Review

Black Hole Entropy in Scalar-Tensor and $f(R)$ Gravity: An Overview

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Abstract: A short overview of black hole entropy in alternative gravitational theories is presented. Motivated by the recent attempts to explain the cosmic acceleration without dark energy, we focus on metric and Palatini $f(R)$ gravity and on scalar-tensor theories.

Keywords: black hole entropy; alternative gravity; metric and Palatini modified gravity

1. Introduction

Alternative theories of gravity have been studied for a long time [1–5]. While there is no experimental evidence of deviations from general relativity at Solar System scales, from the theoretical point of view it is well known that corrections to the Einstein-Hilbert action (such as non-linear terms in the Ricci and Riemann tensors, or extra scalar fields coupling explicitly to gravity) always arise from attempts to renormalize general relativity, to build a theory of quantum gravity or at least some effective action (including the low-energy limit of string theories), or simply from the quantization of scalar fields in curved spacetimes [6], or even from purely classical arguments [7,8]. From the observational point of view, the 1998 discovery that the cosmic expansion accelerates today [9–17] makes it necessary, if one wants to remain within the bounds of general relativity, that 96% of the energy content of the universe be in the exotic form of dark energy with equation of state $P \sim -\rho$ (where ρ and P are the energy density and pressure of the cosmic fluid, respectively), unless one wants to admit a cosmological constant of incredibly small magnitude and then face the cosmological constant problem(s) (for a bibliography on dark energy see Reference [18]). Such explanations are unpalatable to many authors and this fact has provoked a revival of interest in alternative gravity theories: Perhaps the explanation of the cosmic acceleration lies in the fact

that we do not understand gravity at the largest scales. This possibility has led to extensive recent literature on metric $f(R)$ gravity [19–21], Palatini modified gravity [22], and the metric-affine version of these theories [23–27] (see [28,29] for reviews and [2,30–35] for shorter introductions). Metric and Palatini (but not metric-affine) modified gravities can be reduced to scalar-tensor theories [36–39], as explained below.

Black hole thermodynamics [40–42] constitutes an important development of modern theoretical physics and one of the main motivations for its study is the hope that something will be learned about quantum gravity and the construction of the statistical mechanics underlying this macroscopic thermodynamics. Black hole thermodynamics extends well beyond Einstein's theory of gravity; indeed, if there is hope to learn about quantum gravity by studying black hole thermodynamics, it will be necessary to understand it in extensions of Einstein's theory given that quantum corrections, renormalization, effective theories, and the low-energy limit and string theories all introduce extra gravitational scalar fields, non-minimal couplings with the curvature, and higher derivative corrections to general relativity. It is also possible that the stability of black hole thermodynamics with respect to perturbations of the Einstein-Hilbert action selects preferred classes of theories [43]. In this sense, black holes can be regarded as "theoretical laboratories" for quantum gravity.

The thermodynamics of black hole (apparent and event) horizons in general relativity inspired the construction of a thermodynamics of spacetime by Jacobson using local Rindler horizons and assuming the entropy-area relation $S = \frac{A}{4G}$, where G is Newton's constant (we use units in which the speed of light c and the reduced Planck constant $\hbar$ assume the value unity) [44]. Jacobson was able to derive the Einstein equations from such a macroscopic description [44], showing that the field equations of GR are akin to a macroscopic effective equation of state. The obvious implication of this derivation is that, if this picture is correct, quantizing the Einstein equations would make little sense, the same way that it would make no sense to quantize the equation of state of a (monoatomic) hydrogen gas in order to learn about the quantum states of the hydrogen atom.

The derivation of the field equations using the thermodynamics of local Rindler horizons has been performed also for metric $f(R)$ gravity [45]. In this derivation, $f(R)$ corrections to the Einstein-Hilbert action appear to describe non-equilibrium thermodynamics [45] (but see also [46,47]). Viewing the field equations as macroscopic equations and gravity as an emergent phenomenon is possible also in Lanczos-Lovelock and Gauss-Bonnet gravity [48–52].

The first law of black hole thermodynamics for event horizons in general relativity is

$$T\delta S = \delta M - \Omega_H \delta J + \ ... \tag{1}$$

where T and S are the horizon temperature and entropy, M and J are the hole mass and angular momentum measured at spatial infinity, and Ω_H is the angular velocity of the horizon, while the ellipsis denote extra terms which appear if the black hole possesses other charges. The first law relates the quantities M and J measured at infinity with the local quantities S, T, A, and Ω_H on the horizon. This property of the first law still holds true in alternative theories of gravity, as emphasized in [43], but the expression of the entropy S_{BH} must be changed in these theories. The fact that the expression $S_{BH} = \frac{A}{4G}$ does not hold in alternative gravity has been known since the 1980s [53–56,58–60].

Various techniques have been developed to compute black hole entropy, including Wald's Noether charge method [43,58,61–63], field redefinition techniques [43], and the Euclidean path integral approach [64]. Wald's Noether charge method relies on a Lagrangian formulation of the first law of black hole thermodynamics and is applicable to stationary black holes with bifurcate Killing horizons in any relativistic theory of gravity with diffeomorphism invariance in arbitrary spacetime dimension. This method was applied to black holes in Palatini modified gravity [65], metric $f(R)$ gravity, and in other gravitational theories [66,67].

Due to the recent interest in modified gravity coming from cosmology, in the following we focus on the Bekenstein-Hawking entropy [68,69] in scalar-tensor and modified gravity. Before proceeding, we recall the equivalence between metric and Palatini $f(R)$ gravity and scalar-tensor theories in the next subsection. This discussion also serves the purpose of establishing the notations used in the following sections.

1.1. Metric and Palatini $f(R)$ gravity as scalar-tensor theories

The equivalence between metric and Palatini $f(R)$ theories and scalar-tensor gravity has been discussed and rediscovered many times [70–75]. The action of metric $f(R)$ gravity is [19–21]

$$I_{metric} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(R) + I^{(matter)} \tag{2}$$

where $\kappa = 8\pi G$. Variation with respect to the (inverse) metric tensor $g^{\mu\nu}$ yields the field equations

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}\, g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + \kappa\, T_{\mu\nu} \tag{3}$$

where a prime denotes differentiation with respect to R. If $f''(R) \neq 0$, introduce the auxiliary scalar field $\phi = R$ and consider the action

$$I = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, [\psi(\phi)R - V(\phi)] + I^{(matter)} \tag{4}$$

where

$$\psi(\phi) = f'(\phi)\,, \qquad V(\phi) = \phi f'(\phi) - f(\phi) \tag{5}$$

This action reduces trivially to (2) if $\phi = R$. *Vice-versa*, varying (4) with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu} = \frac{1}{\psi}\left(\nabla_\mu \nabla_\nu \psi - g_{\mu\nu}\Box\psi - \frac{V}{2}\, g_{\mu\nu}\right) + \frac{\kappa}{\psi}\, T_{\mu\nu} \tag{6}$$

while the variation with respect to ϕ yields

$$R\frac{d\psi}{d\phi} - \frac{dV}{d\phi} = (R - \phi)\, f''(\phi) = 0 \tag{7}$$

and $\phi = R$ under the assumption $f'' \neq 0$ (in fact, local stability requires $f''(R) > 0$ [28,76–79]). Therefore, the (massive) scalar field $\phi = R$ is a dynamical degree of freedom which satisfies the trace equation

$$3f''(\phi)\Box\phi + 3f'''(\phi)\nabla^\sigma\phi\nabla_\sigma\phi + \phi f'(\phi) - 2f(\phi) = \kappa\, T^\mu{}_\mu \tag{8}$$

It is more convenient to consider $\psi \equiv f'(\phi)$ instead of ϕ; then, ψ satisfies the equation

$$3\Box\psi + 2U(\psi) - \psi\frac{dU}{d\psi} = \kappa\,T^\mu{}_\mu \tag{9}$$

with $U(\psi) = V(\phi(\psi)) - f(\phi(\psi))$, and the action

$$I = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,[\psi R - U(\psi)] + I^{(matter)} \tag{10}$$

is clearly that of a Brans-Dicke theory with Brans-Dicke parameter $\omega = 0$ [36].

In the Palatini approach, both the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha_{\mu\nu}$ are treated as independent variables, *i.e.*, the connection is not the metric connection of $g_{\mu\nu}$. The metric and Palatini variations produce the same field equations in GR [80] and in Lovelock gravity [81] but not for general Lagrangians non-linear in R.

The Palatini action

$$I_{Palatini} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f\left(\mathcal{R}\right) + I^{(matter)}\left[g_{\mu\nu}, \psi^{(m)}\right] \tag{11}$$

implicitly contains two Ricci tensors: the usual $R_{\mu\nu}$ constructed with the metric connection of the physical metric $g_{\mu\nu}$, and $\mathcal{R}_{\mu\nu}$ constructed with the non-metric connection $\Gamma^\alpha_{\mu\nu}$. $\mathcal{R}_{\mu\nu}$ gives rise to the scalar $\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}$. The matter Lagrangian does not depend explicitly from the connection Γ, but only from the metric and the matter fields $\psi^{(m)}$.

The variation of the Palatini action (11) produces the field equation

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{f(\mathcal{R})}{2}\,g_{\mu\nu} = \kappa\,T_{\mu\nu} \tag{12}$$

in which there are no second covariant derivatives of f', in contrast with Equation (3). Varying with respect to the independent connection yields

$$\bar{\nabla}_\alpha\left(\sqrt{-g}\,f'(\mathcal{R})g^{\mu\nu}\right) - \bar{\nabla}_\sigma\left(\sqrt{-g}\,f'(\mathcal{R})g^{\sigma(\mu)}\right)\delta^{\nu)}_\alpha = 0 \tag{13}$$

where $\bar{\nabla}_\mu$ denotes the covariant derivative of the non-metric connection Γ. The trace of equations (12) and (13) yields

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa\,T^\mu{}_\mu \tag{14}$$

and

$$\bar{\nabla}_\gamma\left(\sqrt{-g}\,f'(\mathcal{R})g^{\mu\nu}\right) = 0 \tag{15}$$

$f'(\mathcal{R})$ is non-dynamical, contrary to the scalar degree of freedom of metric $f(R)$ gravity. It is possible to eliminate completely the non-metric connection from the field equations, which are then rewritten as

$$\begin{aligned}
G_{\mu\nu} &= \frac{\kappa}{f'}\,T_{\mu\nu} - \frac{1}{2}\left(R - \frac{f}{f'}\right)g_{\mu\nu} + \frac{1}{f'}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' \\
&\quad - \frac{3}{2(f')^2}\left[\nabla_\mu f'\nabla_\nu f' - \frac{1}{2}g_{\mu\nu}\nabla_\gamma f'\nabla^\gamma f'\right]
\end{aligned} \tag{16}$$

To see the equivalence with a Brans-Dicke theory, proceed as in the metric formalism: Introduce $\phi = \mathcal{R}$ and $\psi \equiv f'(\phi)$ in the action (11). Apart from a boundary term which can be discarded, the action is rewritten in terms of $g_{\mu\nu}$ and $R_{\mu\nu}$ as (see [28] and the references therein for details)

$$I_{Palatini} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[\psi R + \frac{3}{2\psi} \nabla^\mu \psi \nabla_\mu \psi - V(\psi) \right] + I^{(matter)} \tag{17}$$

This is the action of a Brans-Dicke theory with Brans-Dicke parameter $\omega = -3/2$ and non-vanishing potential for ψ [36].

2. Scalar-tensor Gravity

Black hole entropy in Brans-Dicke gravity was analyzed by Kang [82] following numerical studies demonstrating that during the collapse of dust to black holes in this class of theories the area law valid in Einstein's theory (*i.e.*, the horizon area can never decrease) is violated [83–86]. Kang realized that the problem is not in the area law itself but rather in the expression of the black hole entropy, which is not simply one quarter of the area in these theories. The expression for the entropy is rather

$$S_{BH} = \frac{1}{4} \int_\Sigma d^2x \sqrt{g^{(2)}} \, \phi = \frac{\phi A}{4} \tag{18}$$

where ϕ is the Brans-Dicke scalar and $g^{(2)}$ is the determinant of the restriction $g^{(2)}_{\mu\nu} \equiv g_{\mu\nu} \mid_\Sigma$ of the metric $g_{\mu\nu}$ to the horizon surface Σ. This expression can be understood by the simple replacement of the Newton constant G with the effective gravitational coupling

$$G_{eff} = \phi^{-1} \tag{19}$$

in Brans-Dicke theory [82] so that $S_{BH} = A/4G_{eff}$. The quantity S_{BH} is non-decreasing. This philosophy of replacing the gravitational coupling with the effective gravitational coupling that would appear if one were to rewrite the field equations of the theory as effective Einstein equations and read scalar field (or, in $f(R)$ gravity, geometric) terms as an effective form of matter, carries over to more general scalar-tensor gravities and to other gravitational theories. The expression (18) has now been derived using various procedures [43,58,62].

Following [82], one can also consider the Einstein frame representation of Brans-Dicke theory given by the conformal rescaling of the metric

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu} \quad \Omega = \sqrt{G\phi} \tag{20}$$

accompanied by the scalar field redefinition $\phi \to \tilde{\phi}$ with $\tilde{\phi}$ given by

$$d\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi G}} \frac{d\phi}{\phi} \tag{21}$$

The Brans-Dicke action [36]

$$I_{BD} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[\phi R - \frac{\omega}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + \mathcal{L}^{(m)} \right] \tag{22}$$

assumes the Einstein frame form

$$I_{BD} = \int d^4x \, \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \, \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(m)}}{(G\phi)^2} \right] \tag{23}$$

where a tilde denotes Einstein frame (rescaled) quantities and

$$U\left(\tilde{\phi}\right) = \frac{V(\phi(\tilde{\phi}))}{\left(G\phi(\tilde{\phi})\right)^2} \tag{24}$$

with $\phi = \phi(\tilde{\phi})$. In the Einstein frame the gravitational coupling is a true constant but matter couples explicitly to the scalar field and what were massive test particles following timelike geodesics of the Jordan frame metric $g_{\mu\nu}$ do not follow geodesics of the rescaled metric $\tilde{g}_{\mu\nu}$. Null geodesics are left unchanged by the conformal rescaling, as well as null vectors and all forms of conformally invariant matter. A black hole event horizon, being a null surface, is also unchanged. The area of an event horizon is not, and the change in the entropy formula $S_{BH} = \frac{A}{4G} \to \frac{A}{4G_{eff}} = \frac{\phi A}{4}$ can be understood as the change in the area due to the conformal rescaling of $g_{\mu\nu}$. In fact, $\tilde{g}^{(2)}_{\mu\nu} = \Omega^2 \, g^{(2)}_{\mu\nu}$ and, since the horizon surface is unchanged, the Einstein frame area is

$$\tilde{A} = \int_\Sigma d^2x \, \sqrt{\tilde{g}^{(2)}} = \int_\Sigma d^2x \, \Omega^2 \, \sqrt{g^{(2)}} = G\phi \, A \tag{25}$$

assuming that the scalar field is constant on the horizon (if this is not true the surface gravity is unlikely to be constant according to any sensible definition and the zeroth law of black hole thermodynamics fails). Therefore, the entropy-area relation $\tilde{S}_{BH} = \tilde{A}/4G$ still holds in the Einstein frame. This should be expected since, *in vacuo*, the theory reduces to general relativity with varying units of length $\tilde{l}_u \sim \Omega \, l_u$, time $\tilde{t}_u \sim \Omega \, t_u$, and mass $\tilde{m}_u = \Omega^{-1} \, m_u$ (where $t_u, l_u,$ and m_u are the constant units of time, length, and mass in the Jordan frame, respectively), and derived units vary as well [87]. Then, an area scales as $A \sim \Omega^2 = G\phi$ and, in units in which $c = \hbar = 1$ the entropy is dimensionless and is not rescaled. Therefore, the Jordan frame and Einstein frame entropies coincide (a point noted in [82]):

$$\tilde{S} = \frac{\tilde{A}}{4G_{eff}} = \frac{A}{4G} = S \tag{26}$$

The equality between black hole entropies in the Jordan and Einstein frames is not restricted to scalar-tensor gravity but extends to all theories with action $\int d^4x \sqrt{-g} \, f\left(g_{\mu\nu}, R_{\mu\nu}, \phi, \nabla_\alpha \phi\right)$ which admit an Einstein frame representation [88].

A consequence of this equivalence which is worth noting is that the Jordan and the Einstein frames turn out to be physically equivalent again. A debate on whether these two frames are physically equivalent has been going on for years and the issue still causes frequent confusion. It seems now established (although many authors may disagree with this statement) that, at the classical level, the two frames are merely different representations of the same physics (see the discussion in [87,89,90]). There are potential problems arising from the fact that many fundamental properties of physical theories, including the cherished Equivalence Principle, turn out to be dependent on the conformal representation adopted, which means that the fundamental properties of gravitational theories should

be reformulated in a representation-independent manner ([91] and references therein). The classical equivalence is expected to break down at the quantum level, in the same way that the quantization of Hamiltonians related by canonical transformations produce inequivalent energy spectra and eigenfunctions [92–94]. While quantum gravity is expected to definitely break the equivalence between conformal frames, the situation is not so clear at the semiclassical level [89]. Black hole thermodynamics is not purely classical: What ultimately makes it meaningful is the discovery of Hawking radiation, a semiclassical phenomenon. It is widely believed that black holes open a window onto quantum gravity and therefore, implicitly, that at least some aspects of their thermodynamics will be preserved and derived theoretically in the quantum gravity regime. It is therefore significant that the physical equivalence between conformal frames holds for (semiclassical) black hole entropy. Whatever gravity knows about black hole thermodynamics and the underlying microscopic statistical mechanics, it seems to know also about the equivalence between conformal frames.

Another consequence of the discussion above using the Einstein frame representation of Brans-Dicke (and, by extension, of more general scalar-tensor) gravity is that if the scalar field vanishes on the horizon of a Brans-Dicke black hole the latter is attributed zero temperature and, in the light of the previous considerations, zero entropy. These black holes with vanishing ϕ, dubbed "cold black holes", have been the subject of a not insignificant amount of literature [95–105].

If ϕ diverges on the horizon of scalar-tensor black holes, the entropy is infinite there, which seems to rule out the possibility that $\phi \to \infty$ as the horizon is approached. Interestingly, scalar hair always seems to vanish or diverge on the black hole horizon, making the no-hair theorems all the more plausible in scalar-tensor gravity. On the other hand, in the context of Brans-Dicke theory (with zero scalar field potential), there is a well-known theorem by Hawking [106] stating that, barring the situations of scalar field vanishing or diverging on the horizon, all stationary Brans-Dicke black holes are the same as in general relativity in the sense that the scalar field becomes constant outside the horizon and, in this situation, the theory reduces to general relativity. (Note that the limit of scalar-tensor gravity to general relativity as ϕ becomes constant and the explicit dependence of ϕ on the Brans-Dicke parameter ω in this limit are not entirely trivial [84,107–117].) This conclusion is corroborated by numerical studies of black hole collapse in Brans-Dicke gravity [83–86]. According to the discussion above, it seems that these black holes should be discarded as pathological from the thermodynamical point of view, which makes the black holes of general relativity the only possible final state of equilibrium in Brans-Dicke theory. (The use of entropic considerations to select correct gravity theories among the class of metric $f(R)$ models is advocated, e.g., in [66,118].) It seems that, discarding cold black holes and those with diverging horizon entropy, Hawking's theorem [106] should be extendible to all scalar-tensor theories (work is in progress on this subject).

3. Metric $f(R)$ Gravity

It is by now well known that the area formula $S_{BH} = A/4G$ gets corrected as

$$S_{BH} = \frac{f'(R)A}{4G} \tag{27}$$

in metric $f(R)$ gravity [67,119,120,122,124].

The Noether charge method was applied to metric $f(R)$ gravity on various occasions [67,119,120], usually by assuming that the black hole configuration is static and working in D spacetime dimensions. A common result of these studies is that the usual entropy-area relation is still valid provided that Newton's constant G is replaced by a suitable effective gravitational coupling G_{eff}. The identification of G_{eff} with $G/f'(R)$ in metric $f(R)$ gravity is straightforward based on inspection of the action, or of the field equations of the theory rewritten in the form of effective Einstein equations. Those who find this identification too naive should look at the recent work of Brustein and collaborators [67], in which the identification of G_{eff} is made by using the matrix of coefficients of the kinetic terms for metric perturbations [67]. The metric perturbations contributing to the Noether charge in Wald's formula and its generalizations are identified with specific metric perturbation polarizations associated with fluctuations of the area density on the bifurcation surface Σ of the horizon (this is the $(D-2)$-dimensional spacelike cross-section of a Killing horizon on which the Killing field vanishes, and coincides with the intersection of the two null hypersurfaces comprising this horizon). The horizon entropy is

$$S_{BH} = \frac{A}{4G_{eff}} \tag{28}$$

for a theory described by the action

$$I = \int d^4x \, \sqrt{-g} \, \mathcal{L} \left(g_{\mu\nu}, R_{\alpha\beta\rho\lambda}, \nabla_\sigma R_{\alpha\beta\rho\lambda}, \phi, \nabla_\alpha \phi, ... \right) \tag{29}$$

where ϕ is a gravitational scalar field. The Noether charge is

$$S = -2\pi \int_\Sigma d^2x \, \sqrt{g^{(2)}} \left(\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \right)_{(0)} \hat{\epsilon}_{\mu\nu} \hat{\epsilon}_{\rho\sigma} \tag{30}$$

where $\hat{\epsilon}_{\rho\sigma}$ is the (antisymmetric) binormal vector to the bifurcation surface Σ normalized to $\hat{\epsilon}^{ab} \hat{\epsilon}_{ab} = -2$ and the subscript (0) denotes the fact the quantity in brackets is evaluated on solutions of the equations of motion (on the bifurcation surface Σ, the binormal satisfies $\nabla_\mu \chi_\nu = \hat{\epsilon}_{\mu\nu}$, where χ^μ is the Killing field vanishing on the horizon). The effective gravitational coupling is then calculated to be [67]

$$G_{eff}^{-1} = -2\pi \left(\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \right)_{(0)} \hat{\epsilon}_{\mu\nu} \hat{\epsilon}_{\rho\sigma} \tag{31}$$

These prescriptions apply not only to $f(R)$ gravity but to other theories described by the action (29) as well.

For metric $f(R)$ gravity described by the Lagrangian density $\mathcal{L} = f(R)$ this prescription yields $G_{eff} = G/f'(R)$ and the entropy (27). This calculation is consistent with the description of metric $f(R)$ gravity as a scalar-tensor theory with a massive scalar degree of freedom $f'(R)$ and with the corresponding equation. (18) of scalar-tensor gravity.

4. Palatini $f(R)$ Gravity

Wald's Noether charge method can be applied also to Palatini $f(\mathcal{R})$ gravity. In a slightly different notation from the one used earlier, the entropy of a black hole static horizon is given by the (local) Noether charge

$$S_{BH} = \frac{2\pi}{\kappa_g} \int_\Sigma \mathbf{Q} \tag{32}$$

where the $(D-2)$-form **Q** is the Noether potential associated with the diffeomorphisms of the spacetime manifold, Σ is the bifurcation surface of the black hole, and κ_g is the surface gravity on the horizon (the entropy (32) does not change when calculated on any cross-section of the horizon [43]). Vollick [65] considered D-dimensional Palatini $f(R)$ gravity described by the action

$$I_{Palatini} = \int d^D x \sqrt{-g} \left[\frac{f(\mathcal{R})}{16\pi G} + \mathcal{L}^{(m)} \right] \tag{33}$$

Palatini $f(\mathcal{R})$ gravity *in vacuo* is equivalent to general relativity with a cosmological constant and the entropy of a stationary black hole is found to be

$$S_{BH} = \frac{f'(\mathcal{R}) A}{4G} \tag{34}$$

In the presence of matter it is useful to consider the trace (14) of the field equations which is an algebraic (or transcendental) equation, not a differential equation. This fact reflects the non-dynamical nature of the scalar $f'(\mathcal{R})$ present in the theory and has been emphasized many times in the literature (cf. the references in [28,29]). Using equation (14), when the trace $T^\mu{}_\mu$ is constant (and, in particular, for conformally invariant matter for which $T^\mu{}_\mu = 0$), it is possible to eliminate the Ricci curvature $\mathcal{R}$ in terms of $T^\mu{}_\mu$, which becomes a constant and also $f'(\mathcal{R})$ is constant. Then, the theory is again equivalent to general relativity with a cosmological constant, as described by the field equations which can be rewritten as [65]

$$G_{\mu\nu}[g_{\alpha\beta}] = \frac{8\pi G}{f'} T_{\mu\nu} - \left(\frac{D-2}{2D} \right) R g_{\mu\nu} \tag{35}$$

Recasting the field equations in this way allows one to identify the effective gravitational coupling of the theory

$$G_{eff} = \frac{G}{f'(\mathcal{R})} \tag{36}$$

and consequently the black hole entropy given by the Noether charge corresponds simply to the familiar expression with Newton's constant G replaced by G_{eff} or [65]

$$S_{BH} = \frac{A}{4G_{eff}} = \frac{f'(\mathcal{R}) A}{4G} \tag{37}$$

In the presence of matter with non-constant trace $T^\mu{}_\mu$ the situation is more complicated and the black hole entropy depends on the ratio of the effective gravitational couplings on the horizon and at spatial infinity:

$$S_{BH} = \frac{f'_\Sigma}{f'_\infty} \frac{A}{4G} \tag{38}$$

where f'_Σ is the value of $f'(\mathcal{R})$ on the horizon and f'_∞ is the value far away from the black hole [65].

5. Dilaton Gravity (Metric and Palatini)

Theories with action

$$I = \int d^D x \frac{\sqrt{-g}}{16\pi G} f\left(g^{\mu\nu}, R_{(\mu\nu)} \right) \tag{39}$$

were considered in Reference [65]. The variation with respect to $g^{\mu\nu}$ yields the field equations

$$\frac{\partial f}{\partial g^{\mu\nu}} - \frac{f}{2}\, g_{\mu\nu} = 0 \tag{40}$$

while the variation with respect to the independent connection Γ yields

$$\bar{\nabla}_\alpha \left[\sqrt{-g}\, \frac{\partial f}{\partial \mathcal{R}_{(\mu\nu)}} \right] = 0 \tag{41}$$

Again, the *vacuum* theory is equivalent to general relativity with an effective cosmological constant, *i.e.*, the effective matter tensor due to the geometric terms has the form

$$\frac{\partial f}{\partial \mathcal{R}_{(\mu\nu)}} = \lambda g_{\mu\nu} \tag{42}$$

where λ is a constant, the effective gravitational coupling is $G_{eff} = G/\lambda$, and the black hole entropy is [65]

$$S_{BH} = \frac{A}{4G_{eff}} = \frac{\lambda A}{4G} \tag{43}$$

A dilaton gravity in the Palatini approach, described by

$$I = \int d^D x \sqrt{-g}\, \mathrm{e}^{-2\phi} \left[\frac{f(\mathcal{R})}{16\pi G} - \frac{\alpha}{2}\, g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] \tag{44}$$

was also studied in Reference [65]. Here α is a constant (corresponding to $\sim G^{-1}$ in string theory) and ϕ is the dilaton field. The equations of motion

$$f'(\mathcal{R})\, \mathcal{R}_{(\mu\nu)} - \frac{f(\mathcal{R})}{2}\, g_{\mu\nu} = 8\pi G\alpha \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}\, g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right), \tag{45}$$

$$\bar{\nabla}_\alpha \left(\sqrt{-g}\, \mathrm{e}^{-2\phi}\, f' g^{\mu\nu} \right) = 0 \tag{46}$$

lead to the identification of the effective gravitational coupling $G_{eff} = G/f'_\infty$ and the Noether charge entropy is [65]

$$S_{BH} = \left[\mathrm{e}^{-2\phi} f' \right]_\Sigma \frac{A}{4G} = \mathrm{e}^{-2\phi} \big|_\Sigma \frac{f'_\Sigma}{f'_\infty} \frac{A}{4G_{eff}} \tag{47}$$

A similar dilaton gravity in the metric formalism, with action

$$I = \int d^D x \sqrt{-g}\, \frac{\mathrm{e}^{-2\phi} R}{16\pi G} \tag{48}$$

was considered in Reference [67]. In spite of the differences between the Palatini and the metric formalisms, the result is the same: Brustein and collaborators find again [67]

$$S_{BH} = \frac{\mathrm{e}^{-2\phi} \big|_\Sigma A}{4G} \tag{49}$$

6. Conclusions

The entropy-area relation $S = A/4G$ familiar from general relativity is still valid in both metric and Palatini $f(R)$ gravities and in the much larger class of scalar-tensor theories provided that Newton's constant is replaced by a suitable effective gravitational coupling strength G_{eff} (*i.e.*, ϕ^{-1} in scalar-tensor gravity, or G/f' in modified gravities). Black hole entropy has been studied in quantum gravity-inspired theories that depart more radically than scalar-tensor ones from Einstein theory: for example, entropy in Horava-Lifshitz gravity is given by a more complicated expression [121]. The thermodynamics of cosmological horizons is also of interest, see References [120,122–128] for discussions in the context of metric $f(R)$ gravity. We have not discussed studies of the Bekenstein-Hawking entropy in Lovelock [50,55,57,129,130] and Gauss-Bonnet [48–51] gravity, or in theories with Lorentz violation in which thermodynamical considerations have been claimed to allow for the possibility of perpetual motion machines of the second kind [131]—see also [132,133]. This claim has been reconsidered and shown to be invalid in tensor-vector-scalar (TeVeS) theories in Reference [134]. This debate shows once again that black hole thermodynamics in alternative theories of gravity can be quite interesting and may be used in the future to constrain the class of effective actions inspired by low-energy quantum gravity.

Acknowledgements

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