{\phi}_\zeta$ for the reason explained below, as shown in [12,13] there is a new state Ψ_ζ completely defined from the requirement that it is *quasifree* (that is its n -point functions are obtained from the one-point function and the two-point function via Wick expansion) and

$$\langle \Psi_\zeta | \hat{\phi}_\zeta(V, s) \Psi_\zeta \rangle = \zeta \ln |V|, \tag{31}$$

$$\langle \Psi_\zeta | \hat{\phi}_\zeta(V, s) \hat{\phi}_\zeta(V', s') \Psi_\zeta \rangle = -\frac{\delta(s, s')}{4\pi} \ln |V - V'| + \zeta^2 \ln |V| \ln |V'| + R(V) + R(V'), \tag{32}$$

where the rests R are such that they gives no contribution when smearing both the fields with forms as said above. In practice, taking the smearing procedure into account, Ψ_ζ is the Fock vacuum state for the new field operator $\hat{\phi}_0$, with

$$\hat{\phi}_\zeta(V, s) = \hat{\phi}_0(V, s) + \zeta \ln |V| I. \tag{33}$$

Properly speaking the state Ψ_ζ cannot belong to \mathfrak{F}_Ψ because, as shown in [12,13], Ψ_ζ gives rise to a non-unitarily equivalent representation of bosonic commutation relation with respect to the representation given in \mathfrak{F}_Ψ . For this reason we prefer to use the symbol $\hat{\phi}_\zeta$ rather than $\hat{\phi}$ when working with the representation of CCR based on Ψ_ζ instead of Ψ . The picture should be handled in the framework of *algebraic quantum field theory* considering Ψ_ζ as a *coherent state* (see [12,13] for details). Notice that (31) reproduces the requested, spherically symmetric, classical value of $\partial_V \phi|_{\mathbb{F}} = \partial_V \phi^+$.

Remark. Generally speaking, in quantizing gravity one has to discuss how the covariance under diffeomorphisms is promoted at the quantum level. In our picture, the relevant class of diffeomorphisms is restricted to the maps (17). Their action at the quantum level is implemented through (21) and (22) in terms of a class of automorphisms of the algebra generated by the quantum fields $\hat{\phi}$ and $\hat{\rho}$. However, introducing quantum states, the picture becomes more subtle. We shall shortly see that such a symmetry will be broken, the remaining one being described by $PSL(2, \mathbb{R})$ or a subgroup.

2.4. *Properties of Ψ_ζ and $\hat{\phi}_\zeta$: Spontaneous Breaking of Conformal Symmetry, Hawking Temperature, Bose-Einstein Condensate*

Ψ_ζ with $\zeta \neq 0$ involves *spontaneous breaking of $PSL(2, \mathbb{R})$ symmetry*. This breaking of symmetry enjoys an interesting physical meaning we go to illustrate. Let us extend \mathbb{F} to the manifold $\mathbb{S}^1 \times \Sigma$ obtained by adding a point at infinity ∞ to every maximally extended light ray generating the horizon \mathbb{F} . On the circle \mathbb{S}^1 there is a well-known [12,13] natural geometric action $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm$ (called Möbius group of the circle) in terms of diffeomorphisms of the circle. Using global coordinates V, s the circle \mathbb{S}^1 is parametrized by $\theta \in [-\pi, \pi)$ with $V = \tan(\theta/2)$, so that ∞ corresponds to $\pm\pi$ and the bifurcation correspond to $\theta = 0$. Three independent vector fields generating the full action $PSL(2, \mathbb{R})$ group on the extended manifold $\mathbb{S} \times \Sigma$ are

$$\mathcal{D} = V\partial_V = \sin\theta\partial_\theta, \quad \mathcal{K} = \frac{2}{1+V^2}\partial_V = \partial_\theta, \quad \mathcal{H} = \partial_V = (1 + \cos\theta)\partial_\theta. \tag{34}$$

Integrating the transformations generated by linear combinations of these vectors one obtains the action of any $g \in PSL(2, \mathbb{R})$ on $\mathbb{S}^1 \times \Sigma$. g transforms $p \in \mathbb{S}^1 \times \Sigma$ to the point $g(p) \in \mathbb{S}^1 \times \Sigma$. (See [24,25] for the explicit expression of $g(p)$). Finally, the action of $PSL(2, \mathbb{R})$ on $\mathbb{S}^1 \times \Sigma$ induces an active action on fields:

$$\hat{\phi}_\zeta(p) \mapsto \hat{\phi}_\zeta(g^{-1}(p)), \quad \text{for every } g \in PSL(2, \mathbb{R}), \tag{35}$$

which preserves commutation relations. This is valid for any value of ζ , including $\zeta = 0$. Notice that all this structure is quite universal: the vector fields $\mathcal{D}, \mathcal{K}, \mathcal{H}$ do not depend on the state Ψ_ζ characterizing the mass of the black hole, but they depend only on the choice of the preferred coordinate V , that is Υ .

If $\zeta = 0$, it is possible to *unitarily implement* that action (35) of $PSL(2, \mathbb{R})$ on $\hat{\phi}$; in other words [12,13], there is a (strongly continuous) unitary representation U of $PSL(2, \mathbb{R})$ such that

$$U_g \hat{\phi}(p) U_g^\dagger = \hat{\phi}(g^{-1}(p)), \quad \text{for every } g \in PSL(2, \mathbb{R}).$$

Furthermore, it turns out that the state Ψ is *invariant under U* itself, that is

$$U_g \Psi = \Psi, \quad \text{for every } g \in PSL(2, \mathbb{R}).$$

To define U one introduces the *stress tensor*

$$\hat{T}(V, s) = : \partial_V \hat{\phi} \partial_V \hat{\phi} : (V, s).$$

The state Ψ enters the definition by the normal ordering prescription it being defined by subtracting $\langle \Psi | \hat{\phi}(V', s') \hat{\phi}(V, s) \Psi \rangle$ before applying derivatives and then smoothing with a product of delta in V, V' and s, s' . One can smear \hat{T} with a vector field $\mathcal{X} := X(V, s)\partial_V$ obtaining the operator

$$T[\mathcal{X}] = \int_{\mathbb{F}} X(V, s) \hat{T}(V, s) dV \wedge \omega_\Sigma.$$

It is possible to show [12,13] that the three operators, obtained by smearing \hat{T} with $\mathcal{D}, \mathcal{K}, \mathcal{H}$ respectively,

$$T[\mathcal{D}] = \frac{1}{4i} \sum_{j,k>0} :J_{-k}^{(j)} J_{k+1}^{(j)}: - :J_{-k}^{(j)} J_{k-1}^{(j)}:, \tag{36}$$

$$T[\mathcal{K}] = \frac{1}{2} \sum_{k \in \mathbb{Z}, j \in \mathbb{N}} :J_{-k}^{(j)} J_k^{(j)}:, \tag{37}$$

$$T[\mathcal{H}] = \frac{1}{4} \sum_{k \in \mathbb{Z}, j \in \mathbb{N}} 2 :J_{-k}^{(j)} J_k^{(j)}: + :J_{-k}^{(j)} J_{k+1}^{(j)}: + :J_{-k}^{(j)} J_{k-1}^{(j)}:, \tag{38}$$

are the very generators of the unitary representation U of $PSL(2, \mathbb{R})$ which implements the action of $PSL(2, \mathbb{R})$ on $\hat{\phi}(V, s)$ leaving fixed Ψ . They, in fact, generate the one-parameter subgroups of U associated with the diffeomorphisms due to vector fields $\mathcal{D}, \mathcal{K}, \mathcal{H}$ respectively. The fact that $T[\mathcal{D}], T[\mathcal{K}], T[\mathcal{H}]$ enjoy correct commutation relations is not enough to prove the existence of the unitary representation. Rather the existence is consequence of the presence of an invariant and dense space of analytic vectors for $T[\mathcal{D}]^2 + T[\mathcal{K}]^2 + T[\mathcal{H}]^2$ and know nontrivial theorems by Nelson. See [12,13,24–26] for details and references.

The normal ordering prescription for operators P_n is defined by that $:P_k P_h := P_h P_k$ if $h < 0$ and $k > 0$, or $:P_k P_h := P_k P_h$ otherwise.

All that is mathematically interesting, but it is unsatisfactory form a physical point of view if we want to describe classical geometric properties of the horizon as consequences of quantum properties. Indeed, in this way, the quantum picture admits a too large unitary symmetry group which exists anyway, no matter if the manifold is extended by adding the points at infinity or not. This larger group does not correspond to the geometrical shape of the physical manifold \mathbb{F} : The transformations associated with vector fields \mathcal{K} do not preserve the physical manifold \mathbb{F} , they move some points in the physical manifold to infinity. The transformations associated with vector field \mathcal{H} transforms \mathbb{F} into \mathbb{F} itself but they encompass translations of the bifurcation surface Σ which we have assumed to be fixed at the beginning. Only the vector \mathcal{D} may have a completely satisfactory physical meaning as it simply generates dilatations of the coordinate V transforming \mathbb{F} into \mathbb{F} itself and leaving fixed Σ . One expects that there is some way, at quantum level, to get rid of the physically irrelevant symmetries and that the unphysical symmetries are removed from the scenario once one has fixed the quantum state of a black hole. In fact this is the case. Switching on $\zeta \neq 0$ the situation changes dramatically and one gets automatically rid of the unphysical transformations picking out the physical ones. Indeed, the following result can be proved (it is a stronger version than Theorem 3.2 [12,13]).

Theorem 2. *If $\zeta \neq 0$, there is no unitary representation of the whole group $PSL(2, \mathbb{R})$ which unitarily implements the action of $PSL(2, \mathbb{R})$ on the field $\hat{\phi}_\zeta$ (33) referred to Ψ_ζ . Only the subgroup associated with \mathcal{D} admits unitary implementation*

$$e^{-i\tau H_\zeta} \hat{\phi}_\zeta(V, s) e^{i\tau H_\zeta} = \hat{\phi}_\zeta(e^{-\tau} V, s) \tag{39}$$

and Ψ_ζ is invariant under that unitary representation of the group

$$e^{-i\tau H_\zeta} \Psi_\zeta = \Psi_\zeta. \tag{40}$$

Sketch of Proof. If $\zeta \in \mathbb{R}$ is fixed arbitrarily, and ω varies in the class of the admissible real forms used to smear the field operator, the class of all of unitary operators in the Fock space based on Ψ_ζ

$$W_\zeta(\omega) = \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}\omega \right\} = \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}_0\omega + \zeta \ln |V|\omega \right\}$$

turns out to be irreducible (see [12,13]). If $g \in PSL(2, \mathbb{R})$, $W_\zeta(\omega) \mapsto W_\zeta(\omega^{(g)})$ denotes the geometric action of the group on the operators $W_\zeta(\omega)$. We know that, for $\zeta = 0$, this action can be unitarily implemented (Theorem 3.2 in [12,13]). This is equivalent to say that there is a unitary representation U of $PSL(2, \mathbb{R})$ such that

$$U_g \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}_0\omega \right\} U_g^\dagger = \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}_0\omega^{(g)} \right\}. \tag{41}$$

For that representation it holds $U_g\Psi_\zeta = \Psi_\zeta$. Suppose now that the action can be implemented for $\zeta \neq 0$ by means of the unitary representation of $PSL(2, \mathbb{R})$, $V^{(\zeta)}$. In other words

$$V_g^{(\zeta)} \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}_0\omega + \zeta \ln |V|\omega \right\} V_g^{(\zeta)\dagger} = \exp \left\{ i \int_{\mathbb{F}} \hat{\phi}_0\omega^{(g)} \right\} \exp \left\{ i \int_{\mathbb{F}} \zeta \ln |V|\omega^{(g)} \right\}. \tag{42}$$

Consider the unitary operator $S_g = U_g^\dagger V_g^{(\zeta)}$. Due to (41) and (42), one simply gets

$$S_g W_0(\omega) = e^{ic_{g,\omega}} W_0(\omega) S_g, \tag{43}$$

where $c_{g,\omega}$ is the real $\zeta \int_{\mathbb{F}} [\ln |V|(\omega^{(g)} - \omega)]$. From standard manipulations working with the spectral measure of S_g one finds that (43) implies, if P_E is any projector in the spectral measure of S_g :

$$P_E W_0(\omega) = e^{ic_{g,\omega}} W_0(\omega) P_E.$$

Since the spectral measure is complete and $W_0(\omega) \neq 0$, there must be some projector P_E such that $P_E W_0(\omega) \neq 0$ and $W_0(\omega) P_E \neq 0$. For for all those projectors the identity above is possible only for $c_{g,\omega} = 0$. Therefore every projection space (including those whose projectors do not satisfy $P_E W_0(\omega) \neq 0$ and $W_0(\omega) P_E \neq 0$) turns out to be invariant with respect to $W_0(\omega)$. The result is valid for every $W_0(\omega)$. This is impossible (since the considered operator form an irreducible class as said at the beginning) unless $S_g = e^{ia_g} I$ for some real a_g . In other words: $V_g = e^{ia_g} U_g$. Inserting it in (42) and comparing with (41) one finds that the constraints $c_{g,\omega} = 0$ must hold true, that is

$$\int_{\mathbb{F}} [\ln |V|(\omega^{(g)} - \omega)] = 0$$

for every $g \in PSL(2, \mathbb{R})$ and every smearing form ω . It has been established in the proof of Theorem 4.1 of [12,13] that this is possible if and only if g belongs to the one-parameter subgroup of $PSL(2, \mathbb{R})$ generated by \mathcal{D} . The unitary representation of that subgroup has been constructed explicitly finding (39) and (44). Moreover, in the same theorem, it has been similarly proved that Ψ_ζ is invariant under the action of that unitary representation. These results conclude the proof. \square

The thesis shows that the strongest notion of *spontaneously breaking of $(PSL(2, \mathbb{R}))$ symmetry* used in algebraic quantum field theory arises: There is a group of transformations (automorphisms), in our

case associated with $PSL(2, \mathbb{R})$, of the algebra of the fields which cannot be completely implemented unitarily. The self-adjoint generator H_ζ of the surviving group of symmetry turns out to be [12,13]:

$$H_\zeta = \int_{\mathbb{F}} V : \partial_V \hat{\phi}_0 \partial_V \hat{\phi}_0 : (V, s) dV \wedge \omega_\Sigma(s), \tag{44}$$

the normal ordering prescription being defined by subtracting $\langle \Psi_\zeta | \hat{\phi}_0(V', s') \hat{\phi}_0(V, s) \Psi_\zeta \rangle$ before applying derivatives (which is equivalent to subtract $\langle \Psi | \hat{\phi}(V', s') \hat{\phi}(V, s) \Psi \rangle$). This definition is equivalent to that expected by formal calculus:

$$H_\zeta = T_\zeta[\mathcal{D}] = \int_{\mathbb{F}} V \hat{T}_\zeta(V, s) dV \wedge \omega_\Sigma(s), \tag{45}$$

where

$$\hat{T}_\zeta(V, s) = : \partial_V \hat{\phi}_\zeta \partial_V \hat{\phi}_\zeta : (V, s),$$

with the above-defined notion of normal ordering, assuming linearity and $: \hat{\phi}_0 := \hat{\phi}_0$. Indeed, let v be the parameter of the integral curves of \mathcal{D} , so that $v = \ln |V|$ and $v \in \mathbb{R}, s \in \Sigma$ define a coordinate system on both $\mathbb{F}_>$ and $\mathbb{F}_<$ separately. Starting from (45), one has:

$$H_\zeta = \lim_{N \rightarrow +\infty} \left\{ \int_{\mathbb{F}_>} \chi_N(v) : \frac{\partial \hat{\phi}_0}{\partial v} \frac{\partial \hat{\phi}_0}{\partial v} : (V_+(v), s) dv \wedge \omega_\Sigma(s) + \zeta^2 A_0 \int_{\mathbb{R}} \chi_N(v) dv \right. \\ \left. - \int_{\mathbb{F}_<} \chi_N(v) : \frac{\partial \hat{\phi}_0}{\partial v} \frac{\partial \hat{\phi}_0}{\partial v} : (V_-(v), s) dv \wedge \omega_\Sigma(s) - \zeta^2 A_0 \int_{\mathbb{R}} \chi_N(v) dv \right\}, \tag{46}$$

where from now on $A_0 = 4\pi r_0^2$. Moreover $\chi_N(v)$ is a smooth function with compact support in the interior of $F_<$ and $F_>$ separately, which tends to the constant function 1 for $N \rightarrow +\infty$ and $V_\pm(v) = \pm e^v$. We have omitted a term in each line proportional to $(\partial_v \chi_N) \phi_0$ (using derivation by parts). Those terms on, respectively, $\mathbb{F}_<$ and $\mathbb{F}_>$ give no contribution separately as $N \rightarrow \infty$ with our hypotheses on χ_N . The remaining two constant terms at the end of each line in brackets cancel out each other and this computation shows that (45) is equivalent to (44).

Physically speaking, with the given definition of $\zeta > 0$, $\zeta^{-1} \mathcal{D}$ is just the restriction to \mathbb{F} of the Killing vector of the spacetime defining the static time of the external region of black holes. If, as above, v is the parameter of integral curves of \mathcal{D} , ζv itself is the limit of Killing time towards \mathbb{F} . At space infinity this notion of time coincides with Minkowski time. Let us restrict the algebra of observables associated with the field $\hat{\phi}$ to the region $\mathbb{F}_>$ where \mathcal{D} is future directed. This is done by smearing the fields with forms completely supported in $\mathbb{F}_>$. Therein one can adopt coordinates v, s as above obtaining:

$$\langle \Psi_\zeta | \partial_v \hat{\phi}_\zeta(v, s) \Psi_\zeta \rangle = \zeta, \tag{47}$$

$$\langle \Psi_\zeta | \partial_v \hat{\phi}_\zeta(v, s) \partial_{v'} \hat{\phi}_\zeta(v', s') \Psi_\zeta \rangle = -\frac{\delta(s, s')}{4\pi} \frac{e^{v-v'}}{(1 - e^{v-v'})^2}, \tag{48}$$

Take the above-mentioned smearing procedure into account and the fact that one-point and two-point functions reconstruct all n -point functions class as well. Therefore, from (47) and (48), it follows that that the n -point functions are invariant under \mathcal{D} displacements. Furthermore, performing Wick

rotation $v \rightarrow iv$, one obtains 2π periodicity in the variable v . This is nothing but the analytic version of well-known *KMS condition* [27–29]. These fact can be summarized as:

Theorem 3. *Every state Ψ_ζ (including $\zeta = 0$), restricted to the algebra of observables localized at $\mathbb{F}_>$, is invariant under the transformations generated by $\mathcal{D} = \partial_v$ and it is furthermore thermal with respect to the time v with inverse temperature $\beta = 2\pi$. As a consequence, adopting the physical “time coordinate” ζv which accounts for the actual size of the Black hole (enclosed in the parameter ζ), the inverse temperature β turns out to be just Hawking’s value $\beta_H = 8\pi M$.*

It is furthermore possible to argue that the state Ψ_ζ contains a Bose-Einstein condensate of quanta with respect to the generator of v displacements for the theory restricted to $\mathbb{F}_>$. We have provided different reasons for this conclusion in [12,13]. In particular the non-vanishing one-point function (31) is a typical phenomenon in Bose-Einstein condensation (see chapter 6 of [30]). The decomposition (33) of the field operator into a “quantum” $\hat{\phi}_0(v, s)$ part (with vanishing expectation value) and a “classical”, *i.e.* commuting with all the elements of the algebra, part $\zeta v I$, is typical of the theoretical description of a boson system containing a Bose-Einstein condensate; the classical part $\zeta v = \langle \Psi_\zeta | \hat{\phi}(v, s) \Psi_\zeta \rangle$ plays the role of a *order parameter* [30,31]. The classical part is responsible for the macroscopic properties of the state. Considering separately the two disjoint regions of \mathbb{F} , $\mathbb{F}_<$ and $\mathbb{F}_>$ and looking again at (46), H_ζ is recognized to be made of two contributions $H_\zeta^{(<)}$, $H_\zeta^{(>)}$ respectively localized at $\mathbb{F}_<$ and $\mathbb{F}_>$. The two terms have opposite signs corresponding to the fact that the Killing vector ∂_v changes orientation passing from $\mathbb{F}_<$ to $\mathbb{F}_>$. As

$$H_\zeta^{(>)} = \int_{\mathbb{F}_>} VT_\zeta(V, s) dV \wedge \omega_\Sigma(s) + \zeta^2 A_0 \int_{\mathbb{R}} dv \tag{49}$$

it contains the *classical* volume-divergent term

$$\langle \Psi_\zeta | H_\zeta^{(>)} \Psi_\zeta \rangle = \zeta^2 A_0 \int_{\mathbb{R}} dv .$$

This can be interpreted as the “macroscopic energy”, with respect to the Hamiltonian $H_\zeta^{(>)}$, due to the Bose-Einstein condensate localized at $\mathbb{F}_>$, whose density is *finite* and amounts to $\zeta^2 A_0$.

As a final comment we stress that, in [12,13], we have proved that any state Ψ_ζ defines an *extremal state* in the convex set of KMS states on the C^* -algebra of Weyl observable defined on $\mathbb{F}_>$ at inverse temperature 2π with respect to ∂_v and that different choices of ζ individuate not unitarily equivalent representations. The usual interpretation of this couple of results is that the states Ψ_ζ , restricted to the observables in the physical region $\mathbb{F}_>$, coincide with *different thermodynamical phases* of the same system at the temperature 2π (see V.1.5 in [27]).

2.5. Properties of Υ and $\hat{\rho}$: Feigin-Fuchs Stress Tensor

Let us consider the realization of CCR for the field $\hat{\rho}$ in the Fock representation based on the vacuum vector Υ which singles out the preferred admissible null coordinate V . In this case there is no spontaneous breaking of symmetry. However, due to the particular affine transformation rule (21) of

the field $\hat{\rho}$, there are anyway some analogies with the CCR realization for the field $\hat{\phi}$ referred to the state Ψ_ζ . Using the coordinate patches (v, s) on \mathbb{F}_+ with $\partial_v = \mathcal{D}$ and exploiting (21), the field takes the form

$$\hat{\rho}(v, s) = \hat{\rho}(V(v), s) + \ln |V| I . \tag{50}$$

This equation resembles (33) with $\zeta = 1$ and thus one finds in particular:

$$\langle \Upsilon | \partial_v \hat{\rho}(v, s) \Upsilon \rangle = 1 , \tag{51}$$

$$\langle \Upsilon | \partial_v \hat{\rho}(v, s) \partial_{v'} \hat{\rho}(v', s') \Upsilon \rangle = -\frac{\delta(s, s')}{4\pi} \frac{e^{v-v'}}{(1 - e^{v-v'})^2} . \tag{52}$$

As a consequence, analogous comments on the interplay of state Υ and the algebra of fields $\rho(v, s)$ (notice that they are defined in the region $\mathbb{F}_>$) may be stated. In particular:

Theorem 4. *The state Υ restricted to the algebra of observables localized at $\mathbb{F}_>$ turns out to be a thermal (KMS) state with respect to $\partial_{\zeta v}$ at Hawking temperature.*

We want now to focus on the stress tensor generating the action of $SL(2, \mathbb{R})$ on the considered affine field. Fix an admissible global null coordinate frame inducing coordinates (x^+, s) on \mathbb{F} . A stress tensor, called *Feigin-Fuchs stress tensor* [32], can be defined as follows.

$$\hat{\mathbf{T}}(x^+, s) = : \partial_{x^+} \hat{\rho} \partial_{x^+} \hat{\rho} : (x^+, s) - 2\alpha \partial_{x^+} \partial_{x^+} \hat{\rho}_{x^+}(x^+, s) . \tag{53}$$

The normal ordered product with respect to $\Upsilon, : \partial_{x^+} \hat{\rho} \partial_{x^+} \hat{\rho} : (x^+, s)$, is defined by taking the limit for $(x, z) \rightarrow (x^+, s)$ of $: \partial_{x^+} \hat{\rho}(x, z) \partial_{x^+} \hat{\rho}(x^+, s) :$, the latter being defined as

$$\partial_{x^+} \hat{\rho}(x, z) \partial_{x^+} \hat{\rho}(x^+, s) - \langle \Upsilon | \partial_{x^+} \hat{\rho}(x, z) \partial_{x^+} \hat{\rho}(x^+, s) \Upsilon \rangle + \langle \Upsilon | \partial_{x^+} \hat{\rho}(x, z) \Upsilon \rangle \langle \Upsilon | \partial_{x^+} \hat{\rho}(x^+, s) \Upsilon \rangle .$$

The last term, which vanishes with our choice for Υ if working in coordinates V, s , is necessary in the general case to reproduce correct affine transformations for $: \partial_{x^+} \hat{\rho}(x, z) \partial_{x^+} \hat{\rho}(x^+, s) :$ under changes of coordinates for each field in the product separately. The stress tensor can be smeared with vector fields $\mathcal{X} = X(x^+, s) \partial_{x^+}$:

$$\mathbf{T}[\mathcal{X}] = \int_{\mathbb{F}} X(x^+, s) \hat{\mathbf{T}}(x^+, s) dx^+ \wedge \omega_\Sigma .$$

As a consequence one obtains (where it is understood that the fields are smeared with forms as usual)

$$\delta \hat{\rho}(x^+, s) = -i [\mathbf{T}[\mathcal{X}], \hat{\rho}(x^+, s)] = X(x^+, s) \partial_{x^+} \hat{\rho}(x^+, s) + \alpha \partial_{x^+} X(x^+, s) , \tag{54}$$

which is nothing but the infinitesimal version of transformation (21) provided $\alpha = 1$.

It is worth to investigate whether or not $\mathbf{T}[\mathcal{D}], \mathbf{T}[\mathcal{K}], \mathbf{T}[\mathcal{H}]$ are the self-adjoint generators of a unitary representation which implements the active action of $PSL(2, \mathbb{R})$ on the field $\hat{\rho}(V, s)$:

$$\hat{\rho}(V, s) \mapsto \hat{\rho}(g^{-1}(V), s) + \ln \frac{dg^{-1}(V)}{dV} , \quad \text{for any } g \in PSL(2, \mathbb{R}) .$$

The answer is interesting: once again spontaneous breaking of $PSL(2, \mathbb{R})$ symmetry arises but now the surviving subgroup is larger than the analog for $\hat{\phi}$. Indeed, the following set of result can be proved with dealing with similarly to the proof of Theorem 2.

Theorem 5. Working in coordinates V, s and referring to the representation of $\hat{\rho}$ based on Υ :

(1) there is no unitary representation of $PSL(2, \mathbb{R})$ which implements the action of the whole group $PSL(2, \mathbb{R})$ on the field $\hat{\rho}$.

(2) There is anyway a (strongly continuous) unitary representation $U^{(\Delta)}$ of the 2-dimensional subgroup Δ of $PSL(2, \mathbb{R})$ generated by \mathcal{D} and \mathcal{H} together, which implements the action of Δ on the field $\hat{\rho}(V, s)$.

$$U_g^{(\Delta)} \hat{\rho}(V, s) U_g^{(\Delta)\dagger} = \hat{\rho}(g^{-1}(V), s) + \ln \frac{dg^{-1}(V)}{dV}, \quad \text{for any } g \in \Delta.$$

The self-adjoint generators of $U^{(\Delta)}$ are $\mathbf{T}[\mathcal{D}]$ and $\mathbf{T}[\mathcal{H}]$ (with $\alpha = 1$).

(4) Υ is invariant under $U^{(\Delta)}$.

Notice that \mathcal{D} and \mathcal{H} form a sub Lie algebra of that of $PSL(2, \mathbb{R})$, whereas the remaining couples in the triple $\mathcal{D}, \mathcal{H}, \mathcal{K}$ do not. The explicit form of the generators $\mathbf{T}[\mathcal{D}]$ and $\mathbf{T}[\mathcal{H}]$ can be obtained in function of the operators $P_k^{(j)}$. With the same definition of normal ordering for those operators as that given for operators $J_n^{(j)}$, one has:

$$\mathbf{T}[\mathcal{D}] = \frac{1}{4i} \sum_{n \in \mathbb{Z}, j \in \mathbb{N}} :P_{-n}^{(j)} P_{n+1}^{(j)}: - :P_{-n}^{(j)} P_{n-1}^{(j)}:, \tag{55}$$

$$\mathbf{T}[\mathcal{H}] = \frac{1}{4} \sum_{n \in \mathbb{Z}, j \in \mathbb{N}} :P_{-n}^{(j)} P_{n+1}^{(j)}: + :P_{-n}^{(j)} P_{n-1}^{(j)}: + 2 :P_{-n}^{(j)} P_n^{(j)}:. \tag{56}$$

Dropping the dependence on s , $\mathbf{T}(V, s)$ defined in (53) is the stress tensor of a 1-dimensional *Coulomb gas* [32]. As is well known it does not transform as a tensor: By direct inspection one finds that, under changes of coordinates $x^+ \rightarrow x'^+$,

$$\hat{\mathbf{T}}(x'^+, s) = \left(\frac{\partial x^+}{\partial x'^+} \right)^2 \hat{\mathbf{T}}(x^+, s) - 2\alpha^2 \{x^+, x'^+\}, \tag{57}$$

where $\{z, x\}$ is the Schwarzian derivative (which vanishes if $x^+ \rightarrow x'^+$ is a transformation in $PSL(2, \mathbb{R})$)

$$\{z, x\} = \frac{\frac{d^3 z}{dx^3}}{\frac{dz}{dx}} - \frac{3}{2} \left(\frac{\frac{d^2 z}{dx^2}}{\frac{dz}{dx}} \right)^2.$$

The coefficient in front of the Schwarzian derivative in (57) differs from that found in the literature (e.g. see [32]) also because here we use a normal ordering procedure referred to unique reference state, Υ , for all coordinate frames. We stress that, anyway, Υ is the vacuum state only for coordinate V, s . Let us restrict ourselves to $\mathbb{F}_>$ and use coordinate v with $\partial_v = \mathcal{D}$ therein. If $x^+ = V$ and $x'^+ = v$ one finds by (57)

$$\hat{\mathbf{T}}(v, s) = \left(\frac{\partial V}{\partial v} \right)^2 \hat{\mathbf{T}}(V, s) + 1. \tag{58}$$

The formally self-adjoint generator for the field $\hat{\rho}(v, s)$, defined on $\mathbb{F}_>$ and generating the transformations associated with the vector field $\mathcal{D} = V\partial_V = \partial_v$, is

$$\mathbf{H}^{(>)} = \int_{\mathbb{F}_>} \hat{\mathbf{T}}(v, s) dv \wedge \omega_\Sigma .$$

From (58) one finds:

$$\mathbf{H}^{(>)} = \int_{\mathbb{F}_>} V \hat{\mathbf{T}}(V, s) dV \wedge \omega_\Sigma + 1 A_0 \int_{\mathbb{R}} dv , \tag{59}$$

This formula strongly resembles (49) for $\zeta = 1$ also if it has been obtained, mathematically speaking, by a completely different way and using the field $\hat{\rho}$ with property of transformations very different than those of the scalar $\hat{\phi}_\zeta$.

3. Thermodynamical Quantities: Free Energy and Entropy

Assume that the states Υ and Ψ_ζ are given and let $v_\zeta = \zeta v$, v being the integral parameter of \mathcal{D} on $\mathbb{F}_>$. If ϕ^+ denotes the classical field restricted to \mathbb{F} , one has $\phi^+(v) = \langle \hat{\phi}_\zeta(v) \rangle = v_\zeta$ and this is in agreement with the fact that $\partial_{\phi^+} - \partial_{\phi^-}$ is the Killing field defining Schwarzschild time in spacetime (see section 2.2). The temperature of the state Ψ_ζ coincides with Hawking one when referring to the “time” v_ζ . Therefore let us focus attention on the generator of v_ζ displacements $\zeta^{-1}H_\zeta^{(>)}$ whose “density of energy”, due to the condensate, is

$$\langle \Psi_\zeta | \zeta^{-1} H_\zeta^{(>)} | \Psi_\zeta \rangle / \int_{\mathbb{R}} dv = \zeta A_0 .$$

We try to give some physical interpretation to that density of energy. First of all notice that we are considering a system containing Bose-Einstein condensate at temperature $\beta_H^{-1} > 0$. This picture has to be discussed in the approach of grand canonical ensemble in the thermodynamical limit and the chemical potential μ must vanish in this situation. In this context the generator $\zeta^{-1}H_\zeta^{(>)}$ which generate the one-parameter group of transformations verifying KMS conditions is that of a grand canonical ensemble and its averaged value has to be interpreted as the density of *free energy* of the system (see chapter V of [27]) rather than its energy. Notice that the density is computed with respect to the parameter v which is universal, not depending on ζ , and valid for every black hole. We recall the reader that $\beta_H = 8\pi M$ and $\zeta = 4M$. We conclude that

$$F(\beta_H) := \zeta(\beta_H)A_0$$

is a density of free energy. Concerning the densities of energy and entropy one has:

$$E = \frac{\partial}{\partial \beta_H} \beta_H F , \quad S = \beta_H^2 \frac{\partial}{\partial \beta_H} F . \tag{60}$$

where some terms in the right hand side have been dropped because they are proportional to $\mu = 0$. For the case $n = 4$, fixing the universal parameter r_0 as $r_0 = 1/(4\sqrt{\pi})$ one gets, if M is the mass of the black hole and A the area of its horizon:

$$F = \frac{M}{2} , \quad E = M , \quad S = 4\pi M^2 = \frac{A}{4} .$$

4. Final Comments and Open Issues

Even if we have focused on the S -wave sector, which captures just the radial modes of the metric leaving outside, for instance, gravitational waves, the results presented in (60) are suggestive and one may hardly think that they arise by chance. There are anyway two problems to tackle in order to be confident in our approach to understand black hole thermodynamics from a quantum point of view. First of all the parameter r_0 is universal but there is no way to fix it at the beginning, within our approach. However it remains that the densities of energy and entropy scale as the energy and the entropy of black holes modulo r_0 which does not depend on the size of the black hole. The second point concerns the fact that E and S are *densities* of energy and entropy, but they are compared with energy and entropy of black holes. These densities are evaluated with respect to an universal—and dimensionless if introducing dimensions—parameter v , which is proper of the arena where to represent all different black holes (each depending on its own value ζ of the order parameter used to break the conformal symmetry). Notice also that the densities are referred to observables *homogeneously* spread along the Killing horizon, that is the evolution in time of the 2-sphere defining the horizon of the black hole at fixed time. A Cauchy surface for the whole Kruskal spacetime intersect, at every time, the Killing horizon in such a 2-sphere (not necessarily the same). Usually handled quantities of black holes are referred to that 2-sphere. If a relation exists between those two classes of quantities (spread on the whole horizon or defined on the 2-sphere) it is reasonable that quantities defined on the $2D$ sections of \mathbb{F} are the densities of the corresponding ones homogeneously spread along \mathbb{F} . However this issue deserves further investigation.

Further investigation are also necessary to translate horizon quantization proposed here to that presumably existing in the bulk. If this task seems to be straightforward regarding the field ϕ , it seems to be very difficult for the field ρ due to Einstein equations. To this end it is worthwhile stressing that, in the 3-dimensional case, ρ is a Liouville field in the bulk whose quantization is not simple at all. In this case it seems that the horizon fields $\rho_{\mathbb{F}}$ could play the role of a chiral current emerging from canonical quantization of the Liouville fields. In the general case the situation is also more complicated because of the presence of the field η . It enters the equation of motion of ρ , so that, quantization of η needs to be considered as well.

In this paper, we have considered the field ρ and ϕ as almost independent. Actually, on the horizon the following classical equation for classical fields holds:

$$\partial_+ \rho = \zeta^{-1} \partial_+ \phi^+ + \frac{\partial_+^2 \phi^+}{\partial_+ \phi^+}, \quad (61)$$

$$\frac{1}{2} \mathcal{V}(\eta_C) = \zeta^{-1}. \quad (62)$$

These relations are nothing but the Einstein equation on the horizon. The requirement $\phi^+ = \zeta v$ is nothing but a solution of that equation in suitable coordinates. We have considered it as a relation valid for the expectation value of the field. A posteriori (61) and (62) have to be considered as a kind of thermodynamical relations. Their meaning or, more appropriately, the corresponding equations at quantum level controlling the fields $\hat{\rho}$ and $\hat{\phi}$ are not yet understood.

As a final comment we notice that $\hat{\phi}$ may be viewed as a non-commutative light-coordinate on the horizon, in fact on the state implementing symmetry breaking the expectation value of $\langle \hat{\rho}(v) \rangle = \zeta v$ defines a preferred coordinate ζv on the horizon. This issue deserves further investigation.

To conclude we want to make a technical comment concerning the possibility of constructing a unitarizable representation of Virasoro algebra using the Feigin-Fuchs stress tensor (53) and smearing it with the complex vector fields

$$\mathcal{L}_n = i e^{in\theta} \frac{\partial}{\partial \theta}. \tag{63}$$

Here we have added the point at infinity to the light geodesics of \mathbb{F} obtaining the extended (unphysical) manifold $\mathbb{S}^1 \times \Sigma$. If $\mathbb{S}^1 = [-\pi, \pi)$ with $-\pi \equiv \pi$ and θ ranges in \mathbb{S}^1 , $V = \tan(\theta/2)$. The fields $\{\mathcal{L}_n\}_{n \in \mathbb{Z}}$ enjoy Virasoro commutation relations without central charge and satisfy Hermiticity condition with respect to the involution $\iota(X) = -\overline{X}$, respectively

$$\{\mathcal{L}_n, \mathcal{L}_m\} = (n - m)\mathcal{L}_{n+m}, \quad \iota(\mathcal{L}_n) = \mathcal{L}_{-n}, \tag{64}$$

$\{\cdot, \cdot\}$ denoting the usual Lie bracket (see [33] and Section III of [12,13] for further details). However, a direct computation shows that, if $\mathbf{T}[\mathcal{L}_n]$ is that referred to the preferred coordinates V, s defining the vacuum Υ ,

$$\mathbf{T}[\mathcal{L}_n] = \frac{1}{2} \sum_{k \in \mathbb{Z}, j \in \mathbb{N}} P_{-k}^{(j)} P_{k+n}^{(j)} - i\alpha\sqrt{4\pi A_0} \left[nP_n^{(0)} + \sum_{k \in \mathbb{Z}} (-1)^k \sigma_k P_{k+n}^{(0)} \right] \tag{65}$$

where $j = 0$ individuates the constant function among the orthonormal complete set $\{u_j\}_{j \in \mathbb{N}}$, $\sigma_0 = 0$ otherwise σ_k is the sign of $k \in \mathbb{Z}$. It is simply proved that, in general, the commutator of $\mathbf{T}[\mathcal{L}_n]$ and $\mathbf{T}[\mathcal{L}_m]$ produces an infinite constant term among other operatorial terms. In general it is possible to cancel out difficult terms using suitable linear combinations of operators $\mathbf{T}[\mathcal{L}_n]$, in particular those corresponding to $\mathbf{T}[\mathcal{D}]$ and $\mathbf{T}[\mathcal{H}]$. The reason is that not all diffeomorphisms of the circle preserve the physical manifold \mathbb{F}_+ . Only those which do it can be represented by means of \mathbf{T} .

Acknowledgements

The work of N.P. is supported in part by the ERC Advanced Grant 227458 OACFT ‘‘Operator Algebras and Conformal Field Theory.’’

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